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Investigation of simply periodic motions of a heavy solid with a fixed point

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Abstract

Some simply periodical motions of a heavy solid body with $A = B = 3C$ inertia principal moments' ratio near the fixed point in the uniform gravitational field are studied, employing the method of point mappings. The order of motion equations' system is reduced using the Routh method, and the equations are reduced to the form, analogous to the equations of the restricted three bodies problem. Three classes of the revealed simply periodical trajectories are described.

Analytical methods, such as the method of small parameter [4], for example, are widely used nowadays for the analysis of the non-integrable cases of motion of a solid body with a fixed point. The numerical methods, realized on the computer, are employed also.

At the investigation of motions of a dynamic system, its mathematical model W is based on the concept of its state w and operator T, determining its change in time. The current state w of system W may be considered as some point of the system phase space F.

Various aspects of the dynamic system phase trajectories may be elucidated by the examination of the arrangement of points of their crossing with some surface of section M in the phase space F [5]. This sequence characterizes the point transformation T. In other words, the study of system motion is reduced to the consideration of the behaviour not of the whole phase trajectory, but of some of its points on the surface of section. As a result the investigation of motion trajectories becomes, to some extent, more clear and simple. This method is rather effective at the study of system dynamics and is called the method of point mappings, and its origin is connected with H.Poincare name.

The investigation of the dynamic system behaviour in phase space is the simpler the less is the dimensions' number it is characterized by. At present the most simple case, which is in principle reduced to quadrature - the case of two-dimensional systems is studied sufficiently. As for multi-dimensional systems of differential equations, there is little information about them and the elaboration of the theory has been developed intensively during the last decades [6].

The numerical methods were successfully employed even at the investigation of Copenhagen variant of the restricted three-bodies problem. Particularly T.N.Thiele, E.Stromgren and others [7] used them to study the genealogy of classes of simply periodic orbits. In sixties M. Henon [5] and J.Bartlett [2] applied the method of point mappings to the study of this problem and systematized on its base a lot of results, published by the collaborators of Copenhagen observatory, adding some new classes.

In this paper we apply the method of point mappings to the research of simply periodic motions of a heavy body near a fixed point. The Euler-Poisson differential equations describing the rigid body motions have the form [1]:

$$\begin{gathered} A\dot{p} + (C - B)qr = Mg(e_1\gamma_3 - e_3\gamma_3), \\ \dot{\gamma}_1 = r\gamma_2 - q\gamma_3, \\ (A, B, C, p, q, r, 1, 2, 3), \end{gathered} \tag{1}$$

where: A, B, C is the principal moment of inertia;
e_1, e_2, e_3 are the coordinates of the center of mass;
p, q, r are the components of angular velocity;
$\gamma_1, \gamma_2, \gamma_3$ are vertical guiding cosines;
Mg is the body weight.

The motion equations (1) form the six-order system to solve for the variables $\gamma_1, \gamma_2, \gamma_3, p, q, r$, depending on Euler angles: ψ is the precession angle, θ is the nutation angle and φ is the angle of the proper rotation. In the general case they assume the energy integral with constant h, the area integral with constant f, and geometrical relation.

The complete system of first integrals of Euler-Poisson equations was found only for three cases. Kovalevckaya case is one of them, and our case we consider to be its perturbation. It refers to the body having unity weight with inertia ellipsoid for which $A = B = 3C = 1$, and the center of mass is situated on its abscissa axis at the unity distance from the fixed point.

Taking into account that the precession angle ψ is an ignorable coordinate, it might be excluded by Routh's method reducing by two the order of the system of motion equations, and we may pass to coordinates x and y on the inertia ellipsoid [3]. According to work [4], we bring system (1) at $A = B$ to the reduced dynamic system with two degrees of freedom, the motion of which is described as the unity mass point motion in the plane under the action of conservative and gyroscopic forces:

$$\begin{aligned} x'' &= -\Omega y' + \frac{\partial U}{\partial x}, \\ y'' &= \Omega x' + \frac{\partial U}{\partial y}, \end{aligned} \tag{2}$$

where:

$$\Omega = \frac{f\kappa}{A\sqrt{AC}}\sqrt{(1+m\rho^2)}\Big[C + 2(A-C)\rho^2\Big],$$

ρ - real root of the transcendental equation:

$$arth\rho + \mu\ arctg\,\mu\rho = \sqrt{\frac{A}{C}}\ y,$$

$$\kappa = A[1-\rho^2], \qquad m = \frac{A-C}{C}, \qquad \mu^2 = \frac{A-C}{A}.$$

Here the derivative with respect to the variable τ, for which $dt = \kappa d\tau$, is denoted by the prime.

In this case the step to Euler angles φ and θ is conducted directly by formulae:

$$\varphi = \sqrt{\frac{A}{C}}\, x, \tag{3}$$

$$\theta = arctg\left[\sqrt{\frac{C}{A}}\ \ \frac{\sqrt{1-\rho^2}}{\rho}\right], \tag{4}$$

System (2) assumes Jacobi integral

$$x'^2 + y'^2 = 2U. \tag{5}$$

System (2) structure coincides with the structure of equations'system for the motion of the restricted three-bodies problem [7], and, using Jacobi integral, its solutions could be presented in three-dimensional phase space (x, y, x').

Note, that the motion equation (2), as the integral (5), are invariant under substitution

$$x \to x, \qquad t \to t, \qquad y \to -y, \qquad f \to -f.$$

That's why we can choose the plane (x, x') as surface of section M [5]. Such a choice is justified by the fact that φ and x differ by a constant factor only.

Let's define the point mapping, caused by the system (2) solutions, in the following way. We'll choose a point P_0 (Fig.1) in plane (x, x') as an initial one and find the following crossing P_1 of the phase trajectory with this plane. The point P_1, after H. Poincare, is called the consequent of the point P_0.

The problem of search of the system periodical trajectories reduces to determination of the mapping T invariant points' location. The system periodical trajectory, closed after $2n$ crossings with abscissa axis, will be presented in the section plane by the set of n consequent points. In the simplest case at $n = 1$ the trajectory is called simply periodical [7].

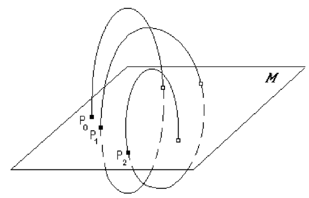

Figure 1

If one periodical trajectory is revealed, then in conformity with the theorem of a continuous dependence of ordinary differential equations' system solutions on initial condititions, for f and h, not greatly different from the original parameters of the system, it is possible to find out such x_0, that correspond to the periodical motion, neighbouring to the given one.

The search of the reduced system periodical trajectories was accomplished with the help of a personal computer by the use of a program, set up for the study of dynamic system motions by the method of point mappings, for initial x_0 and velocity $x'_0 = 0, y' > 0$.

When classifying the discovered trajectories, we assume the tracing of stationary points S_1 and S_2 on Poisson sphere, which are the analogues of libration points in celestial mechanics, as one of the criteria.

In accordance with the relation (3) in order the analysis of the results to be more suitable, the revealed periodical trajectories were recalculated for system (1) with corresponding going to Euler angles φ and θ. The value of the angle $\theta = \pi/2$ conforms the value $y_0 = 0$ of the reduced system, therefore in figures 2 and 3 the trajectories are presented in φ and $\theta_1 = \pi/2 - \theta$ coordinates. The values of angles are given in radians in the tables and in degrees - in figures.

The values of constant integrals of energy h and areas f, the initial value φ_0 and the value φ_1, for the second crossing of the trajectory in configuration space with abscissa axis, and the half-period $T/2$ quantity as well, corresponding to real time t are adduced in applied tables with the aim of characterizing the classes of the revealed simply periodical trajectories.

Here we'll describe the characteristics of three classes, discovered at constant value $f = 0.1$ of simply periodical trajectories.

Simply periodical trajectories of the $first$ class were detected at $-0.9 < h < 0$. Their characteristics are presented in Tabl.1, and their configurations - in Fig.2. All this class trajectories are straight lines and are situated in the vicinity of the stationary point S_2 . They are oval for most values of energy integral constant h in the plane (φ, θ_1). When the value of h increases, these ovals gradually become curves, possessing self-crossings, the sizes and motion periods grow.

Periodical trajectories of the *second* and *third* classes were found out in the range $2. < h < 6$. The characteristics of these classes are offered in Tabl.2 and their configurations - in Fig.3. Simply periodical trajectories of the both classes are straight lines and are situated in the vicinity of the stationary point S_2 and have the oval form without any self-crossings. With the increase of the constant h the small semi-axes of the trajectories of the second-class move slightly to the right along φ, and the trajectories of the third class - to the left, preserving their configuration. As for large semi-axes along θ_1, they have the same deviations and motion periods. The location of the second and third classes'trajectories is such, that they coincide after turn for 180° in plane (φ, θ_1) in respect to the stationary point S_2.

Although the search was limited by the classes of simply periodical motions, we've found out the trajectories of other classes. We are going to conduct additional investigations with the aim of their study.

Table 1

Class I				
f	h			T/2
0.1	-0.9	-1.54681	-1.59478	3.17126
0.1	-0.8	-1.53279	-1.60880	3.21298
0.1	-0.7	-1.51903	-1.62256	3.25734
0.1	-0.6	-1.50439	-1.63720	3.30466
0.1	-0.5	-1.48833	-1.65326	3.35537
0.1	-0.4	-1.47044	-1.67116	3.40997
0.1	-0.3	-1.45037	-1.69122	3.46912
0.1	-0.2	-1.42784	-1.71375	3.53361
0.1	-0.1	-1.40259	-1.73899	3.60449
0.1	0.0	-1.37445	-1.76713	3.68312

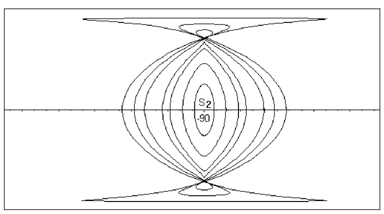

Figure 2

Table 2

f	h	Class II		Class III		T/2
0.1	2.0	-1.01289	-4.46296	1.32136	-2.12871	1.66140
0.1	2.2	-0.99831	-4.43270	1.29110	-2.14329	1.56736
0.1	2.4	-0.98358	-4.40458	1.26295	-2.15802	1.48905
0.1	2.6	-0.96877	-4.37809	1.23650	-2.17281	1.42229
0.1	2.8	-0.95392	-4.35296	1.21139	-2.18765	1.36435
0.1	3.0	-0.93914	-4.32903	1.18744	-2.20248	1.31335
0.1	3.2	-0.92438	-4.30606	1.16445	-2.21722	1.26795
0.1	3.4	-0.90966	-4.28389	1.14224	-2.23201	1.22716
0.1	3.6	-0.89500	-4.26246	1.12082	-2.24665	1.19022
0.1	3.8	-0.88032	-4.24160	1.09997	-2.26131	1.15654
0.1	4.0	-0.86572	-4.22127	1.07959	-2.27596	1.12566
0.1	4.2	-0.85119	-4.20143	1.05979	-2.29046	1.09719
0.1	4.4	-0.83665	-4.18197	1.04034	-2.30500	1.07084
0.1	4.6	-0.82210	-4.16286	1.02126	-2.31950	1.04634
0.1	4.8	-0.80758	-4.14407	1.00252	-2.33397	1.02349
0.1	5.0	-0.79315	-4.12562	0.98398	-2.34850	1.00210
0.1	5.2	-0.77858	-4.10728	0.96563	-2.36309	0.98203
0.1	5.4	-0.76413	-4.08930	0.94768	-2.37746	0.96313
0.1	5.6	-0.74962	-4.07141	0.92981	-2.39198	0.94531
0.1	5.8	-0.73501	-4.05362	0.91208	-2.40657	0.92845
0.1	6.0	-0.72050	-4.03611	0.89449	-2.42110	0.91247

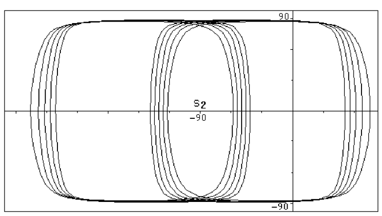

Figure 3

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