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# The problem of the synthesis of a transport network with a single source and the algorithm for its solution

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**Abstract**

The problem of the synthesis of a transport network with a single source is under consideration and the combinatorial algorithm for its solution is proposed.

## 1 Introduction

The problem of the synthesis of a transport network with a single source, where cost functions of flows along the network edges are concave functions is considered in the paper presented. This problem is a generalization of the problems of optimal forest finding in oriented graphs with weight [1]-[3], and has a direct application in research and solving problems of the allocation of points of production and transportation planning in transport networks [4]. For this problem, solving a combinatorial algorithm based on the analysis of optimal flows in the transport network with a single source and a generation of admissible trees with a given source which corresponds to admissible flows in the network is proposed.

## 2 Problem formulation

Let us consider a transport network with a single source and with the cost functions of flows along edges, which is defined in the following way:

An oriented graph $G = (V, E)$ with a set of vertices $V, |V| = n$ and with a set of edges $E, |E| = m$ is given. The graph $G$ has such a property that for each vertex, $v \in V \setminus \{v_0\}$ at least one oriented path exists $P(v_0, v)$ from the source $v_0 \in V$ toward the vertex $v \in V$. For the set of vertices $V \setminus \{v_0\}$ a non-negative function $p : V \setminus \{v_0\} \to R$ is defined; in other words, with every vertex $v \in V \setminus \{v_0\}$ a non-negative number $p(v)$ is associated, which can be interpreted as the corresponding number of consumptions for one consumer allocated in the vertex $v \in V \setminus \{v_0\}$. The source $v_0 \in V$ is associated with the number

$$q(v_0) = \sum_{v \in V \setminus \{v_0\}} p(v)$$

which is interpreted as the whole number of production that is necessary for consumers $v \in V \setminus \{v_0\}$ and must be transported from $v_0$ to consumers along edges of the oriented graph $G$.

It is also supposed that every edge $e = (u, v) \in E$ is associated with a function $\Psi_e(x(e))$ which numerically expresses transportation expenses for the amount of production $x(e)$ along the edge $e \in E$.

So, a transport network is defined with the help of the oriented graph $G = (V, E)$ and characteristic values $p(v), q(v_0), \Psi_e(x(e)), e \in E$.

Remember that a function $x : E \to R$ is called a flow in the transport network $G$ with a single source and with cost functions $\Psi_e(x(e))$ if:

$$\begin{cases} \sum_{e \in O^+(v)} x(e) - \sum_{e \in O^-(v)} x(e) = \begin{cases} p(v), & v \in V \setminus \{v_0\} \\ -q(v_0), & v = v_0 \end{cases} \\ x(e) \geq 0, \forall e \in E \end{cases} \tag{1}$$

where $O^+(v)$ is the set of edges $e \in E$ which have extremes in the vertex $v$; $O^-(v)$ is the set of all edges $e \in E$ which have origins in the vertex $v \in V$.

A problem of the synthesis of a transport network $G$ with a single source is formulated in the following way: it is necessary to find the flow $x^* : E \to R$ for which the functional value

$$F(x) = \sum_{e \in E} \Psi_e(x(e)) \tag{2}$$

reaches its minimum;

In other words, the flow $x^*$ for which holds:

$$F(x^*) = \min_x \sum_{e \in E} \Psi_e(x(e))$$

## 3 The main results

Let $x : E \to R$ be any flow. It is known [1],[2] that if the graph $G$ has a property that for every $v \in V$, an oriented path $P(v_0, v)$ from $v_0$ toward $v$ exists, then a given transport network has a flow; in the other words, a solution to the system (1) exists. Label by $G_x = (V, E_x)$ a subgraph of the graph $G$, which is generated from the set of all edges $e \in E$ for which $x(e) > 0$.

**Theorem 1** *If the functions $\Psi_e(x(e))$ are concave and non-decreasing, then for the problem of the synthesis of a transport network with a single source such a solution $x^* = (x^*(e_1), x^*(e_2), ..., x^*(e_m))$ exists, for which the corresponding graph $G_{x^*} = (V, E^*)$ has a structure of a tree with a source $v_0$.*

**Proof:** First, we will prove that the subgraph $G_{x^*} = (V, E^*)$ does not contain oriented cycles. Indeed, if we suppose that $G_x^*$ contains oriented cycles, then we can reduce for some value the flow along edges of this cycle and as a result, we take a flow with the functional value (2) not bigger than the functional value for the initial flow.

Consequently, the optimal flow $x^*$ in $G$ exists, for which $G_{x^*}$ does not contain oriented cycles. We will prove that in general (non-oriented cycles) cycles $G_{x^*}$ do not exist. Suppose that $G_{x^*}$ contains oriented cycles $C$ with a set of edges $E_C$ . We will fix a certain direction of the motion. Let $\overrightarrow{E_C}$ denote a set of all edges of this cycle for which the direction of the oriented edges coincides with the direction of the motion along the cycle and let $\overleftarrow{E_C}$ denote a set of all edges of this cycle for which the direction of the edges is oriented to the opposite direction of the fixed motions along the cycle.

$$\overleftarrow{E_C} = E_C \setminus \overrightarrow{E_C}$$

Let us construct the following two flows:

$$x^1(e) = \begin{cases} x^*(e), & \text{for } e \in E \setminus E_C \\ x^*(e) - \theta, & \text{for } e \in \overrightarrow{E_C} \\ x^*(e) + \theta, & \text{for } e \in \overleftarrow{E_C} . \end{cases}$$

$$x^2(e) = \begin{cases} x^*(e), & \text{for } e \in E \setminus E_C \\ x^*(e) + \theta, & \text{for } e \in \overrightarrow{E_C} \\ x^*(e) - \theta, & \text{for } e \in \overleftarrow{E_C} . \end{cases}$$

where $\theta = \min_{e \in E_C}$.

Note, that

$$F(x^1) + F(x^2) \leq 2F(x^*). \quad (3)$$

Indeed,

$$F(x^1) + F(x^2) = \sum_{e\in E} \Psi_e(x^1(e)) + \sum_{e\in E} \Psi_e(x^2(e))$$

$$= \sum_{e\in E\setminus E_C} \Psi_e(x^*(e)) + \sum_{e\in \vec{E_C}} \Psi_e(x^*(e) - \theta)$$

$$+ \sum_{e\in \overleftarrow{E_C}} \Psi_e(x^*(e) + \theta) + + \sum_{e\in E\setminus E_C} \Psi_e(x^*(e))$$

$$+ \sum_{e\in \vec{E_C}} \Psi_e(x^*(e) + \theta) + \sum_{e\in \overleftarrow{E_C}} \Psi_e(x^*(e) - \theta)$$

$$= 2\sum_{e\in E\setminus E_C} \Psi_e(x^*(e)) + \sum_{e\in \vec{E_C}} [\Psi_e(x^*(e) - \theta) + \Psi_e(x^*(e) + \theta)]$$

$$+ \sum_{e\in \overleftarrow{E_C}} [\Psi_e(x^*(e) + \theta) + \Psi_e(x^*(e) - \theta)]$$

$$\leq 2\sum_{e\in E\setminus E_C} \Psi_e(x^*(e)) + 2\sum_{e\in \vec{E_C}} \Psi_e(x^*(e))$$

$$+2\sum_{e\in \overleftarrow{E_C}} \Psi_e(x^*(e)) = 2\sum_{e\in E} \Psi_e(x^*(e)) = 2F(x^*)$$

from that it follows (3).

The flow $x^*$ is optimal, so $F(x^*) \leq F(x^1)$, $F(x^*) \leq F(x^2)$. From these relations and from (3) we take $F(x^*) = F(x^1) = F(x^2)$. It means that the flows $x^1$ and $x^2$ also are the optimal flows. From the constructions of the flows $x^1$ and $x^2$ we note that the number of non-oriented elementary cycles is less than the number of non-oriented elementary cycles in the initial graph $G_{x^*}$ at least in one of the corresponding graphs $G_{x^1}$ and $G_{x^2}$. That flow for which this property takes place we denote by $x_1^*$. If $G_{x_1^*}$ does not have cycles, then the theorem is proved. If $G_{x_1^*}$ contains non-oriented cycles, then according to $x_1^*$ we make that construction and as a result, we take the optimal flow $x_2^*$ for which the corresponding graph $G_{x_2^*}$ contains less non-oriented elementary cycles than $G_{x_1^*}$ etc. Continuing this process, we obtain the optimal flow $x_p^*$ for which the corresponding graph $G_{x_p^*}$ does not have non-oriented cycles. The theorem is proved.

**Consequence.** If functions $\Psi_e(x(e)), \forall e \in E$ are rigorously concave, then each optimal solution $x^*$ has such a property that the corresponding graph $G_{x^*} = (V, E^*)$ has a structure of a tree with a source $v_0^*$.

## 4 The algorithm of the problem solving

The algorithm of the formulated problem solving is based on **Theorem 1**. To find the optimal flow, it is enough to generate possible trees in the transport network and for each tree to calculate the flow along edges. It is a fact that it is easy enough to determine a flow along edges by implementing $O(n)$ elementary operations [5]. Knowing the flow for the edges given, we determine a flow along $G$ and calculate the functional value (3). Generating all the trees with a source $v_0$ in $G$, we calculate all the corresponding values of the functional value (3) and after this choose that flow for which the functional value is the least.

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