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Algorithms for generalized fractional programming

D.Solomon V.Pluta

1 Problem formulation

One of the well-known optimization problems is the problem of discrete Min-Max which consists of minimization of a finite number of functions. Consider real-valued functions $\varphi_i(x), i \in I,\ I = \{1, 2, \ldots, m\}$. Then discrete Min-Max problem consists of minimization of the function $f_0(x) = \max_{i \in I} \varphi_i(x)$ where $x \in \mathbb{R}^n$ or $x \in S$, S being a compact set from $\mathbb{R}^n$. But $f_0(x)$ may be defined by maximizing the ratio of two classes of functions $\{\varphi_i(x)\}, i \in I$ and $\{\psi_i(x)\}, i \in I$. Then we shall get a fractional problem of discrete Min-Max

$$\min_{x \in S} \left\{ f(x) = \max_{i \in I} \frac{\varphi_i(x)}{\psi_i(x)} \right\}.$$

The mentioned problem is also encountered in multicriterial programming, where several ratios are to be optimized simultaneously and the overall objective is to minimize the largest of this ratios. Let determine the Pareto optimal point $x^* \in S$ in terms of m given fractional criteria as

$$f_i(x) = \frac{\varphi_i(x)}{\psi_i(x)} \longrightarrow \min .$$

Thus we get the fractional problem of discrete Min-Max

$$\min_{x \in S} \max_{i \in I} f_i(x) = \min_{x \in S} \max_{i \in I} \frac{\varphi_i(x)}{\psi_i(x)}.$$

2 The fractional convex problem of discrete Min-Max

Consider the following fractional convex problem of discrete Min-Max

$$f(x) = \max_{i \in I} \frac{\varphi_i(x)}{\psi_i(x)} \longrightarrow \min \tag{1}$$

$$h_k(x) \leq 0, \quad k = \overline{1,p} \tag{2}$$

where the functions $\{\varphi_i(x), i \in I\}, \{\psi_i(x), i \in I\}$ and $h_k(x),\ k = \overline{1,p}$ are continuous, convex and differentiable on E^n.

The minimization problem of $f_0(x)$ defined by taking maximum of a finite number of functions is called the problem of discrete Min-Max and has been investigated quite actively [1-7]. Among various algorithm modifications of mathematical and convex programming to deal with it must be mentioned the method of quick descent [3], the method of consecutive approximation [7], ε quasigradient methods [5], the method of dual directions and Cebyshev norms [4]. Especially the linearization algorithm of Pshenichny has to be noted, at each step of which several auxiliary problems of linear or square programming are to be solved [6].

Problem (1)–(2) is related to a generalized fractional programming [10-13] for solving of which the nonlinear parametric method of partial linearization is applied, the necessary and sufficient conditions of optimum are known, as well as its dual problem and Lagrange function are examined.

3 Optimum conditions

Denote $S = \{x : h_k(x) \leq 0, k = \overline{1,p}\}$ and assume that $\varphi_i(x) \geq 0$ and $\psi_i(x) > 0$ for any $i \in I$ and $x \in S$.

Lemma 1 *If $\varphi_i(x) \geq 0$ and $\psi_i(x) > 0$ for any $x \in S$, and $\varphi_i(x)$ is convex and $\psi_i(x)$ is concave on S, then the function*

$$f_i(x) = \frac{\varphi_i(x)}{\psi_i(x)}$$

is quasiconvex on S.

Lemma 2 *If functions $f_i(x), i \in I$ are quasiconvex on S, then the function*

$$f(x) = \max_{i \in I} f_i(x)$$

is quasiconvex on S.

Consider the generalized fractional program

$$(P) \quad v^* = f(x^*) = \min_{x \in S} \max_{i \in I} \frac{\varphi_i(x)}{\psi_i(x)}.$$

In [12] is offered an algorithm of nonlinear parametric programming which solves the next problem instead of (P):

$$(P_v) \quad F(v) = \min_{x \in S} \max_{i \in I} \{\varphi_i(x) - v\psi_i(x)\}.$$

To determine the necessary and sufficient optimum conditions for (P) the problem (P_v) is used. For this purpose consider the following problem equivalent to (P_v):

$$(EP_v) \quad \begin{array}{l} t \longrightarrow \min \\ \varphi_i(x) - v\psi_i(x) \leq t, i = \overline{1,m} \\ h_k(x) \leq 0, k = \overline{1,p}. \end{array}$$

The following statements show the relationship between the problems (P), (P_v) and (EP_v).

Lemma 3 *If problem (P) has an optimal solution x^* and v^* is an optimal value then $F(v^*) = 0$, and conversely, if $F(v^*) = 0$ then the solution x^* of (P_v) is also the optimal solution of the problem (P).*

Lemma 4 *If (x, v, t) is a feasible solution of (EP_v), then x is a feasible solution of (P_v). If x is a feasible solution of (P) then there exist a scalars v and t such that (x, v, t) is a feasible solution of (EP_v).*

Lemma 5 *x^* is an optimal solution of (P) and v^* is an optimal value if and only if (x^*, v^*, t^*) is the optimal solution of (EP_v) with the optimal value $t^* = 0$.*

Theorem 1 (Optimum necessary conditions) *Let x^* be the optimal solution of the problem (P) and v be the optimal value. Then there exist $t^* \in R, u^* \in R^m, z^* \in R^p$ such that (x^*, v^*, u^*, z^*) satisfy the conditions*

$$\begin{gathered}
\bigtriangledown\left[\sum_{i=1}^{m} u_i^*(\varphi_i(x^*) - v\psi_i(x^*)) + \sum_{k=1}^{p} z_k^* h_k(x^*)\right] = 0;\\
u_i^*(\varphi_i(x^*) - v^*\psi_i(x^*)) = 0, i = \overline{1,m};\\
z_k^* h_k(x^*) = 0, k = \overline{1,p};\\
\varphi_i(x^*) - v^*\psi_i(x^*) \leq 0, i = \overline{1,m};\\
h_k(x^*) \leq 0, k = \overline{1,p};\\
\sum_{i=1}^{m} u_i^* = 1;\\
t^* = 0;\\
t^* \in R, x^* \in R^n, u^* \in R^m, z^* \in R^p, u^* \geq 0, z^* \geq 0.
\end{gathered} \tag{3}$$

Theorem 2 (Sufficient conditions) *Let (x^*, v^*, u^*, z^*) satisfy conditions (3) and function*

$$A = \sum_{i=1}^{m} u_i^*(\varphi_i(x) - v^*\psi_i(x)) + \sum_{k=1}^{p} z_k^* h_k(x)$$

be pseudoconvex for all x which are feasible solutions of (EP_v). Then x^* is an optimal solution of (P) with the optimal value v^*.

Theorem 3 (Sufficient conditions) *Let (x^*, v^*, u^*, z^*) satisfy conditions (3) and function*

$$B = \sum_{i=1}^{m} u_i^*(\varphi_i(x) - v^*\psi_i(x))$$

be pseudoconvex, and $C = \sum_{k=1}^{p} z_k^* h_k(x)$ be quasiconvex for all x which are feasible solutions of (EP_v). Then x^* is an optimal solution of (P) with the optimal value v^*.

4 Dual problems

In the generalized fractional programming the dual problem comes not for the initial problem (P), but is designed to match its equivalent problem (EP_v). Then (EP_v) has the following dual problem [10,13]:

$$(DEP_v1) \quad \sum_{i=1}^{m} u_i(\varphi_i(x) - v\psi_i(x)) + \sum_{k=1}^{p} z_k h_k(x) \longrightarrow \max$$

$$\nabla \left[\sum_{i=1}^{m} u_i(\varphi_i(x) - v\psi_i(x)) + \sum_{k=1}^{p} z_k h_k(x)\right] = 0$$

$$\sum_{i=1}^{m} u_i = 1$$

$$x \in R^n, u \in R^m, z \in R^p, u \geq 0, z \geq 0$$

Another formulation of dual problem consists of

$$(DEP_v2) \quad \sum_{i=1}^{m} u_i(\varphi_i(x) - v\psi_i(x)) \longrightarrow max$$

$$\nabla \left[\sum_{i=1}^{m} u_i(\varphi_i(x) - v\psi_i(x)) + \sum_{k=1}^{p} z_k h_k(x)\right] = 0$$

$$\sum_{k=1}^{p} z_k h_k(x) \leq 0$$

$$\sum_{i=1}^{m} u_i = 1$$

$$x \in R^n, u \in R^m, z \in R^p, u \geq 0, z \geq 0$$

For the problems (P_v) and (DEP_v1) or (DEP_v2) the theorems of duality hold. Therefore, instead of (P_v) one of the problems (DEP_v1) or (DEP_v2) may be solved.

5 Solving algorithms and methods

To apply the nonlinear parametric method [12] rewrite (1)–(2) as follows:

$$Z(x, v) = \max_{i \in I}\{\varphi_i(x) - v\psi_i(x)\} \longrightarrow \min \tag{4}$$

$$h_k(x) \leq 0, k = \overline{1, p}. \tag{5}$$

The next statements hold [12]:

If problem (1)–(2) has an optimal solution x^*, then $Z(x^*, v^*) = 0$, where $v^* = f(x^*)$;

If $Z(x^*, v^*) = 0$ then x^* is the optimal solution of the problem (1)–(2).

The nonlinear parametric algorithm consists of the followings:

Iteration 0. Let $x^0 \in S$, $v_1 = f(x^0)$ and $r = 1$.

The r-th iteration. Let $x^r \in S$ be the solution of problem (4)–(5) which has been found at the preceding iteration, parameter value v being fixed.

Then:

STEP 1 . Put $v_r = f(x^{r-1})$;

STEP 2 . Determine an optimal solution x^r of (4)–(5) with fixed parameter value $v = v_r$;

STEP 3 . If $Z(x^r, v_r) = 0$, then x^r is the optimal solution of (1)–(2) and STOP.

Step 4 . Replace r by $r + 1$ and go to the Step 1.

To determine the first feasible point $x^0 \in S$ it is sufficient to solve the convex program of minimization of any function $\varphi_i(x)$ on S.

Note that each iteration of nonlinear parametric method deals with the problem (4)–(5) which is as difficult as the initial convex program. As mentioned before, an iterative algorithms are used to solve the problem (4)–(5), each step of which has to solve some auxiliary optimization problems.

Consider another algorithm for generalized fractional programming based on partial linearization method [11]. For this purpose rewrite the problem (1)–(2) in its equivalent form

$$
\begin{array}{ll}
(L) & t \longrightarrow \min \\
& \varphi_i(x) - t\psi_i(x) \leq 0, i = \overline{1,m} \\
& x \in S \\
& t \geq 0.
\end{array}
$$

For any $i \in I$ let define the i-th restriction in the following form:

$$H_i(x,t) = \varphi_i(x) - t\psi_i(x)$$

and consider partial linearizations on t for $H_i(x,t)$ in the point (x^{r-1}, t_r):

$$H_{ir} = H_i(x, t_r) + (t - t_r) \bigtriangledown_i H_i(x^{r-1}, t_r).$$

Then at each step of solving (P) or (L) it is necessary to solve the following problem

$$
\begin{array}{ll}
(L_r) & t \longrightarrow \min \\
& \varphi_i(x) - t_r\psi_i(x) - t \cdot t_r\psi_i(x^{r-1}) \leq 0, i = \overline{1,m} \\
& x \in S, t \geq 0.
\end{array}
$$

Thus, the algorithm of partial linearization consists of the following:

STEP 0. Let $x^0 \in S$ and $t_1 = \max_{i\in I} \frac{\varphi_i(x^0)}{\psi_i(x^0)}$. Note that $t_1 = \inf\{t : H_i(x^0, t) \leq 0, i = \overline{1,m}\}$ and (x,t) is a feasible solution of the problem (P). Let $r = 1$.

STEP 1. Determine an optimal solution (x^r, t'_r) of (L_r).

STEP 2. If $t_r = t'_r$, then x^r is the optimal solution of (P) and t_r is the optimal value. STOP.

STEP 3. Let

$$t_r = \max_{i\in I} \frac{\varphi_i(x^r)}{\psi_i(x^r)} = \inf\{t : H_i(x^r, t) \leq 0, i = \overline{1,m}\}.$$

Replace r by $r + 1$ and repeat Step 1.

As mentioned in [12], the partial linearization algorithm is closed to the parametric method algorithm by its complexity.

6 The algorithm of generalized gradient

Consider another algorithm of solution (1)–(2) based on Lagrange function, decomposition scheme on restrictions and subgradient methods [8,9]. Consider the following problem equivalent to (1)–(2):

$$t \longrightarrow \min \tag{6}$$

$$\varphi_i(x) - t\psi_i(x) \leq 0, i = \overline{1,m}; \tag{7}$$

$$h_k(x) \leq 0, k = \overline{1,p}; \tag{8}$$

$$t \geq 0. \tag{9}$$

The following statements hold:

Theorem 4 *If x^* is an optimal solution of the problem (1)–(2) then there exists scalar $t^* = f(x^*)$ such that (x^*, t^*) is an optimal solution of (6)–(9).*

Proof. Let x^* be an optimal solution of the problem (1)–(2). Then

$$\max_{i\in I} \frac{\varphi_i(x)}{\psi_i(x)} \geq \max_{i\in I} \frac{\varphi_i(x^*)}{\psi_i(x^*)}$$

for any $x \in S$. Simultaneously the following inequalities hold for any $i \in I$:

$$\frac{\varphi_i(x^*)}{\psi_i(x^*)} \leq \max_{i\in I} \frac{\varphi_i(x^*)}{\psi_i(x^*)} = f(x^*) = t^*.$$

Then $\varphi(x^*) - t^*\psi_i(x^*) \leq 0$, i.e. (x^*, t^*) is a feasible solution of the problem (6)–(9).

Let prove that (x^*, t^*) is also the optimal solution of the problem (6)–(9). If not, then there exists scalar $\bar{t}$ such that $t^* > \bar{t}$. Then $(x^*, \bar{t})$ is a feasible solution of the problem (6)–(9) and, as it follows from restrictions (7),

$$\frac{\varphi_i(x^*)}{\psi_i(x^*)} \leq \bar{t} < t^* = \max_{i\in I} \frac{\varphi_i(x^*)}{\psi_i(x^*)}$$

for any $i \in I$. The last strict inequality contradicts that x^* is the optimal solution of the problem (1)–(2). Therefore (x^*, t^*) is the optimal solution of the problem (6)–(9). The theorem is proved.

Theorem 5 *If (x^*, t^*) is an optimal solution of the problem (6)–(9) then x^* is an optimal solution of the problem (1)–(2).*

Proof. Let (x^*, t^*) be an optimal solution of the problem (6)–(9). Then x^* is a feasible solution of the problem (1)–(2). Since t^* is the optimal value of the function (6) and x^* satisfies the restrictions (7), then $t^* < t$ for any t satisfying the relations (7) for any $x \in S$. Thus we have got the following inequalities:

$$\frac{\varphi_i(x^*)}{\psi_i(x^*)} \leq t^*, i \in I \text{ and } \max_{i\in I} \frac{\varphi_i(x^*)}{\psi_i(x^*)} = t^* \leq t.$$

On the other hand, for all $i \in I$ and any $x \in S$ we have $t \geq \frac{\varphi_i(x)}{\psi_i(x)}$ and, by the same, $t = \max_{i\in I} \frac{\varphi_i(x)}{\psi_i(x)}$ for any $x \in S$. Thus, the following inequalities are valid:

$$\max_{i\in I} \frac{\varphi_i(x^*)}{\psi_i(x^*)} = t^* \leq t = \max_{i\in I} \frac{\varphi_i(x)}{\psi_i(x)}$$

for any $x \in S$, i.e.

$$\max_{i\in I} \frac{\varphi_i(x^*)}{\psi_i(x^*)} \leq \max_{i\in I} \frac{\varphi_i(x)}{\psi_i(x)}$$

Hence, x^* is the optimal solution of the problem (1)–(2). The theorem is proved.

Note, that optimum criterion for the problem (6)–(9) may be formulated as follows: to determine the variable value $x \in S$ which minimizes the value of t satisfying restrictions (7), i.e., it is necessary to determine the value x^* of $x \in S$ for which the value of t obtained by the formula

$$t^* = \max_{i\in I} \frac{\varphi_i(x^*)}{\psi_i(x^*)}$$

will be the smallest.

Let us define Lagrange's function for the problem (6)–(9) by the formula

$$\mathcal{L}(x, t, u) = t + \sum_{i\in I} u_i(\varphi_i(x) - t\psi_i(x))$$

where $u = \{u_i, i \in I\}$ are the Lagrange's factors for the restrictions (7), $u_i \geq 0, i \in I$. Then both problem (6)–(9) and (1)–(2) is equivalent to the finding of the saddle point of Lagrange's function $\mathcal{L}(x, t, u^*)$, i.e. to the solving of the problem

$$\mathcal{L}(x^*, t^*, u^*) = \max_{u \geq 0} \min_{t \geq 0} \min_{x \in S} \mathcal{L}(x, t, u). \tag{10}$$

Furthermore, the problem (10) is equivalent to

$$\max_{u \geq 0} \mathcal{L}^*(u) \tag{11}$$

where

$$\mathcal{L}^*(u) = \min_{t \geq 0} \min_{x \in S} \mathcal{L}(x, t, u). \tag{12}$$

Function $\mathcal{L}^*(u)$ is defined for any $u \geq 0$ and is linear on portions, concave and undifferentiable. Therefore, for the solution of the problem (11) a subgradient methods are used, at each step of which the problem (12) must be solved for fixed values of variables u.

Let for solution (11) any subgradient method be used. Then at the r-th iteration it is necessary:

1. To solve (12) with fixed values $u = u^{r-1}$ and to determine an optimal solution $(x^*(u^{r-1}), t^*(u^{r-1}))$.

2. To determine the values of generalized gradient of function $\mathcal{L}^*(u)$ in the point $u = u^{r-1}$ by the formula

$$g_i(u_i^{r-1}) = \varphi_i(x^*(u^{r-1})) - t^*(u^{r-1})\psi_i(x^*(u^{r-1}))), i \in I.$$

3. To calculate new values u^r by the formula

$$u_i^r = \max\{0, u_i^{r-1} + \gamma_r g_i(u_i^{r-1})\}, i \in I$$

   where γ_r is the step value.

Theorem 6 *Let u^* be the optimal solution of (11) and (x^*, t^*) be the optimal solution of the problem (12) when $u = u^*$ are fixed. Then (x^*, t^*) is the optimal solution of the problem (6)–(9) and x^* is the optimal solution of the problem (1)–(2).*

Proof. Since for $u = u^*$ the point (x^*, t^*) is an optimal solution of the problem (12), then we have:

1. $x^* \in S$ and, therefore, x^* satisfies the restrictions (8) and (2);

2. $t^* + \sum_{i \in I} u_i^*(\varphi_i(x^*) - t^*\psi_i(x^*)) \leq t + \sum_{i \in I} u_i(\varphi_i(x) - t\psi_i(x))$ for any $x \in S$;

3. $\varphi_i(x^*) - t^*\psi_i(x^*)) \leq 0$, i.e. (x^*, t^*) is a feasible solution of the problem (6)–(9);

4. $u_i^*(\varphi_i(x^*) - t^*\psi_i(x^*)) = 0, i \in I$.

Then from 3) and 4) we obtain

$$t^* - t \leq \sum_{i \in I} u_i^*(\varphi_i(x) - t\psi_i(x)).$$

Since $u_i^* \geq 0$ for any $i \in I$, then for any x and t which satisfy restrictions (7) and (8) we have $t^* \leq t$, i.e., (x^*, t^*) is the optimal solution of the problem (6)–(9) and consequently (1)–(2). The theorem is proved.

Consider the problem (12), i.e., for fixed values of dual variables $u = \overline{u}$ it is necessary to solve the following problem

$$t + \sum_{i \in I} \overline{u}_i(\varphi_i(x) - t\psi_i(x)) \longrightarrow \min \tag{13}$$

$$h_k(x) \leq 0, k = \overline{1, p} \tag{14}$$

$$t \geq 0. \tag{15}$$

For fixed values of $t \geq 0$ (13)–(15) is a convex program. Therefore, solving (14)–(15) is equivalent to the finding of a such solution $x^* \in S$ which together with the value $t^* \geq 0$ gives us the minimal value of the function (13). To solve out the problem (13)–(15) an approximate algorithm can be used which obtain the optimal solution of (13)–(15) by having the optimal solutions $u = u^*$ of the problem (11). Such an algorithm can be drawn up, if at the next solution process of the problem (13)–(15) the value of parameter t will be fixed by the following formula

$$t = f(x^*(u^{r-1})) = \max_{i \in I} \frac{\varphi_i(x^*(u^{r-1}))}{\psi_i(x^*(u^{r-1}))},$$

where $x^*(u^{r-1}) \in S$ is the solution of the problem (12) found out at the preceding iteration of the subgradient method.

Let have the values of variables $u = u^r$ and the previous solution $(x^*(u^{r-1}), t^*(u^{r-1}))$ of the problem (13)–(15). Then to solve the problem (13)–(15) at the next iteration of the subgradient method the following algorithm will be used:

1. Fix values of variable t by the formula

$$t^*(u^{r-1}) = f(x^*(u^{r-1}))$$

2. Determine the optimal value $x^*(u^{r-1})$ of the convex program

$$\sum_{i \in I} u_i^{r-1}(\varphi_i(x) - t^*(u^{r-1})\psi_i(x)) \longrightarrow \min \tag{16}$$

$$h_k(x) \leq 0, k = \overline{1,p} \tag{17}$$

3. Determine new parameter t value by the formula

$$t^*(u^r) = \max_{i \in I} \frac{\varphi_i(x^*(u^r))}{\psi_i(x^*(u^r))}.$$

Let u^* be the optimal solution of the problem (11). Then find x^* which is the optimal solution of the problem (16)–(17) with fixed value $t = f(x^r(u^r))$ and put $t^* = f(x^*)$.

The offered algorithm was used to solve problems of generalized fractional–linear programming [14] and was tested on the transport problems.

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D.Solomon, V.Pluta,

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D.Solomon,
Institute of Mathematics
Academy of Sciences of Moldova,
Academiei str., 5, Kishinev,
277028, Moldova

V.Pluta,
Academy of Economic Studies of Moldova
B.Bodoni str., 59, Kishinev,
277005, Moldova