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Generalized ABC Energy of Weighted Graphs

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Abstract

In this work, weighted generalized ABC matrix and weighted generalized ABC energy of graphs are considered. Some upper and lower bounds are given for generalized ABC energy of weighted graphs with positive definite matrix edge weights. Related to these bounds some bounds are obtained for number weighted and unweighted graphs.

Keywords: Edge weighted graph, graph energy, ABC energy.

MSC 2010: 05C22, 05C50.

1 Introduction and Preliminaries

Let $G = (V(G), E(G))$ be a graph with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$ and edge set $E(G)$. If v_i, v_j are adjacent vertices, it is denoted by $v_i \sim v_j$ or $v_iv_j \in E(G)$. Degree of a vertex v_i is denoted by d_i. The atom-bond connectivity index ABC of G is introduced by Estrada et al. [7] as

$$ABC = ABC(G) = \sum_{v_iv_j \in E(G)} \sqrt{\frac{d_i + d_j - 2}{d_id_j}}.$$

Estrada et al. [7] presented that ABC can be used for modelling thermodynamic properties of organic chemical compounds. Furtula et al. [10] proposed a generalization of atom-bond connectivity index ABC_α for $\alpha = -3$ as

$$ABC_\alpha = ABC_\alpha(G) = \sum_{v_iv_j \in E(G)} \left(\frac{d_i + d_j - 2}{d_id_j}\right)^\alpha,$$

is called augmented Zagreb index which is a significant predictive index in the study of the heat of formation in heptanes and octanes. Estrada [9] provided a probabilistic interpretation of the term $d_i + d_j - 2/d_i d_j$ which takes place in the definition of ABC indices. He showed that it represents the probability of visiting the nearest neighbor edge from one side or the other of a given edge in a graph. This interpretation can be related to the polarizing capacity of the bond in a molecular context. He represented these probabilities in the form of an ABC matrix as the generalized ABC matrix $S_\alpha = S_\alpha(G) = \left(s_{ij}^{(\alpha)}\right)$ of order n and the $(i,j)-th$ entry of S_α is defined as

$$s_{ij}^{(\alpha)}(G) = \begin{cases} \left(\frac{d_i+d_j-2}{d_i d_j}\right)^\alpha, & \text{if } v_i v_j \in E(G) \\ 0, & \text{otherwise} \end{cases}. \tag{1}$$

In [13], energy of a graph G of order n is defined as

$$E(G) = \sum_{i=1}^{n} |\lambda_i|,$$

where λ_i are the eigenvalues of adjacency matrix $A(G)$. Graph energy is related to many chemical problems, so it is a remarkable concept for chemists and mathematicians ([2], [4], [5], [14], [16], [19], [15]).

In [9], generalized ABC energy of G is defined as

$$E_{ABC}^{(\alpha)} = E_{ABC}^{(\alpha)}(G) = \sum_{i=1}^{n} \left|\nu_i^{(\alpha)}\right|,$$

where $\nu_1^{(\alpha)} \geq \nu_2^{(\alpha)} \geq ... \geq \nu_n^{(\alpha)}$ are the eigenvalues of $S_\alpha(G)$, which are called generalized ABC eigenvalues of G. Notice that if $\alpha = \frac{1}{2}$, $E_{ABC}^{(\alpha)}$ is replaced by $E_{ABC}(G)$ called ABC (total) energy of G with ABC eigenvalues $(\nu_1 \geq \nu_2 \geq ... \geq \nu_n)$ of G.

Let G be a simple connected matrix weighted graph of order n. An edge weighted graph is a graph with numeric labels w_{ij}, which are associated with each edge ij, called the weight (cost) of the edge ij. The weights are usually chosen as nonnegative integers or square matrices. If each edge weight is 1, then the graph is called unweighted graph. Throughout this paper, we consider edge weighted graphs with positive definite matrix edge weights. Let w_{ij} be a positive definite weight matrix of order p of the edge ij. Assume that $w_{ij} = w_{ji}$ and for all $i \in V$, $w_i = \sum\limits_{j:\, j\sim i} w_{ij}$.

In this work, we introduce generalized ABC energy of edge weighted graphs without loops and parallel edges. Accordingly, considering matrix multiplication, we can define weighted generalized ABC matrix $S_{\alpha,w} = S_{\alpha,w}(G) = \left(s_{ij}^{(\alpha,w)}\right)$ of order np, where the (i,j)-*th* entry of $S_{\alpha,w}$ is

$$s_{ij}^{(\alpha,w)}(G) = \begin{cases} \left[(w_i + w_j - 2) . (w_i w_j)^{-1}\right]^{\alpha}, & \text{if } v_i v_j \in E(G) \\ 0 & , \quad \text{otherwise} \end{cases}.$$

0 denotes zero matrix of order p. Obviously, $S_{\alpha,w}(G)$ is a real symmetric matrix, therefore the eigenvalues of $S_{\alpha,w}(G)$ can be arranged as $\nu_{1,w}^{(\alpha)} \geq \nu_{2,w}^{(\alpha)} \geq ... \geq \nu_{np,w}^{(\alpha)}$ and also called weighted generalized ABC eigenvalues of G. Thus, generalized ABC energy of a (positive definite) matrix weighted graph G is defined as

$$E_{ABC_w}^{(\alpha)} = E_{ABC_w}^{(\alpha)}(G) = \sum_{i=1}^{np} \left|\nu_{i,w}^{(\alpha)}\right|. \tag{2}$$

The generalized ABC energy of a weighted graph can be called as weighted generalized ABC energy, briefly. If G is unweighted, then $E_{ABC_w}^{(\alpha)}(G)$ is replaced by $E_{ABC}^{(\alpha)}(G)$.

Some bounds are given for weighted generalized ABC energy for (positive definite) matrix weighted graphs in this work. Through these bounds some results are presented for number weighted and unweighted graphs. In unweighted case, some new bounds are obtained for $E_{ABC}^{(\alpha)}(G)$, thus the weighted generalized ABC energy $E_{ABC_w}^{(\alpha)}(G)$ can also be seen as the general form of generalized ABC energy. Initially, we will give the required lemmas.

Lemma 1 ([18]). *If a_i and b_i $(1 \leq i \leq n)$ are positive real numbers, then*

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \leq \frac{1}{4} \left(\sqrt{\frac{M_1 M_2}{m_1 m_2}} + \sqrt{\frac{m_1 m_2}{M_1 M_2}} \right)^2 \left(\sum_{i=1}^{n} a_i b_i \right)^2, \tag{3}$$

where $M_1 = \max\limits_{1 \leq i \leq n} \{a_i\}$, $M_2 = \max\limits_{1 \leq i \leq n} \{b_i\}$; $m_1 = \min\limits_{1 \leq i \leq n} \{a_i\}$, $m_2 = \min\limits_{1 \leq i \leq n} \{b_i\}$.

Lemma 2 ([17]). *If a_i and b_i $(1 \leq i \leq n)$ are nonnegative real numbers, then*

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i \right)^2 \leq \frac{n^2}{4} \left(M_1 M_2 - m_1 m_2 \right)^2, \tag{4}$$

where M_i and m_i are defined in Lemma 1.

Lemma 3 ([1]). *If a_i and b_i $(1 \leq i \leq n)$ are positive real numbers, then*

$$\left| n \sum_{i=1}^{n} a_i b_i - \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i \right| \leq \beta(n) (A - a)(B - b), \tag{5}$$

where a, b, A, B are real constants and for each i, $1 \leq i \leq n$, $a \leq a_i \leq A$, $b \leq b_i \leq B$ and $\beta(n) = n \left\lfloor \frac{n}{2} \right\rfloor \left(1 - \frac{1}{n} \left\lfloor \frac{n}{2} \right\rfloor \right)$.

Lemma 4 ([6]). *If a_i and b_i $(1 \leq i \leq n)$ are nonnegative real numbers, then*

$$\sum_{i=1}^{n} b_i^2 + rR \sum_{i=1}^{n} a_i^2 \leq (r + R) \sum_{i=1}^{n} a_i b_i, \tag{6}$$

where r and R are real constants and for each i, $1 \leq i \leq n$, $r a_i \leq b_i \leq R a_i$ holds.

2 Main Results

In this section we present some upper and lower bounds for $E^{(\alpha)}_{ABC_w}(G)$, the generalized ABC energy of matrix weighted graphs which have positive definite matrix of order p edge weights. Also, some bounds are obtained for unweighted and number weighted graphs via these bounds. For $\alpha = \frac{1}{2}$ and unweighted case, the obtained bounds are for $E_{ABC}(G)$. Note that some bounds in (12) and (15) are presented in [11] (see Theorem 3.4); the bounds in (28) are given in [3] (see Theorem 4.3-4.8). Now we begin with an essential lemma for proving the theorems as follows.

Lemma 5. *If G is a positive definite matrix (of order p) weighted graph with $n(\geq 3)$ vertices, then*

(1) $\sum\limits_{i=1}^{np} \nu_{i,w}^{(\alpha)} = 0,$

(2) $\sum\limits_{i=1}^{np} \left(\nu_{i,w}^{(\alpha)}\right)^2 = 2 \sum\limits_{\substack{j:\, j\sim i \\ i,j\in\{1,2,\dots,n\}}} \left[(w_i + w_j - 2)\,.\,(w_i w_j)^{-1}\right]^{2\alpha}.$

Proof. (1) Since diagonal entries of $S_{\alpha,w}(G)$ are equal to zero, we have

$$\sum_{i=1}^{np} \nu_{i,w}^{(\alpha)} = tr\left[S_{\alpha,w}(G)\right] = 0,$$

where $tr(.)$ stands for trace of a matrix.

(2) For $i = 1, 2, ..., n$, the (i,i)-th entry of $(S_{\alpha,w}(G))^2$ is

$$\sum_{j:\, j\sim i} \left[(w_i + w_j - 2)\,.\,(w_i w_j)^{-1}\right]^{2\alpha}.$$

Hence

$$\sum_{i=1}^{np} \left(\nu_{i,w}^{(\alpha)}\right)^2 = tr\left[(S_{\alpha,w}(G))^2\right] = \sum_{i=1}^{n} \sum_{j:\, j\sim i} \left[(w_i + w_j - 2)\,.\,(w_i w_j)^{-1}\right]^{2\alpha}$$

$$= 2 \sum_{\substack{j:\, j\sim i \\ i,j\in\{1,2,\dots,n\}}} \left[(w_i + w_j - 2)\,.\,(w_i w_j)^{-1}\right]^{2\alpha}. \tag{7}$$

The proof is completed. □

Remark 1. *(i) If G is an unweighted graph, then $p = 1$, $w_{ij} = 1$ and for all i, j and $i \sim j$, $w_i = d_i$ in (7), then we have*

$$\sum_{i=1}^{n} \left(\nu_i^{(\alpha)}\right)^2 = 2 \sum_{\substack{j:\ j\sim i \\ i,j\in\{1,2,\ldots,n\}}} \left(\frac{d_i + d_j - 2}{d_i d_j}\right)^{2\alpha}. \tag{8}$$

(ii) If we set $\alpha = \frac{1}{2}$ in (8), then

$$\begin{aligned} \sum_{i=1}^{n} \nu_i^2 &= 2 \sum_{\substack{j:\ j\sim i \\ i,j\in\{1,2,\ldots,n\}}} \left(\frac{d_i + d_j - 2}{d_i d_j}\right) \\ &= 2 \sum_{\substack{j:\ j\sim i \\ i,j\in\{1,2,\ldots,n\}}} \left(\frac{d_i + d_j}{d_i d_j} - \frac{2}{d_i d_j}\right) \\ &= 2\left(n - 2R_{-1}(G)\right), \end{aligned} \tag{9}$$

where $R_{-1}(G) = \sum_{uv\in E(G)} \frac{1}{d_u d_v}$ is the general Randić index of G, also known as modified second Zagreb index.

Theorem 1. *If G is a matrix weighted graph with $n(\geq 3)$ vertices and zero is not an eigenvalue of $S_{\alpha,w}(G)$, then*

$$E_{ABC_w}^{(\alpha)} \geq \frac{2\sqrt{\nu_{1,w}^{(\alpha)} \nu_{np,w}^{(\alpha)}} \sqrt{2np \displaystyle\sum_{\substack{j:\ j\sim i \\ i,j\in\{1,2,\ldots,n\}}} \left[(w_i + w_j - 2) \,.\, (w_i w_j)^{-1}\right]^{2\alpha}}}{\nu_{1,w}^{(\alpha)} + \nu_{np,w}^{(\alpha)}}, \tag{10}$$

where $\nu_{1,w}^{(\alpha)}$ and $\nu_{np,w}^{(\alpha)}$ are maximum and minimum of the absolute value of $\nu_{i,w}^{(\alpha)}$'s, respectively.

Proof. If we put $a_i = \left|\nu_{i,w}^{(\alpha)}\right|$ and $b_i = 1$ in (3), then

$$\sum_{i=1}^{np} \left|\nu_{i,w}^{(\alpha)}\right|^2 \sum_{i=1}^{np} 1^2 \leq \frac{1}{4} \left(\sqrt{\frac{\nu_{1,w}^{(\alpha)}}{\nu_{np,w}^{(\alpha)}}} + \sqrt{\frac{\nu_{np,w}^{(\alpha)}}{\nu_{1,w}^{(\alpha)}}}\right)^2 \left(\sum_{i=1}^{np} \left|\nu_{i,w}^{(\alpha)}\right|\right)^2.$$

By (7), we have

$$2np \sum_{\substack{j:\, j\sim i \\ i,j\in\{1,2,\ldots,n\}}} \left[(w_i + w_j - 2)\,.\,(w_i w_j)^{-1}\right]^{2\alpha} \leq$$

$$\leq \frac{1}{4} \frac{\left(\nu_{1,w}^{(\alpha)} + \nu_{np,w}^{(\alpha)}\right)^2}{\nu_{1,w}^{(\alpha)} \nu_{np,w}^{(\alpha)}} \left(E_{ABC_w}^{(\alpha)}\right)^2,$$

then

$$E_{ABC_w}^{(\alpha)} \geq \frac{2\sqrt{\nu_{1,w}^{(\alpha)}\nu_{np,w}^{(\alpha)}} \sqrt{2np \sum\limits_{\substack{j:\, j\sim i \\ i,j\in\{1,2,\ldots,n\}}} \left[(w_i + w_j - 2)\,.\,(w_i w_j)^{-1}\right]^{2\alpha}}}{\nu_{1,w}^{(\alpha)} + \nu_{np,w}^{(\alpha)}},$$

and the proof is completed. $\square$

Corollary 1. *If G is a number weighted graph of order $n(\geq 3)$ with positive number edge weights and zero is not an eigenvalue of $S_{\alpha,w}(G)$, then*

$$E_{ABC_w}^{(\alpha)} \geq \frac{2\sqrt{\nu_{1,w}^{(\alpha)}\nu_{n,w}^{(\alpha)}} \sqrt{2n \sum\limits_{\substack{j:\, j\sim i \\ i,j\in\{1,2,\ldots,n\}}} \left(\frac{w_i + w_j - 2}{w_i w_j}\right)^{2\alpha}}}{\nu_{1,w}^{(\alpha)} + \nu_{n,w}^{(\alpha)}},$$

where $\nu_{1,w}^{(\alpha)}$ and $\nu_{n,w}^{(\alpha)}$ are maximum and minimum of the absolute value of $\nu_{i,w}^{(\alpha)}$'s, respectively.

Proof. If we write $p = 1$ in (10), then the proof is obvious. $\square$

Corollary 2. *If G is an unweighted graph of order $n(\geq 3)$ and zero is not an eigenvalue of $S_\alpha(G)$, then*

$$E_{ABC}^{(\alpha)} \geq \frac{2\sqrt{\nu_1^{(\alpha)}\nu_n^{(\alpha)}} \sqrt{2n \sum\limits_{\substack{j:\, j\sim i \\ i,j\in\{1,2,\ldots,n\}}} \left(\frac{d_i + d_j - 2}{d_i d_j}\right)^{2\alpha}}}{\nu_1^{(\alpha)} + \nu_n^{(\alpha)}}, \tag{11}$$

where $\nu_1^{(\alpha)}$ *and* $\nu_n^{(\alpha)}$ *are maximum and minimum of the absolute value of* $\nu_i^{(\alpha)}$*'s, respectively. If* $\alpha = \frac{1}{2}$*, then*

$$E_{ABC} \geq \frac{2\sqrt{2n\nu_1\nu_n\left(n - 2R_{-1}\left(G\right)\right)}}{\nu_1 + \nu_n}, \tag{12}$$

where ν_1 *and* ν_n *are maximum and minimum of the absolute value of* ν_i*'s, respectively.*

Proof. If G is an unweighted graph, then we write $p = 1$ and take for all i, j and $i \sim j$, $w_i = d_i$ and $w_{ij} = 1$ in (10). Thus

$$E_{ABC}^{(\alpha)} \geq \frac{2\sqrt{\nu_1^{(\alpha)}\nu_n^{(\alpha)}}\sqrt{2n\sum\limits_{\substack{j:\, j\sim i\\ i,j\in\{1,2,\ldots,n\}}}\left(\frac{d_i+d_j-2}{d_id_j}\right)^{2\alpha}}}{\nu_1^{(\alpha)} + \nu_n^{(\alpha)}}$$

yields the result. If we set $\alpha = \frac{1}{2}$ in (11) and use (9), we obtain

$$E_{ABC} \geq \frac{2\sqrt{2n\nu_1\nu_n\left(n - 2R_{-1}\left(G\right)\right)}}{\nu_1 + \nu_n}.$$

which completes the proof. $\square$

Theorem 2. *If* G *is a matrix weighted graph with* $n(\geq 3)$ *vertices, then*

$$E_{ABC_w}^{(\alpha)} \geq \tag{13}$$
$$\geq \sqrt{2np\sum_{\substack{j:\, j\sim i\\ i,j\in\{1,2,\ldots,n\}}}\left[\left(w_i + w_j - 2\right).\left(w_iw_j\right)^{-1}\right]^{2\alpha} - \frac{n^2p^2}{4}\left(\nu_{1,w}^{(\alpha)} - \nu_{np,w}^{(\alpha)}\right)^2},$$

where $\nu_{1,w}^{(\alpha)}$ *and* $\nu_{np,w}^{(\alpha)}$ *are maximum and minimum of the absolute value of* $\nu_{i,w}^{(\alpha)}$*'s, respectively.*

Proof. Setting $a_i = \left|\nu_{i,w}^{(\alpha)}\right|$ and $b_i = 1$ in (4) yields

$$\sum_{i=1}^{np}\left|\nu_{i,w}^{(\alpha)}\right|^2\sum_{i=1}^{np}1^2-\left(\sum_{i=1}^{np}\left|\nu_{i,w}^{(\alpha)}\right|\right)^2\leq\frac{n^2p^2}{4}\left(\nu_{1,w}^{(\alpha)}-\nu_{np,w}^{(\alpha)}\right)^2.$$

From (7), we have

$$2np\sum_{\substack{j:\ j\sim i\\ i,j\in\{1,2,\ldots,n\}}}\left[(w_i+w_j-2)\,.\,(w_iw_j)^{-1}\right]^{2\alpha}-\left(E_{ABC_w}^{(\alpha)}\right)^2\leq$$

$$\leq\frac{n^2p^2}{4}\left(\nu_{1,w}^{(\alpha)}-\nu_{np,w}^{(\alpha)}\right)^2.$$

Hence

$$E_{ABC_w}^{(\alpha)}\geq$$

$$\geq\sqrt{2np\sum_{\substack{j:\ j\sim i\\ i,j\in\{1,2,\ldots,n\}}}\left[(w_i+w_j-2)\,.\,(w_iw_j)^{-1}\right]^{2\alpha}-\frac{n^2p^2}{4}\left(\nu_{1,w}^{(\alpha)}-\nu_{np,w}^{(\alpha)}\right)^2},$$

and the proof is completed. □

Corollary 3. *If G is a number weighted graph of order $n(\geq 3)$ with positive number edge weights, then*

$$E_{ABC_w}^{(\alpha)}\geq\sqrt{2n\sum_{\substack{j:\ j\sim i\\ i,j\in\{1,2,\ldots,n\}}}\left(\frac{w_i+w_j-2}{w_iw_j}\right)^{2\alpha}-\frac{n^2}{4}\left(\nu_{1,w}^{(\alpha)}-\nu_{n,w}^{(\alpha)}\right)^2},$$

where $\nu_{1,w}^{(\alpha)}$ and $\nu_{n,w}^{(\alpha)}$ are maximum and minimum of the absolute value of $\nu_{i,w}^{(\alpha)}$'s, respectively.

Proof. The proof is obvious by setting $p=1$ in (14). □

Corollary 4. *If G is an unweighted graph of order $n(\geq 3)$, then*

$$E_{ABC}^{(\alpha)} \geq \sqrt{2n \sum_{\substack{j:\ j\sim i \\ i,j\in\{1,2,\ldots,n\}}} \left(\frac{d_i+d_j-2}{d_i d_j}\right)^{2\alpha} - \frac{n^2}{4}\left(\nu_1^{(\alpha)} - \nu_n^{(\alpha)}\right)^2}, \quad (14)$$

where $\nu_1^{(\alpha)}$ and $\nu_n^{(\alpha)}$ are maximum and minimum of the absolute value of $\nu_i^{(\alpha)}$'s, respectively. If $\alpha = \frac{1}{2}$, then

$$E_{ABC} \geq \sqrt{2n\left(n - 2R_{-1}(G)\right) - \frac{n^2}{4}\left(\nu_1 - \nu_n\right)^2}, \quad (15)$$

where ν_1 and ν_n are maximum and minimum of the absolute value of ν_i's, respectively.

Proof. If we take $p = 1$ and for all i, j and $i \sim j$, $w_i = d_i$ and $w_{ij} = 1$ in (14) and use (8), the proof of (14) can be seen. If we write $\alpha = \frac{1}{2}$ in (14) and use (9), then it completes the proof of (15). □

Theorem 3. *If G is a matrix weighted graph with $n(\geq 3)$ vertices, then*

$$E_{ABC_w}^{(\alpha)} \geq \quad (16)$$

$$\geq \sqrt{2n \sum_{\substack{j:\ j\sim i \\ i,j\in\{1,2,\ldots,n\}}} \left[(w_i + w_j - 2) \,.\, (w_i w_j)^{-1}\right]^{2\alpha} - \beta\left(\left|\nu_{1,w}^{(\alpha)}\right| - \left|\nu_{np,w}^{(\alpha)}\right|\right)^2},$$

where $\beta = np\left\lfloor \frac{np}{2}\right\rfloor\left(1 - \frac{1}{np}\left\lfloor\frac{np}{2}\right\rfloor\right)$.

Proof. If we set $a_i = \left|\nu_{i,w}^{(\alpha)}\right| = b_i$, $a = \left|\nu_{np,w}^{(\alpha)}\right| = b$, and $A = \left|\nu_{1,w}^{(\alpha)}\right| = B$ in (5), then

$$\left| n\sum_{i=1}^{np}\left|\nu_{i,w}^{(\alpha)}\right|^2 - \left(\sum_{i=1}^{np}\left|\nu_{i,w}^{(\alpha)}\right|\right)^2\right| \leq \beta\left(\left|\nu_{1,w}^{(\alpha)}\right| - \left|\nu_{np,w}^{(\alpha)}\right|\right)^2.$$

Thus, we have

$$2n \sum_{\substack{j:\ j\sim i \\ i,j\in\{1,2,\ldots,n\}}} \left[(w_i + w_j - 2)\,.\,(w_i w_j)^{-1}\right]^{2\alpha} - \left(E^{(\alpha)}_{ABC_w}\right)^2 \leq$$

$$\leq \beta \left(\left|\nu^{(\alpha)}_{1,w}\right| - \left|\nu^{(\alpha)}_{np,w}\right|\right)^2,$$

and the proof is completed. ◻

Corollary 5. *If G is a number weighted graph of order $n(\geq 3)$ with positive number edge weights, then*

$$E^{(\alpha)}_{ABC_w} \geq \sqrt{2n \sum_{\substack{j:\ j\sim i \\ i,j\in\{1,2,\ldots,n\}}} \left(\frac{w_i + w_j - 2}{w_i w_j}\right)^{2\alpha} - \beta \left(\left|\nu^{(\alpha)}_{1,w}\right| - \left|\nu^{(\alpha)}_{n,w}\right|\right)^2}.$$

Proof. The proof is obvious by setting $p = 1$ in (16). ◻

Corollary 6. *If G is an unweighted graph of order $n(\geq 3)$, then*

$$E^{(\alpha)}_{ABC} \geq \sqrt{2n \sum_{\substack{j:\ j\sim i \\ i,j\in\{1,2,\ldots,n\}}} \left(\frac{d_i + d_j - 2}{d_i d_j}\right)^{2\alpha} - \beta \left(\left|\nu^{(\alpha)}_{1}\right| - \left|\nu^{(\alpha)}_{n}\right|\right)^2}. \quad (17)$$

If $\alpha = \frac{1}{2}$, then

$$E^{(\alpha)}_{ABC} \geq \sqrt{2n\,(n - 2R_{-1}(G)) - \beta \left(\left|\nu^{(\alpha)}_{1}\right| - \left|\nu^{(\alpha)}_{n}\right|\right)^2}. \quad (18)$$

Proof. The proof of (17) can be seen by writing $p = 1$ and for all i, j and $i \sim j$, $w_i = d_i$ and $w_{ij} = 1$ in (16) and using (8). The proof of (18) is obvious from taking $\alpha = \frac{1}{2}$ in (17) and from (9) by simple calculation. ◻

Corollary 7. *Since $\beta \leq \frac{n^2p^2}{4}$, (17) is stronger than (14), that is*

$$E_{ABC}^{(\alpha)} \geq \sqrt{2n \sum_{\substack{j:\ j\sim i \\ i,j\in\{1,2,\ldots,n\}}} \left(\frac{d_i+d_j-2}{d_id_j}\right)^{2\alpha} - \beta\left(\left|\nu_1^{(\alpha)}\right| - \left|\nu_n^{(\alpha)}\right|\right)^2}$$
$$\geq \sqrt{2n \sum_{\substack{j:\ j\sim i \\ i,j\in\{1,2,\ldots,n\}}} \left(\frac{d_i+d_j-2}{d_id_j}\right)^{2\alpha} - \frac{n^2}{4}\left(\left|\nu_1^{(\alpha)}\right| - \left|\nu_n^{(\alpha)}\right|\right)^2}.$$

Theorem 4. *If G is a matrix weighted graph with $n(\geq 3)$ vertices and zero is not an eigenvalue of $S_{\alpha,w}(G)$, then*

$$E_{ABC_w}^{(\alpha)} \geq \frac{np\left|\nu_{1,w}^{(\alpha)}\right|\left|\nu_{np,w}^{(\alpha)}\right| + 2\sum\limits_{\substack{j:\ j\sim i \\ i,j\in\{1,2,\ldots,n\}}} \left[(w_i+w_j-2).(w_iw_j)^{-1}\right]^{2\alpha}}{\left|\nu_{1,w}^{(\alpha)}\right| + \left|\nu_{np,w}^{(\alpha)}\right|}. \tag{19}$$

Proof. Setting $b_i = \left|\nu_{i,w}^{(\alpha)}\right|$, $a_i = 1$, $r = \left|\nu_{np,w}^{(\alpha)}\right|$, $R = \left|\nu_{1,w}^{(\alpha)}\right|$ in (6) yields

$$\sum_{i=1}^{np}\left|\nu_{i,w}^{(\alpha)}\right|^2 + \left|\nu_{1,w}^{(\alpha)}\right|\left|\nu_{np,w}^{(\alpha)}\right|\sum_{i=1}^{np}1^2 \leq \left(\left|\nu_{1,w}^{(\alpha)}\right| + \left|\nu_{np,w}^{(\alpha)}\right|\right)\sum_{i=1}^{np}\left|\nu_{i,w}^{(\alpha)}\right|.$$

Thus, from (2) and (7)

$$2\sum_{\substack{j:\ j\sim i \\ i,j\in\{1,2,\ldots,n\}}} \left[(w_i+w_j-2).(w_iw_j)^{-1}\right]^{2\alpha} + np\left|\nu_{1,w}^{(\alpha)}\right|\left|\nu_{np,w}^{(\alpha)}\right|$$
$$\leq (\left|\nu_{1,w}^{(\alpha)}\right| + \left|\nu_{np,w}^{(\alpha)}\right|)E_{ABC_w}^{(\alpha)}$$

which gives the required result. ☐

Corollary 8. *If G is a number weighted graph of order $n(\geq 3)$ with positive number edge weights and zero is not an eigenvalue of $S_{\alpha,w}(G)$,* *then*

$$E^{(\alpha)}_{ABC_w} \geq \frac{n\left|\nu^{(\alpha)}_{1,w}\right|\left|\nu^{(\alpha)}_{n,w}\right| + 2\sum\limits_{\substack{j:\ j\sim i\\ i,j\in\{1,2,\ldots,n\}}}\left(\frac{w_i+w_j-2}{w_iw_j}\right)^{2\alpha}}{\left|\nu^{(\alpha)}_{1,w}\right| + \left|\nu^{(\alpha)}_{n,w}\right|}.$$

Proof. Writing $p = 1$ in (19), the proof can be seen. □

Corollary 9. *If G is an unweighted graph of order $n(\geq 3)$ and zero is not an eigenvalue of $S_\alpha(G)$, then*

$$E^{(\alpha)}_{ABC} \geq \frac{n\left|\nu^{(\alpha)}_{1}\right|\left|\nu^{(\alpha)}_{n}\right| + 2\sum\limits_{\substack{j:\ j\sim i\\ i,j\in\{1,2,\ldots,n\}}}\left(\frac{d_i+d_j-2}{d_id_j}\right)^{2\alpha}}{\left|\nu^{(\alpha)}_{1}\right| + \left|\nu^{(\alpha)}_{n}\right|}. \tag{20}$$

If $\alpha = \frac{1}{2}$, then

$$E_{ABC} \geq \frac{n\left|\nu_1\right|\left|\nu_n\right| + 2\left(n - 2R_{-1}(G)\right)}{\left|\nu_1\right| + \left|\nu_n\right|}. \tag{21}$$

Proof. The proof of (20) can be seen by putting $p = 1$ and for all i, j and $i \sim j$, $w_i = d_i$ and $w_{ij} = 1$ in (19) and using (8) by simple calculation. The proof of (21) is obvious from setting $\alpha = \frac{1}{2}$ in (20) and from (9). □

Theorem 5. *If G is a matrix weighted graph with $n(\geq 3)$ vertices, then*

$$E^{(\alpha)}_{ABC_w} \leq \left|\nu^{(\alpha)}_{1,w}\right| + \tag{22}$$

$$\sqrt{(np-1)\left[2\sum\limits_{\substack{j:\ j\sim i\\ i,j\in\{1,2,\ldots,n\}}}\left[\left(w_i+w_j-2\right).\left(w_iw_j\right)^{-1}\right]^{2\alpha} - \left(\nu^{(\alpha)}_{1,w}\right)^2\right]}.$$

Proof. From (2), we have

$$E^{(\alpha)}_{ABC_w} - \left|\nu^{(\alpha)}_{1,w}\right| = \sum_{i=2}^{nt} \left|\nu^{(\alpha)}_{i,w}\right|.$$

If we set $a_i = 1$ and $b_i = \left|\nu^{(\alpha)}_{i,w}\right|$ in Cauchy-Schwarz inequality and use (7), then

$$\begin{aligned}
&\left(E^{(\alpha)}_{ABC_w} - \left|\nu^{(\alpha)}_{1,w}\right|\right)^2 = \left(\sum_{i=2}^{np} \left|\nu^{(\alpha)}_{i,w}\right| .1\right)^2 \leq \sum_{i=2}^{np} \left(\nu^{(\alpha)}_{i,w}\right)^2 \sum_{i=2}^{np} 1^2 \\
&= (np-1)\left(\sum_{i=1}^{np} \left(\nu^{(\alpha)}_{i,w}\right)^2 - \left(\nu^{(\alpha)}_{1,w}\right)^2\right) \\
&= (np-1)\left[2 \sum_{\substack{j:\, j\sim i \\ i,j\in\{1,2,\ldots,n\}}} \left[(w_i + w_j - 2).(w_i w_j)^{-1}\right]^{2\alpha} - \left(\nu^{(\alpha)}_{1,w}\right)^2\right].
\end{aligned}$$

Thus, the proof is obvious. □

Corollary 10. *If G is a number weighted graph of order $n(\geq 3)$ with positive number edge weights, then*

$$E^{(\alpha)}_{ABC_w} \leq \left|\nu^{(\alpha)}_{1,w}\right| + \sqrt{(n-1)\left[2 \sum_{\substack{j:\, j\sim i \\ i,j\in\{1,2,\ldots,n\}}} \left(\frac{w_i + w_j - 2}{w_i w_j}\right)^{2\alpha} - \left(\nu^{(\alpha)}_{1,w}\right)^2\right]}. \tag{23}$$

Proof. Taking $p = 1$ in (22) completes the proof. □

Corollary 11. *If G is an unweighted graph of order $n(\geq 3)$, then*

$$E^{(\alpha)}_{ABC} \leq \left|\nu^{(\alpha)}_{1}\right| + \sqrt{(n-1)\left[2 \sum_{\substack{j:\, j\sim i \\ i,j\in\{1,2,\ldots,n\}}} \left(\frac{d_i + d_j - 2}{d_i d_j}\right)^{2\alpha} - \left(\nu^{(\alpha)}_{1}\right)^2\right]}. \tag{24}$$

If $\alpha = \frac{1}{2}$, *then*

$$E_{ABC} \leq |\nu_1| + \sqrt{(n-1)\left[2\left(n - 2R_{-1}\left(G\right)\right) - \nu_1^2\right]}.$$

Proof. The proof is obvious by setting $\alpha = \frac{1}{2}$ in (24). □

Theorem 6. *If G is a matrix weighted graph with $n(\geq 3)$ vertices, then*

$$\sqrt{2 \sum_{\substack{j:\ j\sim i \\ i,j\in\{1,2,\ldots,n\}}} \left[\left(w_i + w_j - 2\right).\left(w_i w_j\right)^{-1}\right]^{2\alpha}} \leq E_{ABC_w}^{(\alpha)} \leq \tag{25}$$

$$\leq \sqrt{2np \sum_{\substack{j:\ j\sim i \\ i,j\in\{1,2,\ldots,n\}}} \left[\left(w_i + w_j - 2\right).\left(w_i w_j\right)^{-1}\right]^{2\alpha}}.$$

Proof. If we set $a_i = 1$ and $b_i = \left|\nu_{i,w}^{(\alpha)}\right|$ in Cauchy-Schwarz inequality, then

$$\left(E_{ABC_w}^{(\alpha)}\right)^2 = \left(\sum_{i=1}^{np} 1.\left|\nu_{i,w}^{(\alpha)}\right|\right)^2 \leq \sum_{i=1}^{np} 1^2.\sum_{i=1}^{np}\left(\nu_{i,w}^{(\alpha)}\right)^2 = np\sum_{i=1}^{np}\left(\nu_{i,w}^{(\alpha)}\right)^2.$$

From (7), we have

$$\left(E_{ABC_w}^{(\alpha)}\right)^2 \leq 2np \sum_{\substack{j:\ j\sim i \\ i,j\in\{1,2,\ldots,n\}}} \left[\left(w_i + w_j - 2\right).\left(w_i w_j\right)^{-1}\right]^{2\alpha},$$

which presents the upper bound. On the other hand, if we use (7), then

$$\left(E_{ABC_w}^{(\alpha)}\right)^2 = \left(\sum_{i=1}^{np}\left|\nu_{i,w}^{(\alpha)}\right|\right)^2 \geq \sum_{i=1}^{np}\left(\nu_{i,w}^{(\alpha)}\right)^2$$

$$= 2 \sum_{\substack{j:\ j\sim i \\ i,j\in\{1,2,\ldots,n\}}} \left[\left(w_i + w_j - 2\right).\left(w_i w_j\right)^{-1}\right]^{2\alpha}.$$

So, the proof is completed. □

Corollary 12. *If G is a number weighted graph of order $n(\geq 3)$ with positive number edge weights, then*

$$\sqrt{2\sum_{\substack{j:\ j\sim i\\ i,j\in\{1,2,\ldots,n\}}}\left(\frac{w_i+w_j-2}{w_iw_j}\right)^{2\alpha}}\leq E_{ABC_w}^{(\alpha)}\leq$$
$$\leq\sqrt{2n\sum_{\substack{j:\ j\sim i\\ i,j\in\{1,2,\ldots,n\}}}\left(\frac{w_i+w_j-2}{w_iw_j}\right)^{2\alpha}}. \tag{26}$$

Proof. Setting $p=1$ in (25) completes the proof. □

Corollary 13. *If G is an unweighted graph of order $n(\geq 3)$, then*

$$\sqrt{2\sum_{\substack{j:\ j\sim i\\ i,j\in\{1,2,\ldots,n\}}}\left(\frac{d_i+d_j-2}{d_id_j}\right)^{2\alpha}}\leq E_{ABC}^{(\alpha)}\leq$$
$$\leq\sqrt{2n\sum_{\substack{j:\ j\sim i\\ i,j\in\{1,2,\ldots,n\}}}\left(\frac{d_i+d_j-2}{d_id_j}\right)^{2\alpha}}. \tag{27}$$

If $\alpha=\frac{1}{2}$, then

$$\sqrt{2\left(n-2R_{-1}\left(G\right)\right)}\leq E_{ABC}\leq\sqrt{2n\left(n-2R_{-1}\left(G\right)\right)}. \tag{28}$$

Proof. If we write all i,j and $i\sim j$, $w_i=d_i$ and $w_{ij}=1$ in (25) and use (8), the proof of (27) can be seen. Taking $\alpha=\frac{1}{2}$ in (27) and using (9) completes the proof. □

3 Conclusion

The generalized ABC matrix and its energy $\left(E_{ABC_w}^{(\alpha)}\left(G\right)\right)$ of (positive definite) matrix weighted graphs are considered in this work and some upper and lower bounds are presented for $E_{ABC_w}^{(\alpha)}\left(G\right)$. By means of these bounds, some bounds are obtained for number weighted and unweighted graphs. These bounds are valid for the generalized *ABC* energy $E_{ABC}(G)$ for $\alpha = \frac{1}{2}$ and unweighted case.

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