Computer Science Journal of Moldova, vol.**29**, no.2(86), 2021

Magic Sigma Coloring of a Graph

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Abstract

A sigma coloring of a non-trivial connected graph G is a coloring $c : V(G) \to \mathbb{N}$ such that $\sigma(u) \neq \sigma(v)$ for every two adjacent vertices $u, v \in V(G)$, where $\sigma(v)$ is the sum of the colors of the vertices in the open neighborhood $N(v)$ of $v \in V(G)$. The minimum number of colors required in a sigma coloring of a graph G is called the sigma chromatic number of G, denoted $\sigma(G)$. A coloring $c : V(G) \to \{1, 2, \cdots, k\}$ is said to be a magic sigma coloring of G if the sum of colors of all the vertices in the open neighborhood of each vertex of G is the same. In this paper, we study some of the properties of magic sigma coloring of a graph. Further, we define the magic sigma chromatic number of a graph and determine it for some known families of graphs.

Keywords: Sigma Coloring, open neighborhood sum, magic sigma coloring, sigma chromatic number.

MSC 2010: 05C15.

1 Introduction and preliminaries

Graph coloring is a very important branch of graph theory which has been studied by various authors. While the most commonly studied type of vertex coloring is proper coloring, several of its variations have been introduced and extensively studied. For related work on graph coloring, we refer [11]–[13]. Many of these colorings have been introduced so as to ensure vertex-distinguishing, edge-distinguishing or neighbor-distinguishing properties in a graph. One such coloring is a neighbor-distinguishing coloring, named the sigma coloring of a graph.

As introduced by Chartrand et al. in the year 2010, a k-vertex coloring $c : V(G) \to \mathbb{N}$ of a non-trivial graph G is said to be *sigma coloring* of G if $\sigma_c(u) \neq \sigma_c(v)$ for every two adjacent vertices $u, v \in V(G)$, where $\sigma_c(v) = \sum_{w \in N(v)} c(w)$, called the *open neighborhood sum* of v, is the sum of the colors of all the vertices in the open neighborhood $N(v)$ of $v \in V(G)$. The *sigma chromatic number* of a graph G, denoted $\sigma(G)$, is the minimum number of colors required in a sigma coloring of G. In the paper, it has been proved that the sigma chromatic number of a graph G is bounded by its chromatic number $\chi(G)$. Also, many characterizations of the sigma chromatic number have been established. It is worth mentioning here that the sigma coloring of a graph has been independently studied as lucky labeling by Czerwinski et al. [3] and additive labeling of a graph by Bartnicki et al. [1].

Since its introduction, several studies on the sigma chromatic number have been carried out. In particular, the complexity of sigma partitioning and sigma chromatic number has been discussed by Dehghan et al. [4], [5]. Further, the sigma chromatic number of some particular families of graphs has been obtained by various authors [6], [7].

One interesting question pertaining to graphs G that are not sigma colorable is whether there exists a coloring c of G such that all its vertices receive the same open neighborhood sum, i.e., $\sigma_c(u) = \sigma_c(v)$ for all $u, v \in V(G)$. We introduce the notion of magic sigma coloring to answer this question. Further, we study some of its properties and identify certain families of graphs that admit magic sigma coloring. Also, we obtain the magic sigma chromatic number of some such families of graphs. For standard graph related terminologies, we refer [2], [8], [9].

Definition 1. *Given a simple connected graph $G = (V, E)$, a coloring $c : V(G) \to \{1, 2, \cdots, k\}$ is said to be a magic sigma coloring of G if $\sigma_c(u) = \sigma_c(v)$ for all $u, v \in (G)$. Further, a graph G which admits a magic sigma coloring is said to be magic sigma colorable and $\sigma_c(G) = \sigma_c(v), v \in V(G)$, is called the open neighborhood sum of G w. r. t. the coloring c. Further, a disconnected graph G is said to be magic sigma colorable if each of its components is magic sigma colorable.*

Here, it has to be noted that the magic sigma coloring of a graph *need not be surjective. That is, all the elements in the co-domain, called colors, need not be used in a magic sigma coloring.*

Definition 2. *Let G be a graph with c being a magic sigma coloring of G. The c-color sum of G, denoted $S_c(G)$, is the sum of the colors of all the vertices in G, i.e., $S_c(G) = \sum_{v \in V(G)} c(v)$.*

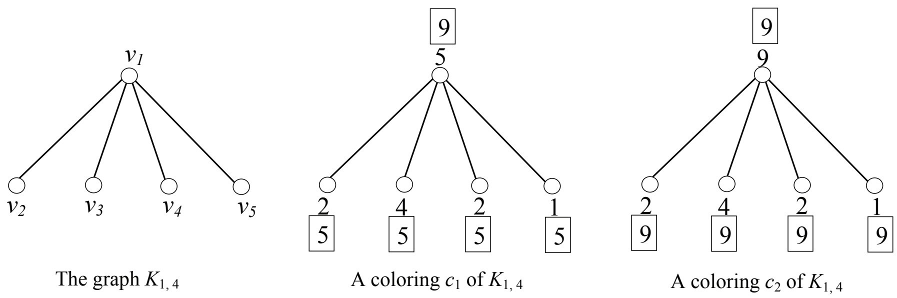

Figure 1. The graph $K_{1,4}$ and its two colorings

To illustrate, consider the graph $K_{1,4}$ and its two colorings c_1 and c_2 in Fig. 1. Since $\sigma_{c_1}(v_1) = 9 \neq 5 = \sigma_{c_1}(v_4), c_1$ is not a magic sigma coloring of $K_{1,4}$. However, c_2 is a magic sigma coloring of $K_{1,4}$ as the open neighborhood sum of each vertex w. r. t. c_2 is the same.

It is easy to observe that some graphs are magic sigma colorable and some others are not. For instance, consider the path P_4 with $V(P_4) = \{v_1, v_2, v_3, v_4\}$ and $E(P_4) = \{v_1v_2, v_2v_3, v_3v_4\}$. Further, for a coloring c to be a magic sigma coloring of P_4, we must have $N_c(v_1) = N_c(v_2) = N_c(v_3) = N_c(v_4)$. However, $N_c(v_1) = c(v_2)$ and $N_c(v_3) = c(v_2) + c(v_4)$. Consequently, we need to have $c(v_2) + c(v_4) = c(v_2)$, i.e., $c(v_4) = 0$. This is not feasible as each color in a magic sigma coloring has to be positive. Thus, P_4 is not magic sigma colorable.

2 Magic colorable graphs

We begin this section with some fundamental results pertaining to the magic sigma colorability of a graph. Further, we discuss about some particular families of graphs which are/are not magic sigma colorable.

Lemma 1. *If a graph G has two vertices such that the open neighborhood of one vertex is a proper subset of the other, then G is not magic sigma colorable.*

Proof. Let G be a graph having two vertices, say u and v, such that $N(u) \subset N(v)$. Then, $\sigma_c(u) < \sigma_c(v)$ in any coloring c of G. Hence, G is not magic sigma colorable. □

Remark 1. *The converse of the above lemma is not true. That is, it is not necessary that, if the graph is not magic sigma colorable, then there exist two of its vertices such that the open neighborhood of one vertex is a proper subset of the other.*

Proof. Consider the graph G_1 in Figure 2. Suppose G_1 is magic sigma

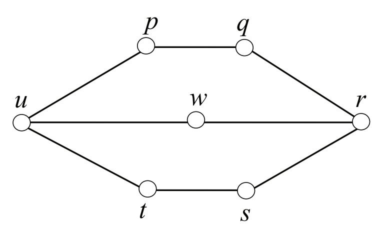

Figure 2. The graph G_1

colorable. Then, there exists a magic sigma coloring, say c of G_1 so that $\sigma_c(v)$ to be the same for all $v \in V(G_1)$. In particular, $\sigma_c(w) = \sigma_c(p) = \sigma_c(q) = \sigma_c(s) = \sigma_c(t)$ implies $c(q) = c(r) = c(s)$ and $c(p) = c(t) = c(u)$.

Similarly, $\sigma_c(u) = \sigma_c(r)$ gives $c(p) + c(t) = c(q) + c(s)$ so that $c(q) = c(r) = c(s) = c(p) = c(t) = c(u)$.

Now, $\sigma_c(u) = \sigma_c(w)$ gives $c(t) + c(w) = c(r)$ which implies that $c(w) = 0$, a contradiction since each color in a magic sigma coloring is positive. Hence, G_1 is not magic sigma colorable.

However, it is easy to observe that no two vertices of G_1 are such that the open neighborhood of one vertex is a proper subset of the other. □

Theorem 2. *The star graph $K_{1,n}, n \geq 1$, is magic sigma colorable.*

Proof. Consider a star graph $K_{1,n}$ with $V(K_{1,n}) = \{v_0, v_1, \cdots, v_n\}$ such that v_0 is the central vertex. Define a coloring $c : V(K_{1,n}) \to \{1, n\}$ as

$$c(v_i) = \begin{cases} n & \text{if } i = 0 \\ 1 & \text{otherwise} \end{cases}.$$

It is easy to observe that $\sigma_c(v_0) = \sigma_c(v_1) = \cdots = \sigma_c(v_n) = n$. Hence, $K_{1,n}$ is magic sigma colorable. □

Lemma 2. *A graph with a pendant vertex has a path of length three if and only if it is not magic sigma colorable.*

Proof. Consider a graph G with a pendant vertex v. Suppose G has a path of length three. Since G is connected, there is a path of length three starting from v. Let the path be $v - w - x - u$. Then, we have $N(v) \subset N(x)$. Therefore, by Lemma 1, G is not magic sigma colorable.

In order to prove the converse, we use the method of contraposition. Suppose G has no path of length three. Then, $diam(G) \leq 2$. Further, G has no cycle. This implies that G is isomorphic to the star graph $K_{1,n}, n \geq 1$ so that it is magic sigma colorable from Theorem 2. □

Corollary 3. *A graph G with $diam(G) \geq 3$ and having a pendant vertex is not magic sigma colorable.*

As a direct consequence of Lemma 2, we have the following result.

Theorem 4. *A non-trivial tree T is magic sigma colorable if and only if $T \cong K_{1,n}, n \geq 1$.*

Theorem 5. *Every k-regular graph with $k \geq 1$ is magic sigma colorable.*

Theorem 6. *The wheel graph W_n is magic sigma colorable.*

Proof. Let W_n be wheel on n vertices $v_1, v_2, \cdots, v_n$, with v_1 as the central vertex and the vertices $v_2 - v_3 - \cdots - v_n - v_2$ forming the cycle. Define a coloring $c : V(W_n) \to \{1, n-3\}$ as

$$c(v_i) = \begin{cases} n-3, & \text{if } i = 1 \\ 1, & \text{otherwise} \end{cases}.$$

It is easy to verify that c is a magic sigma coloring of W_n so that it is magic sigma colorable. □

Theorem 7. *The complete k-partite graph $K_{n_1,n_2,\cdots,n_k}$, $k \geq 2$, is magic sigma colorable, where each $n_i \geq 1$.*

Proof. Consider the complete k-partite graph $K_{n_1,n_2,\cdots,n_k}$, where each $n_i \geq 1$. Without loss in generality, let $n_1 \geq n_2 \geq \cdots \geq n_{k-1} \geq n_k$. Let $V_1 = \{v_{11}, v_{12}, \cdots, v_{1n_1}\}, V_2 = \{v_{21}, v_{22}, \cdots, v_{2n_2}\}, \cdots, V_k = \{v_{k1}, v_{k2}, \cdots, v_{kn_k}\}$ be the k-partite sets of $V(K_{n_1,n_2,\ldots,n_k})$.

Define a coloring $c : V(K_{n_1,n_2,\cdots,n_k}) \to \{1, 2, \cdots, n_1 + n_2 + \cdots + n_k\}$ as

$$c(v_{ij}) = \begin{cases} n_1 - j + 1 & \text{if } j = n_i \\ 1 & \text{otherwise} \end{cases}$$

for each $i = 1, 2, \cdots, k$ and $j = 1, 2, \cdots, n_i$.

By the definition of c, it follows that

$$\sigma_c(v_{ij}) = \sum_{l(\neq i)=1}^{k} \sum_{m=1}^{n_l} c(v_{lm})$$

for each $i = 1, 2, \cdots, k$ and $j = 1, 2, \cdots, n_i$.

In particular, we have the following:

i) For each $j = 1, 2, \cdots, n_1$,

$$\begin{aligned}
\sigma_c(v_{1j}) &= \sum_{l=2}^{k} \sum_{m=1}^{n_l} c(v_{lm}) \\
&= \sum_{l=2}^{k} \sum_{m=1}^{n_l-1} c(v_{lm}) + \sum_{l=2}^{k} c(v_{ln_l}) \\
&= \sum_{l=2}^{k} (n_l - 1) + \sum_{l=2}^{k} (n_1 - n_l + 1) \\
&= \sum_{l=2}^{k} n_1 = (k-1)n_1.
\end{aligned}$$

ii) For each $i = 2, 3, \cdots, k$ and $j = 1, 2, \cdots, n_i$,

$$\begin{aligned}
\sigma_c(v_{ij}) &= \sum_{l(\neq i)=1}^{k} \sum_{m=1}^{n_l} c(v_{lm}) \\
&= n_1 + \sum_{l(\neq i)=2}^{k} \sum_{m=1}^{n_l-1} c(v_{lm}) + \sum_{l(\neq i)=2}^{k} c(v_{ln_l}) \\
&= n_1 + \sum_{l(\neq i)=2}^{k} (n_l - 1) + \sum_{l(\neq i)=2}^{k} (n_1 - n_l + 1) \\
&= n_1 + \sum_{l(\neq i)=2}^{k} n_1 \\
&= n_1 + (k-2)n_1 = (k-1)n_1.
\end{aligned}$$

Thus, we see that the open neighborhood sum of all vertices is the same and is equal to $(k-1)n_1$. As a result, the complete k-partite graph $K_{n_1,n_2,\cdots,n_k}$ is magic sigma colorable. □

Corollary 8. *[10] If a graph G is such that a set consisting of any two adjacent vertices in G forms a minimal dominating set of G, then G is magic sigma colorable.*

Theorem 9. *The k^{th} power graph P_n^k of a path on $n \geq 3$ vertices is magic sigma colorable if and only if $k = n-2$ or $n-1$.*

Proof. Consider the graph P_n^k with $V(P_n^k) = \{v_1, v_2, \cdots, v_n\}$. Since $diam(P_n) = n-1$, we have $k \leq n-1$.

We prove the necessary part by the method of contraposition. Suppose $k \leq n-3$. Then, we see that in the graph P_n^k, $N(v_1) =$

$\{v_2, v_3, \cdots, v_{k+1}\}$ and $N(v_{k+2}) = \{v_2, v_3, \cdots, v_{k+1}, v_{k+3}, v_{k+4}, \cdots, v_{2k+2}\}$ so that $N(v_1) \subset N(v_{k+2})$. Hence, by Lemma 1, P_n^k is not magic sigma colorable.

We prove the converse considering two cases as follows.

Case (1): $k = n-1$. In this case, $P_n^k \cong K_n$ so that by Theorem 5, P_n^k is magic sigma colorable.

Case (2): $k = n-2$. Define a coloring $c : V(P_n^k) \to \{1, 2\}$ as

$$c(v_i) = \begin{cases} 1 & \text{if } i = 1, n \\ 2 & \text{otherwise} \end{cases}.$$

Since $k = n - 2 = diam(P_n) - 1$, v_1 is adjacent to all the vertices except v_n and vice-versa. Further, each of the other vertices is adjacent to all the vertices in P_n^k. Thus, we have $N(v_1) = N(v_n) = \{v_2, v_3, \cdots, v_{n-1}\}$ so that $\sigma_c(v_1) = \sigma_c(v_n) = 2(n-2)$ and $N(v_i) = V(P_n^k)$ so that $\sigma_c(v_i) = 1 + 2(n-3) + 1 = 2(n-2)$ for each $i = 2, 3, \cdots, n-1$.

Consequently, P_n^k is magic sigma colorable. □

Theorem 10. *Two graphs G and H are magic sigma colorable if and only if $G + H$ is magic sigma colorable.*

Proof. Let G and H be magic sigma colorable with c_1 and c_2 being the magic sigma colorings of G and H respectively. Let k and m be any non-negative integers such that $k\,(S_{c_1}(G) - \sigma_{c_1}(G)) = m\,(S_{c_2}(H) - \sigma_{c_2}(H))$.

Let c be a coloring of $G + H$ defined by

$$c(v) = \begin{cases} kc_1(v) & \text{if } v \in V(G) \\ mc_2(v) & \text{if } v \in V(H) \end{cases}.$$

Then, we have, for any vertex $v \in V(G + H)$, the following:

Case (1): If $v \in V(G)$, then $\sigma_c(v) = k\sigma_{c_1}(G) + mS_{c_2}(H)$.

Case (2): If $v \in V(H)$, then $\sigma_c(v) = m\sigma_{c_2}(G) + kS_{c_1}(G)$.

It is easy to verify that the open neighborhood sum of every vertex in $G+H$, w. r. t. c, is the same so that $G+H$ is magic sigma colorable.

Conversely, suppose $G+H$ is magic sigma colorable, with c being a magic sigma coloring of $G+H$. Then, for two vertices $u, v \in V(G)$, we have $\sigma_c(u) = \sigma_c(v)$.

This implies that

$$\sum_{\substack{x \in V(G) \\ (x,u) \in E(G)}} c(x) + S_c(H) = \sum_{\substack{y \in V(G) \\ (y,v) \in E(G)}} c(y) + S_c(H)$$

$$\text{so that} \quad \sum_{\substack{x \in V(G) \\ (x,u) \in E(G)}} c(x) = \sum_{\substack{y \in V(G) \\ (y,v) \in E(G)}} c(y)$$

$$\text{which implies that } \sigma_c(u)\big|_G = \sigma_c(v)\big|_G.$$

Thus, G is magic sigma colorable with the coloring c restricted to its vertices. The magic sigma colorability of H follows similarly. □

Theorem 11. *A graph G is magic sigma colorable if and only if its complement $\bar{G}$ is magic sigma colorable.*

Proof. Suppose G is magic sigma colorable with c being its magic sigma coloring.
Based on the fact that any graph G or its complement $\bar{G}$ is connected, we consider the following cases:

Case (1): Suppose G and $\bar{G}$ are both connected.
Since G is magic sigma colorable, $\sigma_c(u) = \sigma_c(v) \quad \forall\ u, v \in V(G)$ so that
$S_c(G) - \sigma_c(u) = S_c(G) - \sigma_c(v) \ \ \forall\ u, v \in V(G)$.
This implies that c is a magic sigma coloring of $\bar{G}$ with $S_c(G) - \sigma_c(G)$ as its open neighborhood sum. Hence, $\bar{G}$ is magic sigma colorable.

Case (2): Suppose G is connected, but $\bar{G}$ is disconnected.
Let H be an arbitrary component in $\bar{G}$. Then, there exists an edge between every vertex in $V(H)$ and every vertex in $V(G) - V(H)$

in G since H is a disconnected component of $\bar{G}$.
Let $u, v \in V(H)$ be arbitrary.
Then $\sigma_c(u)\big|_G = \sigma_c(v)\big|_G$ so that $S_c(G) - \sigma_c(u)\big|_G = S_c(G) - \sigma_c(v)\big|_G$.
Choosing a coloring c_1 of H such that $\sigma_{c_1}(v) = S_c(G) - \sigma_c(v)\big|_G$, for each $v \in H$, ensures that H is magic sigma colorable, which in turn, implies that $\bar{G}$ is also magic sigma colorable.

Case (3): Suppose G is disconnected, but $\bar{G}$ is connected, and $H_1, H_2, \cdots, H_k$ are the components of G. Then, each H_i is trivial or connected and is magic sigma colorable so that, from cases (1) and (2), its complement $\bar{H}_i$ is magic sigma colorable too. Further, $\bar{G} = H_1 + H_2 + \cdots + H_k$, so that, by Theorem 10, $\bar{G}$ is magic sigma colorable.

□

3 Magic sigma chromatic number of some graphs

In this section, we define the magic sigma chromatic number of a graph which is magic sigma colorable. Further, we determine this parameter for some classes of graphs.

Definition 3. *Suppose a graph G is magic sigma colorable. Then, the least k for which G admits a magic sigma coloring is called the magic sigma chromatic number of G, denoted by $\sigma_m(G)$.*

Observation 12. *If a graph G is magic sigma colorable, then $\sigma_m(G) \geq 1$.*

Theorem 13. *A graph G is regular if and only if $\sigma_m(G) = 1$.*

Proof. Suppose G is a regular graph. Then, by Theorem 5, G is magic sigma colorable. Further, the coloring $c : V(G) \to \{1\}$ is a magic sigma coloring of G so that $\sigma_m(G) = 1$.

Conversely, let G be a graph with $\sigma_m(G) = 1$. Then, there exists a magic sigma coloring, say $c : V(G) \to \{1\}$ i. e., every vertex is given the same color 1 in G. Suppose G is not regular. Then, there exist at least two vertices, say u and v, such that $deg(u) \neq deg(v)$. Then, $\sigma_c(u) \neq \sigma_c(v)$, a contradiction to the fact that c is a magic sigma coloring of G. Thus, G is a regular graph. □

Lemma 3. *For a star graph $K_{1,n}, n \geq 1$, $\sigma_m(K_{1,n}) = n$.*

Proof. Let $V(K_{1,n}) = \{v_0, v_1, \cdots, v_n\}$ such that v_0 is the central vertex. By Theorem 2, $K_{1,n}$ is magic sigma colorable and uses n colors. Thus, $\sigma_m(K_{1,n}) \leq n$.

For $n = 1$, we have $K_{1,1} \cong P_2$ so that $\sigma_m(K_{1,1}) = 1$ by Theorem 7. Consider the case $n \geq 2$. For a coloring c of $K_{1,n}$ to be a magic sigma coloring, we must have $\sigma_c(v_i) = \sigma_c(v_j)$ for all $i, j = 0, 1, \cdots, n$. Also, we have $\sigma_c(v_0) = \sum\limits_{i=1}^{n} c(v_i)$ and $\sigma_c(v_i) = c(v_0)$ for each $i = 1, 2, \cdots, n$. Thus, $c(v_0) = \sum\limits_{i=1}^{n} c(v_i) \geq n$ since each color is positive. Consequently, $\sigma_m(K_{1,n}) \geq n$. Therefore, $\sigma_m(K_{1,n}) = n$. □

Theorem 14. *The magic sigma chromatic number of the complete k-partite graph $K_{n_1,n_2,\cdots,n_k}$ with $k \geq 2$ and $n_i \geq n_{i+1} \geq 1, i = 1, 2, \cdots k-1$, is $\lceil \frac{n_1}{n_k} \rceil$.*

Proof. Consider the complete k-partite graph $K_{n_1,n_2,\cdots,n_k}$ with $1 \geq n_i \geq n_{i+1}, i = 1, 2, \cdots k-1$. Let V be the vertex set of $K_{n_1,n_2,\cdots,n_k}$ with $V_1 = \{v_{11}, v_{12}, \cdots, v_{1n_1}\}$, $V_2 = \{v_{21}, v_{22}, \cdots, v_{2n_2}\}$, ..., $V_k = \{v_{k1}, v_{k2}, \cdots, v_{kn_k}\}$ being its k-partite sets.

By Theorem 7, $K_{n_1,n_2,\cdots,n_k}$ is magic sigma colorable. For a coloring c to be a magic sigma coloring of $K_{n_1,n_2,\cdots,n_k}$, we should have $\sigma_c(u) = \sigma_c(v)$ for all $u, v \in V$. In particular, $\sigma_c(u) = \sigma_c(v)$ for all $u \in V_1$ and $v \in V_k$. This implies that

$$\sum_{v_{ij} \in V(i \neq 1)} c(v_{ij}) = \sum_{v_{ij} \in V(i \neq k)} c(v_{ij}).$$

Simplifying, we get $\sum\limits_{l=1}^{n_k} c(v_{kl}) = \sum\limits_{m=1}^{n_1} c(v_{1m}) \geq n_1$ since each color is positive. Thus, by the generalized pigeon hole principle, we see that there exists at least one vertex $v \in V_k$ with $c(v) \geq \lceil \frac{n_1}{n_k} \rceil$. We therefore conclude that $\sigma_m(K_{n_1,n_2,\cdots,n_k}) \geq \lceil \frac{n_1}{n_k} \rceil$.

To prove the reverse inequality, define $c : V \to \{1, 2, \cdots, \lceil \frac{n_1}{n_k} \rceil\}$ as

$$c(v_{ij}) = \begin{cases} \lceil \frac{n_1}{n_i} \rceil, & \text{if } j = 1 \\ \left\lceil \frac{n_1 - \sum\limits_{m=1}^{j-1} c(v_{im})}{n_i - j + 1} \right\rceil, & \text{otherwise} \end{cases}$$

for each $i = 1, 2, \cdots, k$.
By the definition of c, we see that $c(v_{k1}) = \lceil \frac{n_1}{n_k} \rceil \geq c(v_{ij})$ for all i, j so that the greatest color used in c is $\lceil \frac{n_1}{n_k} \rceil$. Further, it is easy to observe that c is a magic sigma coloring of $K_{n_1,n_2,\cdots,n_k}$. Consequently, $\sigma_m(K_{n_1,n_2,\cdots,n_k}) \leq \lceil \frac{n_1}{n_k} \rceil$.

Therefore, $\sigma_m(K_{n_1,n_2,\cdots,n_k}) = \lceil \frac{n_1}{n_k} \rceil$. □

Conclusion The concept of magic sigma coloring of graphs has been introduced in this paper. Some families of graphs which are/are not magic sigma colorable have been identified. Further, the magic sigma chromatic number of some such graphs have been established. As a continuation of the work carried out in the paper, one can attempt to characterize magic sigma colorable graphs. Also, graphs with a specific magic sigma chromatic number can be constructed. Further, forbidden graphs pertaining to magic sigma coloring can be identified.

Acknowledgment The authors are thankful to the managements of Tumkur University, Tumakuru and Dr. Ambedkar Institute of Technology, Bengaluru for their constant support and encouragement during the preparation of this paper. The authors are indebted to the learned referees for their thoughtful comments and suggestions.

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Received May 31, 2020
Accepted April 27, 2021

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