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Connected Domination Number and a New Invariant in Graphs with Independence Number Three

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Abstract

Adding a connected dominating set of vertices to a graph G increases its number of Hadwiger $h(G)$. Based on this obvious property in [2] we introduced a new invariant $\eta(G)$ for which $\eta(G) \leq h(G)$. We continue to study its property. For a graph G with independence number three without induced chordless cycles C_7 and with $n(G)$ vertices, $\eta(G) \geq n(G)/4$.

Keywords: dominating set, number of Hadwiger, clique number, independence number.

1 Introduction

All graphs considered in this paper are undirected, simple and finite. Let G be a graph with vertex set $V(G)$. We denote $|V(G)|$ by $n(G)$. Let $X \subseteq V(G)$, X is connected if the subgraph $G[X]$ induced by X is connected. Further, $G-X = G[V(G)-X]$. X is dominating in a graph G if every vertex of G is in X or has a neighbor in X. We will write $v \sim u$ ($v \nsim u$) when vertices v and u are (are not) adjacent. If every pair of vertices in X are adjacent, then $G[X]$ is a complete subgraph or a clique K_n, where $n = |X|$. The clique number $\omega(G)$ of a graph G is the number of vertices in a maximum clique in G. The degree of a vertex v is $deg(v)$, the number of edges that are incident to the vertex. The maximum degree $\Delta(G)$ and the minimum degree $\delta(G)$ of a graph G are the maximum and the minimum degree of its vertices. A k-colouring of G is a function that assigns one of k colours to each vertex of G such that adjacent vertices receive distinct colours. The chromatic number $\chi(G)$ is the minimum integer k such that G is k-colourable. A graph H is a minor of the graph G if H can be formed from G by deleting edges and vertices and by contacting edges. The Hadwiger number $h(G)$ is the maximum integer n such that the complete graph K_n is a minor of G. Further, a cycle $C_n = (v_1, v_2, \ldots, v_l)$ is a chordless cycle of length n, and any dominating set is a connected dominating set.

In [2] we introduced a new invariant $\eta = \eta(G)$ of a graph G as a maximum length of a sequence of subsets of its vertices $V_1, V_2, V_3, \ldots$, where $V_i \cap V_j = \varnothing$ $(i \neq j)$ and V_k is a dominating set in a graph $G[V_1 \cup V_2 \cup \ldots \cup V_k]$, $k = 1, 2, \ldots, \eta$. In the mentioned paper we have proved some properties of $\eta(G)$:

(i) $\omega(G) \leq \eta(G) \leq h(G)$,
(ii) If D is any dominating set in G, then $\eta(G) \geq \eta(G - D) + 1$,
(iii) $\eta(G) \leq \Delta(G) + 1$,
(iv) $\eta(G) \geq \chi(G)$ if $\chi(G) \leq 4$,

and we have posed the stronger than Hadwiger's conjecture:

Conjecture 1. *For all graphs G, $\chi(G) \leq \eta(G)$.*

2 Vertex cut sets and one more property of the new invariant

We say that a graph G is η-critical if $\eta(G) = \eta$ and $\eta(H) < \eta$ for every proper subgraph H of G. It is obvious that any η-critical graph is connected. 2-critical graph is an edge, 3-critical graph is a cycle. It's clear that if graph G is η-critical, $\eta(G - D) = \eta - 1$ and D is a dominating set in G, then $G - D$ is $(\eta - 1)$-critical.

A vertex cut set of a connected graph G is a subset $S \subseteq V(G))$ such that $G - S$ has more than one connected component. A subgraph induced by a vertex cut set is a cut subgraph.

Let S be a vertex cut set of a connected graph G, and the components of $G - S$ have vertex sets $U_1, U_2, \ldots, U_m$, $m \geq 2$. Denote $G_i = G[U_i \cup S]$.

Theorem 1. *Let $G[S]$ be complete, then for all G_i and for each its induced subgraph G'_i, the following is true: if $D \subseteq V(G) - V(G'_i)$ and D is a dominating set in a graph $G[D \cup V(G'_i)]$, then there exists D_i such that $D_i \subseteq D$, $D_i \subseteq V(G_i)$ and D_i is a dominating set in a graph $G[D_i \cup V(G'_i)]$.*

Proof. $D = A \cup S_i \cup B$, where $A \subseteq U_i$ $S_i \subseteq S$, $B \cap V(G_i) = \varnothing$. Since $G[S_i]$ is complete, then $G[A \cup S_i]$ is connected and all vertices of subgraph G'_i are joined with at least one vertex of the set $A \cup S_i$. So we can take $D_i = A \cup S_i$. ■

Theorem 2. *If S is a vertex cut set of a connected graph G and $G[S]$ is complete, then $\eta(G) = max_{1 \leq i \leq m} \eta(G_i)$.*

Proof. From definition $\eta(G) \geq max_{1 \leq i \leq m} \eta(G_i)$. Let $\eta = \eta(G)$ and $V_1, V_2, \ldots, V_\eta$ be a longest sequence of the sets of vertices in the definition of $\eta(G)$. If $\eta = |S|$, then $\eta(G_i) \geq \eta$ for all i, and the theorem is true. If $\eta > |S|$, then at least one set V_j contains vertices from set $V(G) - S$. Let V_{j_0} be the first such set from the sequence and let V_{j_0} contains vertices from U_{i_0}. By Theorem 1, there exists a dominating set $V'_{j_0}, V'_{j_0} \subseteq V_{j_0}$ containing only vertices of the set V_{i_0}. Therefore, there exists a sequence $V_1, \ldots, V_{j_0 - 1}, V'_{j_0}, V'_{j_0+1}, \ldots, V'_\eta$ of dominating sets of vertices of the graph $G_{i_0} = G[U_{i_0} \cup S]$ and therefore, $\eta(G_{i_0}) \geq \eta = \eta(G)$. ■

Corollary 1. *If G is η-critical, then G does not contain complete cut subgraphs.*

Proof. Let G be η-critical, $G[S]$ is complete. If we suppose that graph $G - S$ has components with not empty vertex sets U_1, U_2 and $\eta(G_1) \geq \eta(G_2)$, where $G_1 = G[U_1 \cup S]$ and $G_2 = G[U_2 \cup S]$, then, by Theopem 2, $\eta(G) = \eta(G_!)$ and therefore, G is not η-critical.

In [2] we proved that $\eta(G) \geq 4$ for graphs G with $\delta(G) \geq 3$. Using Theorem 2, we can slightly strengthen this result.

Corollary 2. *If degrees of all vertices of a graph G are at least three, except maybe one case from three: (a) one vertex of degree one, (b) one vertex of degree two, (c) two adjacent vertices of degree two, then $\eta(G) \geq 4$.*

Proof. (a) Let in graph G $deg(v) = 1$, $v \sim u$ and graph G' is isomorphic to G. Let in graph G' $deg(v') = 1$, $v' \sim u'$. From disjoint union of G and G' we form a new graph H by identifying v and u', u and v'. Since $\delta(H) \geq 3$, $\eta(H) \geq 4$ and by Theorem 2, $\eta(G) = \eta(H)$.

(b) Let in graph G $deg(v) = 2$, G' is isomorphic to G and in graph G' $deg(v') = 2$. We form a new graph H by identifying v and v'. Since $\delta(H) \geq 3$, $\eta(G) = \eta(H) \geq 4$.

(c) Let in graph G $deg(v) = deg(u) = 2$, $v \sim u$ and G' is isomorphic to G. Let in graph G' $deg(v') = deg(u') = 2$ and $v' \sim u'$. We form a new graph H by identifying v and v', u and u'. Since $\delta(H) \geq 3$, $\eta(G) = \eta(H) \geq 4$.

3 Domination and independence number

We need to introduce more notations. In a graph G, the independence number $\alpha(G)$ is the maximum cardinality of an independent set. The connected domination number $\gamma_c(G)$ is the number of vertices in the minimum connected dominating set. The neighborhood of vertex $v \in V(G)$, denoted by $N(v)$, is a set of all vertices adjacent to v. The closed neighborhood of v is $N[v] = N(v) \cup v$. A simplicial vertex of a graph G is a vertex v for which $G[N(v)]$ is complete.

Duchet and Meyniel [3] proved that $\gamma_c(G) \leq 2\alpha(G) - 1$ for any graph G. It is clear that $\alpha(C_{2l+1}) = l$, $\gamma_c(C_{2l+1}) = 2l - 1$, and for these graphs $\gamma_c(C_{2l+1}) = 2\alpha(C_{2l+1}) - 1$. Other upper bounds include additional graph parameters or conditions (see [1, 4, 5]). Plummer, Stiebitz and Toft [6] proved for any connected graph G that if $\alpha(G) = 2$ and G does not contain C_5, then for any non-simplicial vertex v graph G contains a dominating edge vu.

A claw $K_{1,3}$ is a graph with four vertices for which one vertex has three pairwise nonadjacent neighbors. It is clear that if $\alpha(G) = 3$ and graph G has a claw $K_{1,3}$ as an induced subgraph, then $V(K_{1,3})$ is a dominating set.

Theorem 3. *If graph G is connected, claw-free, $\alpha(G) = 3$, and G does not contain an induced C_7, then for any non-simplicial vertex v there* *exists connected dominating set D, such that $v \in D$ and $n(D) \leq 4$.*

Proof. Let $v \sim v_1, v \sim v_2, v_1 \nsim v_2$. Denote by V_1 the set of neighbors of v_1 which are not neighbors of v_2, by V_2 – the set of neighbors of v_2 which are not neighbors of v_1 and by V_{12} – the set of vertices adjacent to both v_1 and v_2. Denote by G' subgraph $G - N[v]$ and by V'_{12} – the set of vertices $V(G') - (V_1 \cup V_{12} \cup V_2)$. Since $\alpha(G) = 3$, $\alpha(G') \leq 2$, and subgraph $G[V'_{12}]$ is complete or $V'_{12} = \varnothing$.

If $V'_{12} = \varnothing$, then $D = v, v_1, v_2$. If $V'_{12} \neq \varnothing$ and $V_1 \cup V_2 \cup V_{12} = \varnothing$, then $D = v, a, b$, where a is any vertex adjacent to v and to vertex $b \in V'_{12}$. Let $V'_{12} \neq \varnothing$ and $V_1 \cup V_2 \cup V_{12} \neq \varnothing$. Since G is claw-free, induced subgraphs $G[V_1 \cup V_{12}]$ and $G[V_2 \cup V_{12}]$ are complete (one of them can be null graph), and if vertex $u \in V_{12}$, then u does not have neighbors in $G[V'_{12}]$.

Case 1. G' is connected.

1.1 G' contains $C_5 = (u_1, u_2, u_3, u_4, u_5)$.
Sets $V_1 \cup V_{12}$, $V_2 \cup V_{12}$, V'_{12} contain at most two consecutive vertices of C_5, and V_{12} contains at most one. Since G is claw-free, if V_{12} does not contain vertex of C_5, then V_1 and V_2 contain two vertices each. There are two possible cases:

1.1.1 $u_1 \in V_1$, $u_2 \in V_{12}$, $u_3 \in V_2$, $\{u_4, u_5\} \subseteq V'_{12}$. See Figure 1.

1.1.2 $\{u_1, u_2\} \subseteq V_1$, $\{u_3, u_4\} \in V_2$, $u_5 \subseteq V'_{12}$.

Since $\alpha(G') = 2$, any pair of nonadjacent vertices from V_1 and V_2 is adjacent to all vertices from V'_{12}. If u_4 is connected to all vertices V'_{12}, then $D = \{v, v_1, v_2, u_4\}$. If $u \in V'_{12}$ and $u_4 \nsim u$, then $u_2 \sim u$. See Figure 2.

1.2 G' does not contain C_5.

1.2.1 G' is complete.
In this case there exists a dominating set $D = \{v, a, b\}$, where a is any vertex such that $a \sim v$, $a \sim b$, where b is any vertex from $V(G')$.

1.2.2 G' is not complete.

In subgraph G' for any non-simplicial vertex $b \in V(G')$ there exists an edge bc dominating in G' (see [6]). If b is adjacent to $a \in N(v)$, then $D = \{v, a, b, c\}$. Now let all vertices of G', which are connected to $N(v)$, be simplicial.

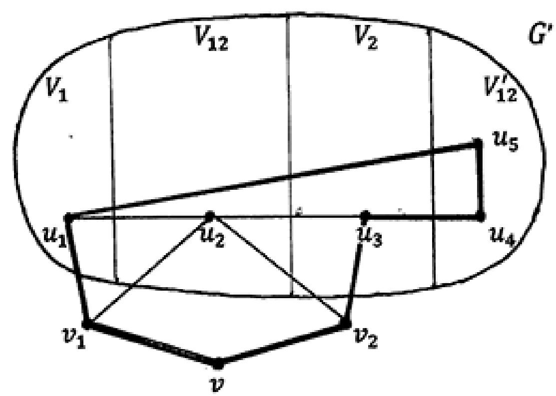

Figure 1. $C_7 = (v, v_2, u_3, u_4, u_5, u_1, v_1)$

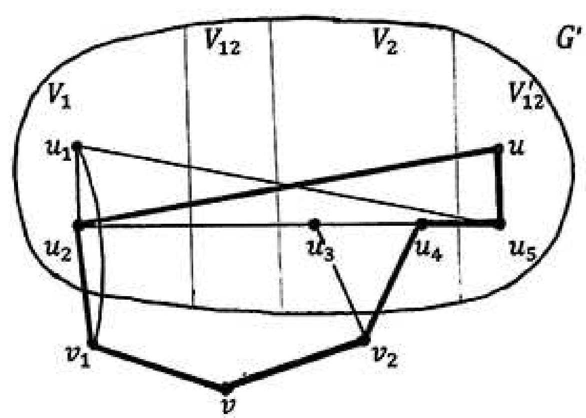

Figure 2. $C_7 = (v, v_2, u_4, u_5, u, u_2, v_1)$

1.2.2.1 $V_{12} \neq \varnothing$.

In this case subgraph $G[V_1 \cup V_2 \cup V_{12}]$ is complete and, since G' is connected, there exists an edge ab, where $b \in V'_{12}$ and $a \in V_1$ (or $a \in V_2$). Therefore, $D = \{v, v_1, a, b\}$ (or $D = \{v, v_2, a, b\}$).

1.2.2.2 $V_{12} = \varnothing$.

Since all vertices of the sets V_1 and V_2 are simplicial, $G[V_1 \cup V_2]$ is complete (and we have the same dominating set as in case 1.2.2.1) or V_1 does not have neighbors in V_2. In the last case any two vertices $u_1 \in V_1$ and $u_2 \in V_2$ are adjacent to all vertices V'_{12}. If one of these two vertices is adjacent to all V'_{12}, then $D = \{v, v_1, v_2, u_1\}$ or $D = \{v, v_1, v_2, u_2\}$. Otherwise, there exist two vertices $u_3, u_4 \in V'_{12}$ such that $u_1 \nsim u_3$ and $u_2 \nsim u_4$. See Figure 3.

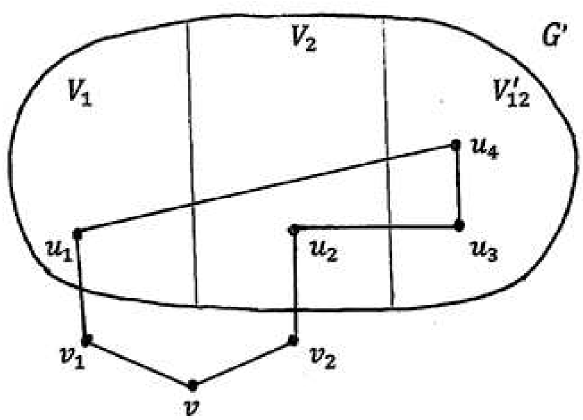

Figure 3. $C_7 = (v, v_1, u_1, u_4, u_3, u_2, v_2)$

Case 2. G' is not connected.

Since $V_1 \cup V_2 \cup V_{12} \neq \varnothing$, $V'_{12} \neq \varnothing$, $\alpha(G') = 2$, subgraph G' is a disjoint union of two complete subgraphs $G'_1 = G[V_1 \cup V_2 \cup V_{12}]$ and $G'_2 = G[V'_{12}]$. Since G is claw-free, any vertex $a \in N(v)$ is adjacent to at least one vertex v_1 or v_2, and if a has neighbors in G'_2, then a is adjacent to exactly one. Since $\alpha(G) = 3$, if a has neighbors in G'_2, $a \sim v_1$, $a \nsim v_2$, then a is adjacent to all vertices V'_{12} or $V_1 = \varnothing$. If a is adjacent to all vertices V'_{12}, then we can take $D = \{a, v, v_2, b\}$, where $b \in V_2 \cup V_{12}$ or $D = \{a, v, v_1\}$ if $V_2 \cup V_{12} = \varnothing$. If $V_1 = \varnothing$, then $D = \{v_2, v, a, c\}$, where $c \sim a$ and $c \in V'_{12}$. ■

Remark 1. *The graph G shown in Figure 4 does not contain $C_7, \alpha(G) = 3$, and the set of vertices $\{a, v_1, v_2, b\}$ induced a claw. In this graph any connected dominating set with vertex v, contains at least five vertices.*

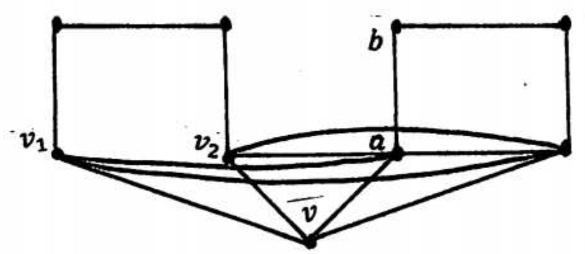

Figure 4. v is a non-simplicial vertex, $\{a, v_1, v_2, b\}$ is a vertex set of a claw

Corollary 3. *Let G be a graph with $\alpha(G) = 3$. If G does not contain an induced C_7, then $h(G) \geq n(G)/4$.*

Proof. We proceed by induction on $n = n(G)$. For $n \leq 4$, the result is clear. Suppose $n \geq 5$ and suppose the result is true for all graphs with fewer than n vertices and let G be a graph with n vertices. If G contains a claw, then the set D of vertices of this claw is dominating in G, if not, by Theorem 3, we can build a dominating set D with $n(D) \leq 4$. In both cases

$$h(G) \geq h(G - D) + 1 \geq \frac{n(G - D)}{4} + 1 \geq \frac{n(G) - 4}{4} + 1 = n(G)/4.$$

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