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Properties of Finitely Supported Self - Mappings on the Finite Powerset of Atoms

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Abstract

The theory of finitely supported algebraic structures represents a reformulation of Zermelo-Fraenkel set theory in which every classical structure is replaced by a finitely supported structure according to the action of a group of permutations of some basic elements named atoms. It provides a way of representing infinite structures in a discrete manner, by employing only finitely many characteristics. In this paper we present some (finiteness and fixed point) properties of finitely supported self-mappings defined on the finite power set of atoms.

Keywords: finitely supported structures, atoms, finite powerset, injectivity, surjectivity, fixed points.

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1 Introduction

Finitely Supported Mathematics (FSM) is a general name for the theory of finitely supported sets equipped with finitely supported internal operations or with finitely supported relations [2]. Finitely supported sets are related to the recent development of the Fraenkel-Mostowski axiomatic set theory, to the theory of admissible sets of Barwise (particularly by generalizing the theory of hereditary finite sets) and to the theory of nominal sets. Fraenkel-Mostowski set theory (FM) represents an axiomatization of the Fraenkel Basic Model of the Zermelo-Fraenkel set theory with atoms (ZFA); its axioms are the ZFA axioms together with an axiom of finite support claiming that any set-theoretical construction has to be finitely supported modulo a canonical hierarchically defined permutation action. An alternative approach for FM set theory that works in the classical Zermelo-Fraenkel (ZF) set theory (i.e. without being necessary to consider an alternative set theory obtained by weakening the ZF axiom of extensionality) is related to the theory of nominal sets that are defined as usual ZF sets equipped with canonical permutation actions of the group of all one-to-one and onto transformations of a fixed infinite, countable ZF set formed by basic elements (i.e. by elements whose internal structure is not taken into consideration, called 'atoms') satisfying a finite support requirement (meaning that 'for every element x in a nominal set there should exist a finite subset of basic elements S such that any one-to-one and onto transformation of basic elements that fixes S pointwise also leaves x invariant under the effect of the permutation action with who the nominal set is equipped').

Nominal sets [5] are related to binding, freshness and renaming in the computation of infinite structures containing enough symmetries such that they can be concisely manipulated. Ignoring the requirement regarding the countability of A in the definition of a nominal set, and motivated by Tarski's approach regarding logicality (a logical notion is defined by Tarski as one that is invariant under the one-to-one transformations of the universe of discourse onto itself), we introduce *invariant sets*. A finitely supported set is defined as a finitely supported element in the power set of an invariant set. Equipping finitely supported sets with finitely supported mappings and relations, we get finitely supported algebraic structures that form FSM.

In this paper we collect specific properties of finitely supported mappings defined of the finite power set of atoms [2]–[4] and we present some other new properties. We are particularly focused on proving the equivalence between injectivity and surjectivity for such mappings, together with some fixed point properties. Therefore, although the finite power set of atoms is infinite, it has some finiteness properties. Furthermore, although the finite power set of atoms is not a complete lattice in FSM, some fixed points of Tarski type hold. Particularly, finitely supported self-mappings defined on the finite powerset of atoms have infinitely many fixed points if they satisfy some properties (such as strict monotony, injectivity or surjectivity).

2 Preliminary Results

A finite set (without other specification) is referred to a set that can be represented as $\{x_1, \dots, x_n\}$ for some $n \in \mathbb{N}$. An infinite set (without other specification) means "a set which is not finite". We consider a fixed infinite ZF set A (called 'the set of atoms' by analogy with ZFA set theory; however, despite classical set theory with atoms, we do not need to modify the axiom of extensionality in order to define A). The atoms are entities whose internal structure is ignored and which are considered as basic for a higher-order construction. This means atoms can be checked only for equality.

A *transposition* is a function $(a\,b) : A \to A$ that interchanges only a and b. A *permutation* of A in FSM is a bijection of A generated by composing finitely many transpositions. We denote by S_A the group of all permutations of A. According to Proposition 2.11 and Remark 2.2 in [2], an arbitrary bijection on A is finitely supported if and only if it is a permutation of A.

Definition 1.

1. *Let X be a ZF set. An* S_A-action *on X is a group action $\cdot$ of S_A on X. An* S_A-set *is a pair $(X, \cdot)$, where X is a ZF set, and $\cdot$ is an S_A-action on X.*

2. *Let $(X, \cdot)$ be an S_A-set. We say that $S \subset A$* supports x *whenever for each $\pi \in Fix(S)$ we have $\pi \cdot x = x$, where $Fix(S) = \{\pi \,|\, \pi(a) = a, \forall a \in S\}$. The least finite set (w.r.t. the inclusion relation) supporting x (which exists according to [2]) is called* the support of x *and is denoted by $supp(x)$. An empty supported element is called* equivariant.

3. *Let $(X, \cdot)$ be an S_A-set. We say that X is an* invariant set *if for each $x \in X$ there exists a finite set $S_x \subset A$ which supports x.*

Proposition 1. *[2], [5] Let $(X, \cdot)$ and $(Y, \diamond)$ be S_A-sets.*

1. *The set A of atoms is an invariant set with the S_A-action $\cdot : S_A \times A \to A$ defined by $\pi \cdot a := \pi(a)$ for all $\pi \in S_A$ and $a \in A$. Furthermore, $supp(a) = \{a\}$ for each $a \in A$.*

2. *Let $\pi \in S_A$. If $x \in X$ is finitely supported, then $\pi \cdot x$ is finitely supported and $supp(\pi \cdot x) = \{\pi(u) \,|\, u \in supp(x)\} := \pi(supp(x))$.*

3. *The Cartesian product $X \times Y$ is also an S_A-set with the S_A-action $\otimes : S_A \times (X \times Y) \to (X \times Y)$ defined by $\pi \otimes (x, y) = (\pi \cdot x, \pi \diamond y)$ for all $\pi \in S_A$ and all $x \in X$, $y \in Y$. If $(X, \cdot)$ and $(Y, \diamond)$ are invariant sets, then $(X \times Y, \otimes)$ is also an invariant set.*

4. *The powerset $\wp(X) = \{Z \,|\, Z \subseteq X\}$ is also an S_A-set with the S_A-action $\star : S_A \times \wp(X) \to \wp(X)$ defined by $\pi \star Z := \{\pi \cdot z \,|\, z \in Z\}$ for all $\pi \in S_A$, and all $Z \subseteq X$. For each invariant set $(X, \cdot)$, we denote by $\wp_{fs}(X)$ the set of elements in $\wp(X)$ which are finitely supported according to the action $\star$. $(\wp_{fs}(X), \star|_{\wp_{fs}(X)})$ is an invariant set.*

5. *The finite powerset of X denoted by $\wp_{fin}(X) = \{Y \subseteq X \,|\, Y \text{finite}\}$ and the cofinite powerset of X denoted by $\wp_{cofin}(X) = \{Y \subseteq X \,|\, X \setminus Y \text{finite}\}$ are both S_A-sets with the S_A-action $\star$ defined as in the previous item. If X is an invariant set, then both $\wp_{fin}(X)$ and $\wp_{cofin}(X)$ are invariant sets.*

6. *We have $\wp_{fs}(A) = \wp_{fin}(A) \cup \wp_{cofin}(A)$. If $X \in \wp_{fin}(A)$, then $supp(X) = X$. If $X \in \wp_{cofin}(A)$, then $supp(X) = A \setminus X$.*

7. *Any ordinary (non-atomic) ZF-set X (such as $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$ for example) is an invariant set with the single possible S_A-action $\cdot : S_A \times X \to X$ defined by $\pi \cdot x := x$ for all $\pi \in S_A$ and $x \in X$.*

Definition 2. *Let $(X, \cdot)$ be an S_A-set. A subset Z of X is called* finitely supported *if and only if $Z \in \wp_{fs}(X)$. A subset Z of X is* uniformly supported *if all the elements of Z are supported by the same set S (and so Z is itself supported by S).*

From Definition 1, a subset Z of an invariant set $(X,\cdot)$ is finitely supported by a set $S \subseteq A$ if and only if $\pi \star Z \subseteq Z$ for all $\pi \in Fix(S)$, i.e. if and only if $\pi \cdot z \in Z$ for all $\pi \in S_A$ and all $z \in Z$. This is because any permutation of atoms should have finite order, and so the relation $\pi \star Z \subseteq Z$ is equivalent to $\pi \star Z = Z$.

Proposition 2. *[2] Let X be a uniformly supported (particularly, a finite) subset of an invariant set $(U,\cdot)$. Then X is finitely supported and $supp(X) = \cup\{supp(x) \,|\, x \in X\}$.*

Definition 3. *Let X and Y be invariant sets.*

1. *A function $f : X \to Y$ is* finitely supported *if $f \in \wp_{fs}(X \times Y)$.*

2. *Let Z be a finitely supported subset of X and T a finitely supported subset of Y. A function $f : Z \to T$ is* finitely supported *if $f \in \wp_{fs}(X \times Y)$. The set of all finitely supported functions from Z to T is denoted by T^Z_{fs}.*

Proposition 3. *[2], [5] Let $(X,\cdot)$ and $(Y,\diamond)$ be two invariant sets.*

1. *Y^X (i.e. the set of all functions from X to Y) is an S_A-set with the S_A-action $\widetilde{\star} : S_A \times Y^X \to Y^X$ defined by $(\pi \widetilde{\star} f)(x) = \pi \diamond (f(\pi^{-1} \cdot x))$ for all $\pi \in S_A$, $f \in Y^X$ and $x \in X$. A function $f : X \to Y$ is finitely supported (in the sense of Definition 3) if and only if it is finitely supported with respect the permutation action $\widetilde{\star}$.*

2. *Let Z be a finitely supported subset of X and T a finitely supported subset of Y. A function $f : Z \to T$ is supported by a finite set $S \subseteq A$ if and only if for all $x \in Z$ and all $\pi \in Fix(S)$ we have $\pi \cdot x \in Z$, $\pi \diamond f(x) \in T$ and $f(\pi \cdot x) = \pi \diamond f(x)$.*

3 Finitely Supported Self-Mappings on the Finite Powerset of A

This section collects surprising finiteness and fixed point properties of finitely supported self mappings defined on $\wp_{fin}(A)$. We involve specific

FSM proving techniques, especially properties of uniformly supported sets. Details regarding these aspects can be found in [2]–[4].

Theorem 1. *A finitely supported function $f : \wp_{fin}(A) \to \wp_{fin}(A)$ is injective if and only if it is surjective.*

Proof. 1. For proving the direct implication, assume, by contradiction, that $f : \wp_{fin}(A) \to \wp_{fin}(A)$ is a finitely supported injection having the property that $Im(f) \subsetneq \wp_{fin}(A)$. This means that there exists $X_0 \in \wp_{fin}(A)$ such that $X_0 \notin Im(f)$. We can construct a sequence of elements from $\wp_{fin}(A)$ which has the first term X_0 and the general term $X_{n+1} = f(X_n)$ for all $n \in \mathbb{N}$. Since $X_0 \notin Im(f)$, it follows that $X_0 \neq f(X_0)$. Since f is injective and $X_0 \notin Im(f)$, according to the injectivity of f we obtain that $f^n(X_0) \neq f^m(X_0)$ for all $n, m \in \mathbb{N}$ with $n \neq m$. Furthermore, X_{n+1} is supported by $supp(f) \cup supp(X_n)$ for all $n \in \mathbb{N}$. Indeed, let $\pi \in Fix(supp(f) \cup supp(X_n))$. According to Proposition 3, $\pi \star X_{n+1} = \pi \star f(X_n) = f(\pi \star X_n) = f(X_n) = X_{n+1}$. Since $supp(X_{n+1})$ is the least set supporting X_{n+1}, we obtain $supp(X_{n+1}) \subseteq supp(f) \cup supp(X_n)$ for all $n \in \mathbb{N}$. By induction on n, we have $supp(X_n) \subseteq supp(f) \cup supp(X_0)$ for all $n \in \mathbb{N}$. Thus, all X_n are supported by the same set of atoms $S = supp(f) \cup supp(X_0)$, which means the family $(X_n)_{n\in\mathbb{N}}$ is infinite and uniformly supported, contradicting the fact that $\wp_{fin}(A)$ has only finitely many elements supported by S, namely the subsets of S.

2. In order to prove the reverse implication, let us consider a finitely supported surjection $f : \wp_{fin}(A) \to \wp_{fin}(A)$. Let $X \in \wp_{fin}(A)$. Then $supp(X) = X$ and $supp(f(X)) = f(X)$ according to Proposition 2. Since $supp(f)$ supports f and $supp(X)$ supports X, for any π fixing pointwise $supp(f) \cup supp(X) = supp(f) \cup X$ we have $\pi \star f(X) = f(\pi \star X) = f(X)$ which means $supp(f) \cup X$ supports $f(X)$, that is $f(X) = supp(f(X)) \subseteq supp(f) \cup X$ (1).

For a fixed $m \geq 1$, let us fix m (arbitrarily considered) atoms $b_1, \ldots, b_m \in A \setminus supp(f)$. Let $\mathcal{F} = \{\{a_1, \ldots, a_n, b_1, \ldots, b_m\} \,|\, a_1, \ldots, a_n \in supp(f), n \geq 1\} \cup \{\{b_1, \ldots, b_m\}\}$. The set $\mathcal{F}$ is finite since $supp(f)$ is finite and the elements $b_1, \ldots, b_m \in A \setminus supp(f)$ are fixed. Let us consider an arbitrary $Y \in \mathcal{F}$, that is $Y \setminus supp(f) = \{b_1, \ldots, b_m\}$. There exists

$Z \in \wp_{fin}(A)$ such that $f(Z) = Y$. According to (1), Z must be either of form $Z = \{c_1, \ldots, c_k, b_{i_1}, \ldots, b_{i_l}\}$ with $c_1, \ldots, c_k \in supp(f)$ and $b_{i_1}, \ldots, b_{i_l} \in A \setminus supp(f)$ or of form $Z = \{b_{i_1}, \ldots, b_{i_l}\}$ with $b_{i_1}, \ldots, b_{i_l} \in A \setminus supp(f)$. In both cases we have $\{b_1, \ldots, b_m\} \subseteq \{b_{i_1}, \ldots, b_{i_l}\}$. We should prove that $l = m$ and hence the above sets are equal. Assume, by contradiction, that there exists b_{i_j} with $j \in \{1, \ldots, l\}$ such that $b_{i_j} \notin \{b_1, \ldots, b_m\}$. Then $(b_{i_j}\ b_1) \star Z = Z$ since both $b_{i_j}, b_1 \in Z$ and Z is a finite subset of A (b_{i_j} and b_1 are interchanged in Z under the effect of the transposition $(b_{i_j}\ b_1)$, while the other atoms belonging to Z are left unchanged, meaning that the entire Z is left invariant under the action $\star$). Furthermore, since $b_{i_j}, b_1 \notin supp(f)$, we have that the transposition $(b_{i_j}\ b_1)$ fixes $supp(f)$ pointwise, and, because $supp(f)$ supports f, from Proposition 3, we get $f(Z) = f((b_{i_j}\ b_1) \star Z) = (b_{i_j}\ b_1) \star f(Z)$, which is a contradiction, because $b_1 \in f(Z)$, while $b_{i_j} \notin f(Z)$. Thus, $\{b_{i_1}, \ldots, b_{i_l}\} = \{b_1, \ldots, b_m\}$, and so $Z \in \mathcal{F}$. Therefore, $\mathcal{F} \subseteq f(\mathcal{F})$, which means $|\mathcal{F}| \leq |f(\mathcal{F})|$. However, because f is a function and $\mathcal{F}$ is a finite set, we obtain $|f(\mathcal{F})| \leq |\mathcal{F}|$. We finally get $|\mathcal{F}| = |f(\mathcal{F})|$ and, because $\mathcal{F}$ is finite with $\mathcal{F} \subseteq f(\mathcal{F})$, we obtain $\mathcal{F} = f(\mathcal{F})$ (2), which means that $f|_{\mathcal{F}} : \mathcal{F} \to \mathcal{F}$ is surjective. Since $\mathcal{F}$ is finite, $f|_{\mathcal{F}}$ should be injective, i.e. $f(F_1) \neq f(F_2)$ whenever $F_1, F_2 \in \mathcal{F}$ with $F_1 \neq F_2$ (3).

Whenever $d_1, \ldots, d_u \in A \setminus supp(f)$ with $\{d_1, \ldots, d_u\} \neq \{b_1, \ldots, b_m\}$, $u \geq 1$, and considering $\mathcal{U} = \{\{a_1, \ldots, a_n, d_1, \ldots, d_u\} \,|\, a_1, \ldots, a_n \in supp(f), n \geq 1\} \cup \{\{d_1, \ldots, d_u\}\}$, we conclude that $\mathcal{F}$ and $\mathcal{U}$ are disjoint. Whenever $F_1 \in \mathcal{F}$ and $U_1 \in \mathcal{U}$, we have $f(F_1) \in \mathcal{F}$ and $f(U_1) \in \mathcal{U}$ by using the same arguments used to prove (2), and so $f(F_1) \neq f(U_1)$ (4). If $\mathcal{T} = \{\{a_1, \ldots, a_n\} \,|\, a_1, \ldots, a_n \in supp(f)\}$ and $Y \in \mathcal{T}$, then there is $T' \in \wp_{fin}(A)$ such that $Y = f(T')$. Similarly as in (2), we should have $T' \in \mathcal{T}$. Otherwise, if T' belonged to some $\mathcal{U}$ considered above, i.e. if T' contained an element outside $supp(f)$, we would get the contradiction $Y = f(T') \in \mathcal{U}$. Hence $\mathcal{T} \subseteq f(\mathcal{T})$ from which $\mathcal{T} = f(\mathcal{T})$ since $\mathcal{T}$ is finite (using similar arguments as those involved to prove (3) from $\mathcal{F} \subseteq f(\mathcal{F})$). Thus, $f|_{\mathcal{T}} : \mathcal{T} \to \mathcal{T}$ is surjective. Since $\mathcal{T}$ is finite, $f|_{\mathcal{T}}$ should be also injective, namely $f(T_1) \neq f(T_2)$ whenever $T_1, T_2 \in \mathcal{T}$ with $T_1 \neq T_2$ (5). The case $supp(f) = \emptyset$ is contained in the above analysis; it leads to $f(\emptyset) = \emptyset$ and $f(X) = X$ for all $X \in \wp_{fin}(A)$. We also have $f(T_1) \neq f(U_1)$ whenever $T_1 \in \mathcal{T}$ and $U_1 \in \mathcal{U}$ since $f(T_1) \in \mathcal{T}$, $f(U_1) \in \mathcal{U}$ and $\mathcal{T}$ and $\mathcal{U}$ are disjoint (6). Since $b_1, \ldots, b_m$ and $d_1, \ldots, d_u$ were arbitrarily chosen from $A \setminus supp(f)$, the injectivity of f leads from the claims (3), (4), (5) and (6) covering all the possible cases for two different finite subsets of atoms and comparison of the values of f over the related subsets of atoms. □

Proposition 4. *Let $f : \wp_{fin}(A) \to \wp_{fin}(A)$ be finitely supported and injective. For each $X \in \wp_{fin}(A)$ we have $X \setminus supp(f) \neq \emptyset$ if and only if $f(X) \setminus supp(f) \neq \emptyset$. Furthermore, $X \setminus supp(f) = f(X) \setminus supp(f)$. Moreover, if f is monotone (i.e. order preserving), then $X \setminus supp(f) = f(X \setminus supp(f))$ for all $X \in \wp_{fin}(A)$, and $f(supp(f)) = supp(f)$.*

Proof. Let us consider $Y \in \wp_{fin}(A)$. Then we have $supp(Y) = Y$. According to Proposition 3, for any permutation $\pi \in Fix(supp(f) \cup supp(Y)) = Fix(supp(f) \cup Y)$ we have $\pi \star f(Y) = f(\pi \star Y) = f(Y)$ meaning that $supp(f) \cup Y$ supports $f(Y)$, that is $f(Y) = supp(f(Y)) \subseteq supp(f) \cup Y$ (1). If $Y \subseteq supp(f)$, we have $f(Y) \subseteq supp(f)$ (2). Let $X \in \wp_{fin}(X)$ with $X \subseteq supp(f)$. From (2) we get $f(X) \subseteq supp(f)$. Conversely, assume $f(X) \subseteq supp(f)$. By successively applying (2), we obtain $f^n(X) \subseteq supp(f)$ for all $n \in \mathbb{N}^*$ (3). Since $supp(f)$ is finite, there should exist $l, m \in \mathbb{N}^*$ with $l \neq m$ such that $f^l(X) = f^m(X)$. Assume $l > m$. Since f is injective, we obtain $f^{l-m}(X) = X$, and so by (3) we conclude that $X \subseteq supp(f)$. Therefore, $X \subseteq supp(f)$ if and only if $f(X) \subseteq supp(f)$, and hence $X \setminus supp(f) \neq \emptyset$ if and only if $f(X) \setminus supp(f) \neq \emptyset$.

Let $T \in \wp_{fin}(A)$ such that $f(T) \setminus supp(f) \neq \emptyset$ or, equivalently, $T \setminus supp(f) \neq \emptyset$. Thus, T should have the form $T = \{a_1, \ldots, a_n, b_1, \ldots, b_m\}$ with $a_1, \ldots, a_n \in supp(f)$ and $b_1, \ldots, b_m \in A \setminus supp(f)$, $m \geq 1$, or the form $T = \{b_1, \ldots, b_m\}$ with $b_1, \ldots, b_m \in A \setminus supp(f)$, $m \geq 1$. According to (1), we should have $f(T) = \{c_1, \ldots, c_k, b_{i_1}, \ldots, b_{i_l}\}$ with $c_1, \ldots, c_k \in supp(f)$ and $b_{i_1}, \ldots, b_{i_l} \in A \setminus supp(f)$, or $f(T) = \{b_{i_1}, \ldots, b_{i_l}\}$ with $b_{i_1}, \ldots, b_{i_l} \in A \setminus supp(f)$, having in any case the property that $\{b_{i_1}, \ldots, b_{i_l}\}$ is non-empty (i.e. it should contain at least one element, say b_{i_1}) and $\{b_{i_1}, \ldots, b_{i_l}\} \subseteq \{b_1, \ldots, b_m\}$. If $m = 1$, then $l = 1$, $b_{i_1} = b_1$, and we are done, so let $m > 1$.

Assume by contradiction that there exists $j \in \{1, \ldots, m\}$ such that $b_j \notin \{b_{i_1}, \ldots, b_{i_l}\}$. Then $(b_{i_1}\ b_j) \star T = T$ since both $b_{i_1}, b_j \in T$ and T is a finite subset of atoms (b_{i_1} and b_j are interchanged in T under the effect of the transposition $(b_{i_1}\ b_j)$, but the whole T is left invariant). Furthermore, since $b_{i_1}, b_j \notin \mathit{supp}(f)$ we have that the transposition $(b_{i_1}\ b_j)$ fixes $\mathit{supp}(f)$ pointwise, and hence by Proposition 3 we obtain $f(T) = f((b_{i_1}\ b_j) \star T) = (b_{i_1}\ b_j) \star f(T)$ which is a contradiction because $b_{i_1} \in f(T)$ while $b_j \notin f(T)$. Thus, $\{b_{i_1}, \ldots, b_{i_l}\} = \{b_1, \ldots, b_m\}$, and so $T \setminus \mathit{supp}(f) = f(T) \setminus \mathit{supp}(f)$.

Assume now that f is monotone. Let us fix $X \in \wp_{fin}(A)$, and consider the case $X \setminus \mathit{supp}(f) \neq \emptyset$, that is $X = \{a_1, \ldots, a_n, b_1, \ldots, b_m\}$ with $a_1, \ldots, a_n \in \mathit{supp}(f)$ and $b_1, \ldots, b_m \in A \setminus \mathit{supp}(f)$, $m \geq 1$, or $X = \{b_1, \ldots, b_m\}$ with $b_1, \ldots, b_m \in A \setminus \mathit{supp}(f)$, $m \geq 1$. Therefore we get $X \setminus \mathit{supp}(f) = \{b_1, \ldots, b_m\}$, and by involving the above arguments, we should have $f(X \setminus \mathit{supp}(f)) = \{x_1, \ldots, x_i, b_1, \ldots, b_m\}$ with $x_1, \ldots, x_i \in \mathit{supp}(f)$ or $f(X \setminus \mathit{supp}(f)) = \{b_1, \ldots, b_m\}$. In either case we obtain $X \setminus \mathit{supp}(f) \subseteq f(X \setminus \mathit{supp}(f))$, and since f is monotone we construct an ascending chain $X \setminus \mathit{supp}(f) \subseteq f(X \setminus \mathit{supp}(f)) \subseteq \ldots \subseteq f^k(X \setminus \mathit{supp}(f)) \subseteq \ldots$. Since for any $k \in \mathbb{N}$ we have that $f^k(X \setminus \mathit{supp}(f))$ is supported by $\mathit{supp}(f) \cup \mathit{supp}(X \setminus \mathit{supp}(f)) = \mathit{supp}(f) \cup \mathit{supp}(X)$ and $\wp_{fin}(A)$ does not contain an infinite uniformly supported subset, the related chain should be stationary, that is there exists $n \in \mathbb{N}$ such that $f^n(X \setminus \mathit{supp}(f)) = f^{n+1}(X \setminus \mathit{supp}(f))$, which, according to the injectivity of f, leads to $X \setminus \mathit{supp}(f) = f(X \setminus \mathit{supp}(f))$.

It remains to analyze the case $X \subseteq \mathit{supp}(f)$ or, equivalently, $X \setminus \mathit{supp}(f) = \emptyset$. We have $f(\emptyset) \subseteq \mathit{supp}(f)$. In the finite set $\mathit{supp}(f)$ we can define the chain of subsets $\emptyset \subseteq f(\emptyset) \subseteq f^2(\emptyset) \subseteq \ldots \subseteq f^m(\emptyset) \subseteq \ldots$ which is uniformly supported by $\mathit{supp}(f)$. Therefore the related chain should be stationary, meaning that there should exist $k \in \mathbb{N}$ such that $f^k(\emptyset) = f^{k+1}(\emptyset)$. According to the injectivity of f, we get $X \setminus \mathit{supp}(f) = \emptyset = f(\emptyset) = f(X \setminus \mathit{supp}(f))$.

According to (2), we have $f(\mathit{supp}(f)) \subseteq \mathit{supp}(f)$, and because f preserves the inclusion relation, we construct in $\mathit{supp}(f)$ the chain $\ldots \subseteq f^m(\mathit{supp}(f)) \subseteq \ldots \subseteq f(\mathit{supp}(f)) \subseteq \mathit{supp}(f)$. Since $\mathit{supp}(f)$ is finite, the chain should be stationary, and so $f^{k+1}(\mathit{supp}(f)) = f^k(\mathit{supp}(f))$

for some positive integer k, which, because f is injective, conduces to $f(supp(f)) = supp(f)$. $\square$

Remark 1. *From the proof of Proposition 4, if* $f : \wp_{fin}(A) \to \wp_{fin}(A)$ *is finitely supported (even if it is not injective) with* $X \subseteq supp(f)$*, we have* $f(X) \subseteq supp(f)$*. If* $f(X) \setminus supp(f) \neq \emptyset$*, then* $X \setminus supp(f) = f(X) \setminus supp(f)$*.*

Corollary 1. *Let* $f : \wp_{fin}(A) \to \wp_{fin}(A)$ *be finitely supported and surjective. Then for each* $X \in \wp_{fin}(A)$ *we have* $X \setminus supp(f) \neq \emptyset$ *if and only if* $f(X) \setminus supp(f) \neq \emptyset$*. In either of these cases* $X \setminus supp(f) = f(X) \setminus supp(f)$*. If, furthermore,* f *is monotone, then* $X \setminus supp(f) = f(X \setminus supp(f))$ *for all* $X \in \wp_{fin}(A)$*, and* $f(supp(f)) = supp(f)$*.*

Proof. From Theorem 1, a finitely supported surjective function $f : \wp_{fin}(A) \to \wp_{fin}(A)$ should be injective. The result now follows from Proposition 4. $\square$

Theorem 2. *Let* $f : \wp_{fin}(A) \to \wp_{fin}(A)$ *be finitely supported and strictly monotone (i.e.* f *has the property that* $X \subsetneq Y$ *implies* $f(X) \subsetneq f(Y)$*). Then we have* $X \setminus supp(f) = f(X \setminus supp(f))$ *for all* $X \in \wp_{fin}(A)$*.*

Proof. Let $X \in \wp_{fin}(A)$. According to Proposition 2, we have $supp(X) = X$ and $supp(f(X)) = f(X)$. According to Proposition 3, for any permutation $\pi \in Fix(supp(f) \cup supp(X)) = Fix(supp(f) \cup X)$ we get $\pi \star f(X) = f(\pi \star X) = f(X)$ meaning that $supp(f) \cup X$ supports $f(X)$, that is $f(X) = supp(f(X)) \subseteq supp(f) \cup X$ (1).

If $supp(f) = \emptyset$, we obtain $f(X) \subseteq X$ for all $X \in \wp_{fin}(A)$. If there exists $Y \in \wp_{fin}(A)$ with $f(Y) \subsetneq Y$, then we can construct the sequence $\ldots \subsetneq f^k(Y) \subsetneq \ldots \subsetneq f^2(Y) \subsetneq f(Y) \subsetneq Y$ which is infinite and uniformly supported by $supp(Y) \cup supp(f)$. This is a contradiction because the finite set Y cannot contain infinitely many distinct subsets, and so $f(X) = X$ for all $X \in \wp_{fin}(A)$.

Assume now that $supp(f)$ is non-empty. If $X \subseteq supp(f)$, then $f(X \setminus supp(f)) = f(\emptyset) = \emptyset = X \setminus supp(f)$. The second identity follows because f is strictly monotone; otherwise we could construct an infinite strictly ascending chain in $\wp_{fin}(A)$, uniformly supported by $supp(f)$, namely $\emptyset \subsetneq f(\emptyset) \subsetneq \ldots \subsetneq f^k(\emptyset) \subsetneq \ldots$, contradicting the fact that $\wp_{fin}(A)$ does not contain an infinite uniformly supported subset.

Now we prove the following intermediate result. Let us consider an arbitrary set $T = \{b_1, \ldots, b_n\}$ such that $b_1, \ldots, b_n \in A \setminus supp(f)$, $n \geq 1$ and $f(T) \setminus supp(f) \neq \emptyset$. We prove that $f(T) = T$ (2). According to (1), $f(T)$ should be $f(T) = \{c_1, \ldots, c_k, b_{i_1}, \ldots, b_{i_l}\}$ with $c_1, \ldots, c_k \in supp(f)$ and $b_{i_1}, \ldots, b_{i_l} \in A \setminus supp(f)$, or $f(T) = \{b_{i_1}, \ldots, b_{i_l}\}$ with $b_{i_1}, \ldots, b_{i_l} \in A \setminus supp(f)$. In both cases we have that $\{b_{i_1}, \ldots, b_{i_l}\}$ is non-empty (i.e. it should contain at least one element, say b_{i_1}, because we assumed that $f(T)$ contains at least one element outside $supp(f)$) and $\{b_{i_1}, \ldots, b_{i_l}\} \subseteq \{b_1, \ldots, b_n\}$. If $n = 1$, then $l = 1$ and $b_{i_1} = b_1$. Now let us consider $n > 1$. Assume by contradiction that there is $j \in \{1, \ldots, n\}$ such that $b_j \notin \{b_{i_1}, \ldots, b_{i_l}\}$. Then $(b_{i_1}\ b_j) \star T = T$ since both $b_{i_1}, b_j \in T$ and T is a finite subset of atoms (b_{i_1} and b_j are interchanged in T under the effect of the transposition $(b_{i_1}\ b_j)$, while the other atoms belonging to T are left unchanged, which means the entire T is left invariant under the effect of the related transposition under the induced action $\star$). Furthermore, since $b_{i_1}, b_j \notin supp(f)$ we have the transposition $(b_{i_1}\ b_j)$ fixes $supp(f)$ pointwise, and by Proposition 3 we get $f(T) = f((b_{i_1}\ b_j) \star T) = (b_{i_1}\ b_j) \star f(T)$ which is a contradiction because $b_{i_1} \in f(T)$ while $b_j \notin f(T)$. Thus, $\{b_{i_1}, \ldots, b_{i_l}\} = \{b_1, \ldots, b_n\}$. Now we prove that $f(T) = T$. Assume, by contradiction, that we are in the case $f(T) = \{c_1, \ldots, c_k, b_1, \ldots, b_n\}$ with $c_1, \ldots, c_k \in supp(f)$. Then $T \subsetneq f(T)$, and since f is strictly monotone we can construct a strictly ascending chain $T \subsetneq f(T) \subsetneq \ldots \subsetneq f^l(T) \subsetneq \ldots$. Since for any $i \in \mathbb{N}$ we have that $f^l(T)$ is supported by $supp(f) \cup supp(T)$ (this follows by induction on l involving Proposition 3) and $\wp_{fin}(A)$ does not contain an infinite uniformly supported subset (the elements of $\wp_{fin}(A)$ supported by $supp(f) \cup supp(T)$ are exactly the subsets of $supp(f) \cup supp(T)$), we get a contradiction. Thus, $f(T) = T$.

We return to the proof of our theorem and we consider the remaining case $X \setminus supp(f) \neq \emptyset$. We should have that $X = \{a_1, \ldots, a_p, d_1, \ldots, d_m\}$ with $a_1, \ldots, a_p \in supp(f)$ and $d_1, \ldots, d_m \in A \setminus supp(f)$, $m \geq 1$, or $X = \{d_1, \ldots, d_m\}$ with $d_1, \ldots, d_m \in A \setminus supp(f)$,

$m \geq 1$. We have that $X \setminus supp(f) = \{d_1, \ldots, d_m\}$. Denote by $U = X \setminus supp(f)$. If $f(U) \setminus supp(f) \neq \emptyset$, then $f(U) = U$ according to (2). Assume, by contradiction, that $f(U) \setminus supp(f) = \emptyset$, that is, $f(U) = \{x_1, \ldots, x_k\}$ with $x_1, \ldots, x_k \in supp(f)$, $k \geq 1$ (we cannot have $f(U) = \emptyset$ because f is strictly monotone $f(\emptyset) = \emptyset$ and $\emptyset \subsetneq U$). Since $supp(f)$ has only finitely many subsets, A is infinite and f is strictly monotone, there should exist $V \in \wp_{fin}(A)$, $V \subsetneq A \setminus supp(f)$ such that $U \subsetneq V$ and $f(V)$ contains at least one element outside $supp(f)$; for example, we can choose finitely many distinct atoms $d_{m+1}, \ldots, d_{m+2^{|supp(f)|}+1} \in A \setminus (supp(f) \cup \{d_1, \ldots, d_m\})$, and consider $V = \{d_1, \ldots, d_m, d_{m+1}, \ldots, d_{m+2^{|supp(f)|}+1}\}$; since $\{d_1, \ldots, d_m\} \subsetneq \{d_1, \ldots, d_m, d_{m+1}\} \subsetneq \ldots \subsetneq \{d_1, \ldots, d_m, \ldots, d_{m+2^{|supp(f)|}+1}\}$ and f is strictly monotone, we get that $f(V)$ should contain at least one element outside the finite set $supp(f)$. However, in this case, $f(V) = V$ according to (2), and since $f(U) \subsetneq f(V) = V$, we get $\{x_1, \ldots, x_k\} \subseteq V$, i.e. $x_1, \ldots, x_k$ are outside $supp(f)$, a contradiction. Therefore, we necessarily have $f(U) \setminus supp(f) \neq \emptyset$, and hence $f(U) = U$, that is $X \setminus supp(f) = f(X \setminus supp(f))$ for all $X \in \wp_{fin}(A)$. □

Theorem 3. *Let $f : \wp_{fin}(A) \to \wp_{fin}(A)$ be a finitely supported progressive function (i.e. f has the property that $X \subseteq f(X)$ for all $X \in \wp_{fin}(A)$). There are infinitely many fixed points of f, namely the finite subsets of A containing all the elements of $supp(f)$.*

Proof. Let $X \in \wp_{fin}(A)$. Since the support of a finite subset of atoms coincides with the related subset (see Proposition 2 and use the trivial remark that any finite set is uniformly supported), we have $supp(X) = X$ and $supp(f(X)) = f(X)$. According to Proposition 3, for any permutation π fixing $supp(f) \cup supp(X) = supp(f) \cup X$ pointwise we have $\pi \star f(X) = f(\pi \star X) = f(X)$ meaning that $supp(f) \cup X$ supports $f(X)$, that is $f(X) = supp(f(X)) \subseteq supp(f) \cup X$ (1). Since we also have $X \subseteq f(X)$, we obtain $X \setminus supp(f) \subseteq f(X) \setminus supp(f) \subseteq X \setminus supp(f)$, that is $X \setminus supp(f) = f(X) \setminus supp(f)$ (2). If $supp(f) = \emptyset$, the result follows immediately. Let us consider the case $supp(f) = \{a_1, \ldots, a_k\}$. According to (1) and to the hypothesis, we have $supp(f) \subseteq f(supp(f)) \subseteq supp(f)$, and so

$f(supp(f)) = supp(f)$. If X has the form $X = \{a_1, \dots, a_k, b_1, \dots, b_n\}$ with $b_1, \dots, b_n \in A \setminus supp(f)$, $n \geq 1$, we should have by hypothesis that $a_1, \dots, a_k \in f(X)$, and by (2) $f(X) \setminus supp(f) = X \setminus supp(f) = \{b_1, \dots, b_n\}$. Since no other elements different from $a_1, \dots, a_k$ are in $supp(f)$, from (1) we obtain $f(X) = \{a_1, \dots, a_k, b_1, \dots, b_n\} = X$. $\square$

Theorem 4. *Let $f : \wp_{fin}(A) \to \wp_{fin}(A)$ be a finitely supported function having the properties that $f(X) \subseteq X$ for all $X \in \wp_{fin}(A)$ and $f(X) \neq \emptyset$ for all $X \neq \emptyset$. Then $f(Y) = Y$ for all $Y \in \wp_{fin}(A)$ with $Y \cap supp(f) = \emptyset$.*

Proof. Let $Y \in \wp_{fin}(A)$ with $Y \cap supp(f) = \emptyset$. Thus, Y is either equal to the empty set or Y is of form $Y = \{b_1, \dots, b_m\}$ with $b_1, \dots, b_m \in A \setminus supp(f)$, $m \geq 1$. Obviously, $f(\emptyset) = \emptyset$ from our hypothesis. Furthermore, from the hypothesis we should have $f(Y) = \{b_{i_1}, \dots, b_{i_n}\}$ with $b_{i_1}, \dots, b_{i_n} \in A \setminus supp(f)$, $n \geq 1$ and $\{b_{i_1}, \dots, b_{i_n}\} \subseteq \{b_1, \dots, b_m\}$.

Assume by contradiction that there exists $b_j \in \{b_1, \dots, b_m\}$ such that $b_j \notin \{b_{i_1}, \dots, b_{i_n}\}$. Hence $b_j \neq b_{i_1}$ and $(b_{i_1}\ b_j) \star Y = Y$ because we have $b_{i_1}, b_j \in Y$ and $Y \in \wp_{fin}(A)$. Moreover, since $b_{i_1}, b_j \notin supp(f)$, we have that $(b_{i_1}\ b_j) \in Fix(supp(f))$. From Proposition 3, we obtain $\{b_{i_1}, \dots, b_{i_n}\} = f(\{b_1, \dots, b_m\}) = f((b_{i_1}\ b_j) \star \{b_1, \dots, b_m\}) = (b_{i_1}\ b_j) \star f(\{b_1, \dots, b_m\}) = (b_{i_1}\ b_j) \star \{b_{i_1}, b_{i_2}, \dots, b_{i_n}\} = \{b_j, b_{i_2}, \dots, b_{i_n}\}$, which is a contradiction. Thus, $f(Y) = Y$. $\square$

Theorem 5. *Let $f : \wp_{fin}(A) \to \wp_{fin}(A)$ be a finitely supported function and let $X \in \wp_{fin}(A)$ such that $X \subseteq f(X)$. If f is monotone or progressive, then there exists $n \in \mathbb{N}^*$ such that $f^l(X)$ is a fixed point of f for all $l \geq n$.*

Proof. Since $X \subseteq f(X)$ and f is monotone (i.e. order preserving) or progressive, we can define the ascending sequence $X \subseteq f(X) \subseteq f^2(X) \subseteq \ldots \subseteq f^m(X) \subseteq \ldots$.

We prove by induction that the sequence $(f^m(X))_{m \in \mathbb{N}^*}$ is uniformly supported by $supp(f) \cup supp(X)$, that is, $supp(f^m(X)) \subseteq supp(f) \cup supp(X)$ for each $m \in \mathbb{N}^*$. Let $m = 1$. For any permutation π fixing $supp(f) \cup supp(X)$ pointwise, from Proposition 3 we have $\pi \star f(X) = f(\pi \star X) = f(X)$ meaning that $supp(f) \cup supp(X)$

supports $f(X)$, that is $supp(f(X)) \subseteq supp(f) \cup supp(X)$. Let us suppose that $supp(f^k(X)) \subseteq supp(f) \cup supp(X)$ for some $k \in \mathbb{N}^*$. We have to prove that $supp(f^{k+1}(X)) \subseteq supp(f) \cup supp(X)$. Equivalently, we have to prove that each permutation π fixing $supp(f) \cup supp(X)$ pointwise also fixes $f^{k+1}(X)$. Let $\pi \in Fix(supp(f) \cup supp(X))$. From the inductive hypothesis, we have $\pi \in Fix(supp(f^k(X)))$, and hence $\pi \star f^k(X) = f^k(X)$. According to Proposition 3, we have $\pi \star f^{k+1}(X) = \pi \star f(f^k(X)) = f(\pi \star f^k(X)) = f(f^k(X)) = f^{k+1}(X)$. Therefore, $(f^m(X))_{m \in \mathbb{N}^*}$ is uniformly supported by $supp(f) \cup supp(X)$. Therefore, this sequence should be stationary because $\wp_{fin}(A)$ does not contain an infinite uniformly supported subset. Thus, there exists $n \in \mathbb{N}$ such that $f^n(X) = f^l(X)$ for all $l \geq n$. Fix some $l \geq n$. We have $f(f^l(X)) = f^{l+1}(X) = f^n(X) = f^l(X)$, and so $f^l(X)$ is a fixed point of f. □

Corollary 2. *Let $f : \wp_{fin}(A) \to \wp_{fin}(A)$ be a finitely supported monotone function. Then there exists a least $X_0 \in \wp_{fin}(A)$ supported by $supp(f)$ such that $f(X_0) = X_0$.*

Proof. Since $\emptyset \subseteq f(\emptyset)$ and f is monotone (order preserving), from Theorem 5 we have that there exists $m_0 \in \mathbb{N}^*$ such that $f^{m_0}(\emptyset)$ is a fixed point of f. This fixed point is supported by $supp(f) \cup supp(\emptyset)$. However, we prove that $supp(\emptyset) = \emptyset$. Indeed, from the definition of $\emptyset$, we have $\emptyset \subseteq \pi \star \emptyset$ and $\emptyset \subseteq \pi^{-1} \star \emptyset$ for each π, which means $\emptyset = \pi \star \emptyset$ and $supp(\emptyset) = \emptyset$.

If T is another fixed point of f, then from $\emptyset \subseteq T$, we get $f^n(\emptyset) \subseteq f^n(T)$ for all $n \in \mathbb{N}$. Therefore, $f^{m_0}(\emptyset) \subseteq f^{m_0}(T) = T$, and so $f^{m_0}(\emptyset)$ is the least fixed point of f. □

Theorem 6. *Let $f : \wp_{fin}(A) \to \wp_{fin}(A)$ be a finitely supported function.*

1. *We have $f(supp(f)) \subseteq supp(f)$;*

2. *If f is monotone, then there exists $n \in \mathbb{N}^*$ such that $f^l(supp(f))$ is a fixed point of f for all $l \geq n$.*

Proof. According to Proposition 3, for any permutation π fixing $supp(f)$ pointwise we have $\pi \star supp(f) = supp(f)$ and $\pi \star f(supp(f)) = f(\pi \star supp(f)) = f(supp(f))$ meaning that $supp(f)$ supports $f(supp(f))$, that is, $supp(f(supp(f))) \subseteq supp(f)$. Since the support of a finite subset of atoms coincides with the related subset, we obtain $supp(f(supp(f))) = f(supp(f))$, and so $f(supp(f)) \subseteq supp(f)$.

Assume now that f is monotone. According to the previous item, we can construct the sequence $\ldots \subseteq f^m(supp(f)) \subseteq \ldots \subseteq f^2(supp(f)) \subseteq f(supp(f)) \subseteq supp(f)$. Since $supp(f)$ is finite, the related sequence should be finite, and so there exists $n \in \mathbb{N}$ such that $f^n(supp(f)) = f^l(supp(f))$ for all $l \geq n$. Fix some $l \geq n$. We have $f(f^l(supp(f))) = f^{l+1}(supp(f)) = f^n(supp(f)) = f^l(supp(f))$, and so $f^l(supp(f))$ is a fixed point of f. □

Proposition 5. *Let $f : \wp_{fin}(A) \to \wp_{fin}(A)$ be a finitely supported injective and progressive function. Then $f(Y) = Y$ for all $Y \in \wp_{fin}(A)$.*

Proof. Let $Y \in \wp_{fin}(A)$. As in the proof of Theorem 5, the ascending sequence $Y \subseteq f(Y) \subseteq f^2(Y) \subseteq \ldots \subseteq f^m(Y) \subseteq \ldots$ is uniformly supported by $supp(f) \cup supp(Y)$. Therefore, this sequence should be stationary because $\wp_{fin}(A)$ does not contain an infinite uniformly supported subset. Thus, there exists $n \in \mathbb{N}$ such that $f^n(Y) = f^{n+1}(Y) = f^n(f(Y))$. Since f is injective (and so is f^n), we obtain $f(Y) = Y$. □

Corollary 3. *Let $f : \wp_{fin}(A) \to \wp_{fin}(A)$ be a finitely supported surjective and progressive function. Then $f(Y) = Y$ for all $Y \in \wp_{fin}(A)$.*

Proof. According to Theorem 1, f should be injective. The result now follows from Proposition 5. □

Proposition 6. *Let $f : \wp_{fin}(A) \to \wp_{fin}(A)$ be a finitely supported injective function having the property that $f(X) \subseteq X$ for all $X \in \wp_{fin}(A)$. Then $f(Y) = Y$ for all $Y \in \wp_{fin}(A)$.*

Proof. Let $Y \in \wp_{fin}(A)$. The sequence $\ldots \subseteq f^i(Y) \subseteq \ldots \subseteq f^2(Y) \subseteq f(Y) \subseteq Y$ should be finite since Y is finite. Therefore, there exists

$n \in \mathbb{N}$ such that $f^n(f(Y)) = f^{n+1}(Y) = f^n(Y)$. Since f is injective (and so is f^n), we obtain $f(Y) = Y$. $\square$

Corollary 4. *Let $f : \wp_{fin}(A) \to \wp_{fin}(A)$ be a finitely supported surjective function having the property that $f(X) \subseteq X$ for all $X \in \wp_{fin}(A)$. Then $f(Y) = Y$ for all $Y \in \wp_{fin}(A)$.*

Proof. According to Theorem 1, f should be injective. The result now follows from Proposition 6. $\square$

Theorem 7. *Let $f : \wp_{fin}(A) \to \wp_{fin}(A)$ be a finitely supported function having the property that $f(X \cup Y) = f(X) \cup f(Y)$ for all $X, Y \in \wp_{fin}(A)$. If $X_0 \in \wp_{fin}(A)$ and $k \in \mathbb{N}^*$ such that $X_0 \subseteq f^k(X_0)$, then $\underset{n\in\mathbb{N}}{\cup} f^n(X_0)$ is a finite subset of A and a fixed point of f.*

Proof. As in the proof of Theorem 5, the sequence $(f^n(X_0))_{n\in\mathbb{N}} \subseteq \wp_{fin}(A)$ is uniformly supported by $supp(f) \cup supp(X_0)$. Therefore, this sequence should be finite, and so there exist $\underset{n\in\mathbb{N}}{\cup} f^n(X_0)$ and $\underset{n\in\mathbb{N}}{\cup} f(f^n(X_0))$ which are proved to be supported by $supp(f) \cup supp(X_0)$. Clearly, $\{f^{n+1}(X_0) \,|\, n \in \mathbb{N}\} = \{f^n(X_0) \,|\, n \in \mathbb{N}^*\} \subseteq \{f^n(X_0) \,|\, n \in \mathbb{N}\}$, and so $\underset{n\in\mathbb{N}}{\cup} f^{n+1}(X_0) \subseteq \underset{n\in\mathbb{N}}{\cup} f^n(X_0)$. Since $f^0(X_0) = X_0 \subseteq f^k(X_0)$ with $k \geq 1$, we have $f^0(X_0) \subseteq \underset{n\in\mathbb{N}}{\cup} f^{n+1}(X_0)$. However, obviously, $f^i(X_0) \subseteq \underset{n\in\mathbb{N}^*}{\cup} f^n(X_0) = \underset{n\in\mathbb{N}}{\cup} f^{n+1}(X_0)$ for all $i \in \mathbb{N}^*$, and so $\underset{n\in\mathbb{N}}{\cup} f^n(X_0) \subseteq \underset{n\in\mathbb{N}}{\cup} f^{n+1}(X_0)$. Therefore, $\underset{n\in\mathbb{N}}{\cup} f^{n+1}(X_0) = \underset{n\in\mathbb{N}}{\cup} f^n(X_0)$, and so, according to the hypothesis, $f(\underset{n\in\mathbb{N}}{\cup} f^n(X_0)) = \underset{n\in\mathbb{N}}{\cup} f(f^n(X_0)) = \underset{n\in\mathbb{N}}{\cup} f^{n+1}(X_0) = \underset{n\in\mathbb{N}}{\cup} f^n(X_0)$, which means $\underset{n\in\mathbb{N}}{\cup} f^n(X_0)$ is a fixed point of f. $\square$

Theorem 8. *Let $f : \wp_{fin}(A) \to \wp_{fin}(A)$ be a finitely supported injective function. Then for any $X \in \wp_{fin}(A)$ there exists $n \in \mathbb{N}^*$ such that X is a fixed point of f^n.*

Proof. Let $X \in \wp_{fin}(A)$. As in the proof of Theorem 5, the sequence $(f^m(X))_{m\in\mathbb{N}} \subseteq \wp_{fin}(A)$ is uniformly supported by $supp(f) \cup supp(X)$.

Therefore, this sequence should be finite, and so there exist $l, k \in \mathbb{N}$, $l > k$, such that $f^l(X) = f^k(X)$. Since f is injective, we get $X = f^{l-k}(X)$ and so the result follows by denoting $n = l - k$. $\square$

Corollary 5. *Let $f : \wp_{fin}(A) \to \wp_{fin}(A)$ be a finitely supported surjective function. Then for any $X \in \wp_{fin}(A)$ there exists $n \in \mathbb{N}^*$ such that X is a fixed point of f^n.*

Proof. From Theorem 1, the surjective function $f : \wp_{fin}(A) \to \wp_{fin}(A)$ should be injective, and the result follows from Theorem 5. $\square$

4 Conclusion

This paper is the extended and revised version of the conference paper [1] presented at MFOI 2020. We are able to prove that for finitely supported self-mappings (self-functions) defined on $\wp_{fin}(A)$ the injectivity is equivalent with the surjectivity. These mappings also satisfy some fixed point properties if some particular requirements (such as injectivity, surjectivity, monotony or progressivity) are introduced.

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