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Total energy of signed digraphs

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Abstract

The energy of a sidigraph is defined as the sum of absolute values of real parts of its eigenvalues. The iota energy of a sidigraph is defined as the sum of absolute values of imaginary parts of its eigenvalues. Recently a new notion of energy of digraphs is introduced which is called the total energy of digraphs. In this paper, we extend this concept of total energy to sidigraphs. We compute total energy formulas for negative directed cycles and show that the total energy of negative directed cycles with fixed order increases monotonically. We introduce complex adjacency matrix to give the integral representation for total energy of sidigraphs. We discuss the increasing property of total energy over some particular subfamilies of $S_{n,h}$, where $S_{n,h}$ contains n-vertex sidigraphs with each cycle having length h. Using the Cauchy-Schwarz inequality, we find upper bound for the total energy of sidigraphs. Finally, we find the class of noncospectral equienergetic sidigraphs.

Keywords: Signed Digraphs, Total energy, Increasing property, T-equienergetic sidigraphs.

MSC 2010: 05C35, 05C50.

1 Introduction

A signed digraph (or sidigraph, for short) is a pair $S = (D, \alpha)$, where $D = (\mathcal{V}, \mathcal{A})$ is the underlying digraph of S and $\alpha : \mathcal{A} \longrightarrow \{-1, 1\}$ is the signing function. Elements of $\mathcal{V}$ are called vertices and elements of $\mathcal{A}$ are called arcs. An arc from a vertex u to a vertex v is denoted by uv. A positive (respectively, a negative) arc is an arc with a $+1$ (respectively, a -1) sign. The product of signs of all arcs of a sidigraph

S is called the sign of S. The sets $\mathcal{A}^+(S)$ and $\mathcal{A}^-(S)$ are the sets of positive and negative arcs of S, respectively. Thus, the set of all signed arcs of S is $\mathcal{A}(S) = \mathcal{A}^+(S) \cup \mathcal{A}^-(S)$. If the direction of all arcs of the underlying digraph D is removed, then $S = (D, \alpha)$ is called a sigraph.

A signed directed path of length $n \geq 1$ is a sidigraph on n vertices $v_1, v_2, \dots v_n$ with $n-1$ signed arcs $v_i v_{i+1}$, $i = 1, 2, \dots, n-1$. A signed directed cycle of length $n \geq 2$ is a sidigraph with vertices $v_1, v_2, \dots, v_n$ and signed arcs $v_i v_{i+1}$, $i = 1, 2, \dots, n-1$ and $v_n v_1$. If each directed cycle of a sidigraph S has positive sign, then it is said to be cycle-balanced; otherwise non cycle-balanced. If for each pair of vertices u and v of a sidigraph S, there is a path from u to v and a path from v to u, then S is called a strongly connected sidigraph. The strong components of a sidigraph are maximally connected subsidigraphs. The indegree and outdegree of a vertex u of S, denoted by $d^+(u)$ and $d^-(u)$, respectively, is the number of arcs with tail u and the number of arcs with head u, respectively. A linear sidigraph is a sidigraph in which $d^+(u) = 1 = d^-(u)$ for every vertex $u \in \mathcal{V}(S)$. A sidigraph is positive (respectively, negative) if its sign is positive (respectively, negative). A sidigraph in which the number of vertices equals the number of arcs and has a unique directed cycle is called a unicyclic sidigraph. Throughout this paper, we denote a positive directed cycle by C_n and a negative directed cycle by $\boldsymbol{C}_n$, where n is the length of cycle.

The adjacency matrix $A(S) = [a_{ij}]_{n \times n}$ of an n-vertex sidigraph S is defined as

$$a_{ij} = \begin{cases} \alpha(v_i, v_j) & \text{if there is an arc from } v_i \text{ to } v_j, \\ 0 & \text{otherwise.} \end{cases}$$

The characteristic polynomial of a sidigraph S is the polynomial

$$\phi_S(\lambda) = \det(A(S) - \lambda I_n),$$

where I_n is the identity matrix of order n. The eigenvalues of a sidigraph S are the eigenvalues of its adjacency matrix $A(S)$. The multiset of eigenvalues of a sidigraph S is called the spectrum of S. It is denoted by spec(S).

A sidigraph is said to be symmetric, if for an arc $uv \in \mathcal{A}(S)$ an arc $vu \in \mathcal{A}(S)$ also holds, where $u, v \in \mathcal{V}(S)$. A one to one correspondence between sigraphs and symmetric sidigraphs is given by $S \rightsquigarrow \overleftrightarrow{S}$, where $\overleftrightarrow{S}$ is the sidigraph with the same vertex set as that of sigraph S and each signed edge is replaced by a pair of symmetric arcs, uv and vu, both with the same sign as that of edge uv. Under this correspondence, a sigraph can be identified with a symmetric sidigraph.

Let $S_1 = (\mathcal{V}_1, \mathcal{A}_1, \alpha_1)$ and $S_2 = (\mathcal{V}_2, \mathcal{A}_2, \alpha_2)$ be two sidigraphs. The cartesian product $S_1 \times S_2$ is the sidigraph $S = (\mathcal{V}, \mathcal{A}, \alpha)$, where $\mathcal{V} = \mathcal{V}_1 \times \mathcal{V}_2$, the arc set $\mathcal{A}$ is that of the Cartesian product of underlying unsigned digraphs, and the signing function is defined by:

$$\alpha((x_1, x_2)\,(y_1, y_2) = \begin{cases} \alpha_1(x_1 y_1) & \text{if } x_2 = y_2 \\ \alpha_2(x_2 y_2) & \text{if } x_1 = y_1. \end{cases}$$

In 1978, Gutman [7] introduced the concept of energy of a simple graph. He defined the energy of a graph as the sum of the absolute values of its eigenvalues. The concept of energy in signed graphs was introduced by Germina et al. [6] in 2010. Bhat and Pirzada [3] finds the unicyclic sigraphs with minimal energy. The authors show that even and odd coefficients of the characteristic polynomial of a unicyclic sigraph respectively alternate in sign. The concept of energy was extended to digraphs by Peña and Rada [12] in 2008. Since the adjacency matrix of a digraph is not necessarily symmetric, its eigenvalues may be complex. Pirzada and Bhat [13] extended the concept of energy of digraphs to sidigraphs. Khan et al. [10] extend the concept of energy of digraphs to iota energy of digraphs and defined iota energy as the sum of absolute values of the imaginary parts of its eigenvalues. Khan et al. [9] introduced the notion of total energy of digraphs. The authors find the uncyclic digraphs with minimal and maximal total energy. Motivated by Khan et al. [9], we extend this concept of total energy to sidigraphs. Among all n-vertex unicyclic sidigraphs, we find unicyclic sidigraphs with minimal and maximal total energy. We show that total energy increases over the sets $S^1_{n,h}$ and $S^2_{n,h}$ with respect to the quasi-order relation when $h \equiv 4 (\text{mod } 8)$ and $h \equiv 0 (\text{mod } 8)$, respectively. The classes $S^1_{n,h}$ and $S^2_{n,h}$ are defined in section 4. We find upper bound for the total energy of sidigraphs using the Cauchy-Schwarz inequality. Finally, we find the class of noncospectral equienergetic sidigraphs.

2 Total energy of sidigraphs

The total energy of a digraph is defined by Khan et al. [9] as the sum of absolute values of both real and imaginary part of its eigenvalues. In this section, we extend the concept of total energy to sidigraphs.

The following theorem gives the coefficients of the characteristic polynomial of the adjacency matrix of sidigraphs.

Theorem 1 (Acharya et al. [1]). *Let S be an n-vertex sidigraph with characteristic polynomial given by*

$$\phi_S(\lambda) = \lambda^n + \sum_{k=1}^{n} c_k \, \lambda^{n-k}.$$

Then

$$c_k = \sum_{L \in \mathcal{L}_k} (-1)^{p(L)} \prod_{Z \in c(L)} s(Z)$$

for all $k = 1, 2, \ldots, n$, where $\mathcal{L}_k$ is the set of all linear subdigraphs L of S of order k, $p(L)$ denotes the number of components of L, $c(L)$ denotes the set of all cycles and $s(Z)$ the sign of cycle Z.

Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of an n-vertex sidigraph S. Then energy and iota energy is defined as

$$E(S) = \sum_{k=1}^{n} |\mathrm{Re}(\lambda_k)|, \quad (1)$$

$$E_c(S) = \sum_{k=1}^{n} |\mathrm{Im}(\lambda_k)|. \quad (2)$$

Now we define total energy of S as follows:

$$E_t(S) = \sum_{k=1}^{n} |\mathrm{Re}(\lambda_k) + \mathrm{Im}(\lambda_k)|, \quad (3)$$

where $\mathrm{Re}(\lambda_{\mathrm{k}})$ and $\mathrm{Im}(\lambda_{\mathrm{k}})$ are the real part and imaginary part of eigenvalue λ_k, respectively.

Example 1. *By Theorem 1, the characteristic polynomial of an acyclic sidigraph of order n is given as $\phi_S(\lambda) = \lambda^n$. So its total energy is $E_t(S) = 0$.*

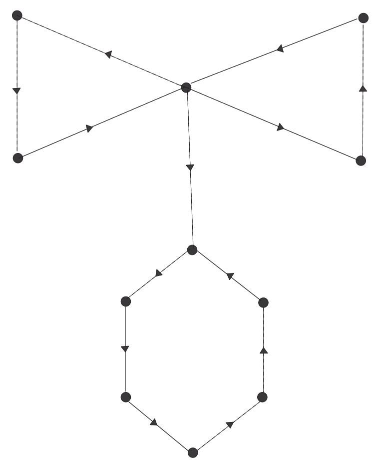

Figure 1. A sidigraph with three signed directed cycles

Example 2. *Consider the sidigraph S shown in Figure 1. Negative arcs and positive arcs are represented by dotted lines and solid lines, respectively. By Theorem 1, the characteristic polynomial of S is given by*

$$\phi_S(\lambda) = \lambda^{11} + \lambda^5 = \lambda^5(\lambda^6 + 1).$$

The spectrum of S is given by

$$spec(S) = \left\{ 0, 0, 0, 0, 0, \pm i, \ \frac{-i \pm \sqrt{3}}{2}, \ \frac{i \pm \sqrt{3}}{2} \right\}.$$

Thus,

$$E_t(S) = 2 + 2\sqrt{3}.$$

The following theorem gives the relation between total energy of sidigraph and its strong components.

Theorem 2. *Total energy of a sidigraph is the sum of total energies of its strong components.*

Proof. Proof is similar to the proof of Theorem 2.7 [13]. □

The next two lemmas will be useful in proving several results.

Lemma 1. (Khan et al. [8]) *Let x, a, b be real numbers such that $x \geq a > 0$ and $b > 0$. Then we have*

$$\frac{\pi\, x}{b\, x^2 - \pi^2} \leq \frac{\pi\, a}{b\, a^2 - \pi^2}.$$

Lemma 2. (Farooq et al. [4]) *For $x \in (0, \frac{\pi}{2}]$, the following inequality holds:*

$$\frac{1}{x} - 0.429\, x \leq \cot\, x \leq \frac{1}{x} - \frac{x}{3}.$$

For any real number x with $0 < x < \frac{\pi}{2}$, sine function satisfies the following:

$$\sin x \leq x, \quad \sin x \geq x - \frac{x^3}{3!}. \tag{4}$$

3 Computation of total energy of signed directed cycles

In this section, we calculate the total energy formulae for negative directed cycles. If S is an n-vertex sidigraph with a unique signed directed cycle of length m, where $2 \leq m \leq n$, then by Theorem 1, $\phi_S(\lambda) = \lambda^n + (-1)^p \lambda^{n-m} = \lambda^{n-m}(\lambda^m + (-1)^p)$, where $p = 1$ or $p = 0$ according to whether S is cycle-balanced or non cycle-balanced. Clearly, total energy equals to the total energy of the unique cycle.

Using Theorem 1, the characteristic polynomial of C_n is given by:

$$\phi_{C_n}(x) = x^n - 1.$$

Thus, the eigenvalues of C_n are $\exp^{\frac{2k\pi i}{n}}$, where $k = 1, \dots, n-1$. Therefore, the energy and total energy of C_n are given by:

$$E(C_n) = \sum_{k=1}^{n} \left| \cos \frac{2k\pi}{n} \right|, \qquad (5)$$

$$E_t(C_n) = \sum_{k=1}^{n} \left| \cos \frac{2k\pi}{n} + \sin \frac{2k\pi}{n} \right|. \qquad (6)$$

Using (5), Pirzada and Bhat [13] calculated the following energy formulas for cycle C_n, $n \geq 2$.

$$E(C_n) = \begin{cases} 2\cot\frac{\pi}{n} & \text{if } n \equiv 0 (\text{mod} 4) \\ 2\csc\frac{\pi}{n} & \text{if } n \equiv 2 (\text{mod} 4) \\ \csc\frac{\pi}{2n} & \text{if } n \equiv 1 (\text{mod} 2). \end{cases} \qquad (7)$$

For a directed cycle C_n, $n \geq 2$, Khan et al. [9] gave the following total energy formulae,

$$E_t(C_n) = \begin{cases} 2\sqrt{2}\csc\frac{\pi}{n} & \text{if } n \equiv 4 (\text{mod } 8) \\ 2\sqrt{2}\cot\frac{\pi}{n} & \text{if } n \equiv 0 (\text{mod } 8) \\ \sqrt{2}\csc\frac{\pi}{2n} & \text{if } n \equiv 2 (\text{mod } 4) \\ \frac{1}{\sqrt{2}}\csc\frac{\pi}{4n} & \text{if } n \equiv 1 (\text{mod } 2). \end{cases} \qquad (8)$$

Again, using Theorem 1, the characteristic polynomial of $\boldsymbol{C}_n$ is given by:

$$\phi_{\boldsymbol{C}_n}(x) = x^n + 1.$$

Thus, the eigenvalues of $\boldsymbol{C}_n$ are $\exp^{\frac{(2k+1)\pi i}{n}}$, where $k = 1, \dots, n-1$. Therefore, the energy and total energy of $\boldsymbol{C}_n$ are computed by:

$$E(\boldsymbol{C}_n) = \sum_{k=1}^{n} \left| \cos \frac{(2k+1)\pi}{n} \right|, \qquad (9)$$

$$E_t(\boldsymbol{C}_n) = \sum_{k=1}^{n} \left| \cos \frac{(2k+1)\pi}{n} + \sin \frac{(2k+1)\pi}{n} \right|. \qquad (10)$$

Using (9), Pirzada and Bhat [13] calculated the energy of a cycle $\boldsymbol{C}_n$, $n \geq 2$ as follows:

$$E(\boldsymbol{C}_n) = \begin{cases} 2\csc\frac{\pi}{n} & \text{if } n \equiv 0(\text{mod} 4) \\ 2\cot\frac{\pi}{n} & \text{if } n \equiv 2(\text{mod} 4) \\ \csc\frac{\pi}{2n} & \text{if } n \equiv 1(\text{mod} 2). \end{cases} \tag{11}$$

Next, we calculate the total energy formulae of $\boldsymbol{C}_n$, by considering the following four cases:

Case 1. If $n \equiv 4(\text{mod } 8)$, then

$$\begin{aligned} E_t(\boldsymbol{C}_n) &= \sum_{k=0}^{n-1} \left| \cos\frac{(2k+1)\pi}{n} + \sin\frac{(2k+1)\pi}{n} \right| \\ &= 2\sum_{k=0}^{\frac{n}{2}-1} \left| \cos\frac{(2k+1)\pi}{n} + \sin\frac{(2k+1)\pi}{n} \right| \\ &= 2\Bigg(\sum_{k=0}^{\frac{n}{4}-1} \left| \cos\frac{(2k+1)\pi}{n} + \sin\frac{(2k+1)\pi}{n} \right| \\ &\quad + \sum_{k=0}^{\frac{n}{4}-1} \left| \cos\frac{(2k+1)\pi}{n} - \sin\frac{(2k+1)\pi}{n} \right| \Bigg) \\ &= 2\Bigg(\sum_{k=0}^{\frac{n}{4}-1} \left(\cos\frac{(2k+1)\pi}{n} + \sin\frac{(2k+1)\pi}{n} \right) \\ &\quad + \sum_{k=0}^{\frac{n-12}{8}} \left(\cos\frac{(2k+1)\pi}{n} - \sin\frac{(2k+1)\pi}{n} \right) \\ &\quad \sum_{k=\frac{n+4}{8}}^{\frac{n}{4}-1} \left(\sin\frac{(2k+1)\pi}{n} - \cos\frac{(2k+1)\pi}{n} \right) \Bigg). \end{aligned} \tag{12}$$

Using geometric series sum formula and some basic trigonometric identities, we obtain

$$\begin{aligned}&\sum_{k=0}^{\frac{n}{4}-1}\left(\cos\frac{(2k+1)\pi}{n}+\sin\frac{(2k+1)\pi}{n}\right)\\&=\left(\cos\frac{\pi}{4}+\sin\frac{\pi}{4}\right)\sin\frac{\pi}{4}\csc\frac{\pi}{n}=\csc\frac{\pi}{n}.\end{aligned}\tag{13}$$

$$\begin{aligned}&\sum_{k=0}^{\frac{n-12}{8}}\left(\cos\frac{(2k+1)\pi}{n}-\sin\frac{(2k+1)\pi}{n}\right)\\&=\frac{1}{2}\left(\sqrt{2}\cos\frac{\pi}{n}-1\right)\csc\frac{\pi}{n}.\end{aligned}\tag{14}$$

$$\begin{aligned}&\sum_{k=\frac{n+4}{8}}^{\frac{n}{4}-1}\left(\sin\frac{(2k+1)\pi}{n}-\cos\frac{(2k+1)\pi}{n}\right)\\&=\frac{1}{2}\left(\sqrt{2}\cos\frac{\pi}{n}-1\right)\csc\frac{\pi}{n}.\end{aligned}\tag{15}$$

Using (13)∼(15), equation (12) can be written as:

$$E_t(\boldsymbol{C}_n)=\sum_{k=0}^{n-1}\left|\cos\frac{(2k+1)\pi}{n}+\sin\frac{(2k+1)\pi}{n}\right|=2\sqrt{2}\cot\frac{\pi}{n}.$$

Case 2. If $n \equiv 0 (\text{mod } 8)$, then

$$\begin{aligned}
E_t(\boldsymbol{C}_n) &= \sum_{k=0}^{n-1} \left|\cos\frac{(2k+1)\pi}{n} + \sin\frac{(2k+1)\pi}{n}\right| \\
&= 2\sum_{k=0}^{\frac{n}{2}-1} \left|\cos\frac{(2k+1)\pi}{n} + \sin\frac{(2k+1)\pi}{n}\right| \\
&= 2\Bigg(\sum_{k=0}^{\frac{n}{4}-1} \left|\cos\frac{(2k+1)\pi}{n} + \sin\frac{(2k+1)\pi}{n}\right| \\
&\quad + \sum_{k=0}^{\frac{n}{4}-1} \left|\cos\frac{(2k+1)\pi}{n} - \sin\frac{(2k+1)\pi}{n}\right|\Bigg) \\
&= 2\Bigg(\sum_{k=0}^{\frac{n}{4}-1} \left(\cos\frac{(2k+1)\pi}{n} + \sin\frac{(2k+1)\pi}{n}\right) \\
&\quad + \sum_{k=0}^{\frac{n-8}{8}} \left(\cos\frac{(2k+1)\pi}{n} - \sin\frac{(2k+1)\pi}{n}\right) \\
&\quad \sum_{k=\frac{n}{8}}^{\frac{n}{4}-1} \left(\sin\frac{(2k+1)\pi}{n} - \cos\frac{(2k+1)\pi}{n}\right)\Bigg). \qquad (16)
\end{aligned}$$

Using geometric series sum formulas and some basic trigonometric identities, we get

$$\sum_{k=0}^{\frac{n-8}{8}} \left(\cos\frac{(2k+1)\pi}{n} - \sin\frac{(2k+1)\pi}{n}\right) = \frac{1}{2}\left(\sqrt{2}-1\right)\csc\frac{\pi}{n}. \qquad (17)$$

$$\sum_{k=\frac{n}{8}}^{\frac{n}{4}-1} \left(\sin\frac{(2k+1)\pi}{n} - \cos\frac{(2k+1)\pi}{n}\right) = \frac{1}{2}\left(\sqrt{2}-1\right)\csc\frac{\pi}{n}. \qquad (18)$$

Using (13), (17) and (18), equality (16) becomes

$$E_t(\boldsymbol{C}_n) = \sum_{k=0}^{n-1} \left|\cos\frac{(2k+1)\pi}{n} + \sin\frac{(2k+1)\pi}{n}\right| = 2\sqrt{2}\csc\frac{\pi}{n}.$$

Similary, one can prove that $E_t(\boldsymbol{C}_n) = \sqrt{2}\csc\frac{\pi}{2n}$ when $n \equiv 2(\text{mod } 4)$ and $E_t(\boldsymbol{C}_n) = \frac{1}{\sqrt{2}}\csc\frac{\pi}{4n}$ when $n \equiv 1(\text{mod } 2)$.

In brief, we get

$$E_t(\boldsymbol{C}_n) = \begin{cases} 2\sqrt{2}\cot\frac{\pi}{n} & \text{if } n \equiv 4(\text{mod } 8) \\ 2\sqrt{2}\csc\frac{\pi}{n} & \text{if } n \equiv 0(\text{mod } 8) \\ \sqrt{2}\csc\frac{\pi}{2n} & \text{if } n \equiv 2(\text{mod } 4) \\ \frac{1}{\sqrt{2}}\csc\frac{\pi}{4n} & \text{if } n \equiv 1(\text{mod } 2). \end{cases} \tag{19}$$

To find minimal and maximal total energy among all non cycle-balanced unicyclic sidigraphs, the following lemma will be useful. We would like to mention that the idea of proof is taken from the proof of Lemma 3.5 [2].

Lemma 3. *For $n \geq 2$, the sequence $\langle a_n \rangle$ given by*

$$a_n = \begin{cases} 2\sqrt{2}\cot\frac{\pi}{n} & \textit{if } n \equiv 4(\text{mod } 8) \\ 2\sqrt{2}\csc\frac{\pi}{n} & \textit{if } n \equiv 0(\text{mod } 8) \\ \sqrt{2}\csc\frac{\pi}{2n} & \textit{if } n \equiv 2(\text{mod } 4) \\ \frac{1}{\sqrt{2}}\csc\frac{\pi}{4n} & \textit{if } n \equiv 1(\text{mod } 2) \end{cases}$$

is strictly increasing sequence.

Proof. To prove that $\langle a_n \rangle$ is strictly increasing sequence, we need to show that for $k \geq 1$, the following inequalities hold true.
$2\csc\frac{\pi}{2(4k-2)} < \csc\frac{\pi}{4(4k-1)} < 4\cot\frac{\pi}{4k} < \csc\frac{\pi}{4(4k+1)} < 2\csc\frac{\pi}{2(4k+2)} < \csc\frac{\pi}{4(4k+3)} < 4\csc\frac{\pi}{4k+4}$.

Using (4), we get

$$\begin{aligned} 2\csc\frac{\pi}{8k-4} &\leq 2\left(\frac{1}{\frac{\pi}{8k-4}(1-\frac{\pi^2}{6(8k-4)^2})}\right) \\ &= \frac{2(8k-4)}{\pi}\left(1+\frac{\pi^2}{6(8k-4)^2-\pi^2}\right) \\ &= \frac{2(8k-4)}{\pi} + \frac{2(8k-4)\pi}{6(8k-4)^2-\pi^2}. \end{aligned}$$

By Lemma 1, we obtain

$$\begin{aligned}\frac{2(8k-4)}{\pi}+\frac{2(8k-4)\pi}{6(8k-4)^2-\pi^2} &\leq \frac{2(8k-4)}{\pi}+\frac{2(4)\pi}{6(4)^2-\pi^2}\\ &\leq \frac{16k}{\pi}-2.2547.\end{aligned}$$

Thus

$$2\csc\frac{\pi}{8k-4}\leq\frac{16k}{\pi}-2.2547. \tag{20}$$

On the other hand, using (4), we have

$$\csc\frac{\pi}{4(4k-1)}\geq\frac{4(4k-1)}{\pi}\geq\frac{16k}{\pi}-1.2732. \tag{21}$$

(20) and (21) give $2\csc\frac{\pi}{2(4k-2)}<\csc\frac{\pi}{4(4k-1)}$, thereby proving the first inequality.

Next, we need to show that $\csc\frac{\pi}{4(4k-1)}<4\cot\frac{\pi}{4k}$. It is equivalent to show
$4\cos\frac{\pi}{n}\sin\frac{\pi}{4n-4}-\sin\frac{\pi}{n}>0$, where $n=4k$.

We use Taylor series expansion for $\sin x$ and $\cos x$ to prove this. Now

$$\begin{aligned}& 4\cos\frac{\pi}{n}\sin\frac{\pi}{4n-4}-\sin\frac{\pi}{n}\\ = \; & 4\,[1-\frac{1}{2!}(\frac{\pi}{n})^2+\dots]\,[\frac{\pi}{4n-4}-\frac{1}{3!}(\frac{\pi}{4n-4})^2+\dots]\\ & -[\,\frac{\pi}{n}-\frac{1}{3!}(\frac{\pi}{n})^3+\dots]\\ = \; & \frac{\pi}{n-1}-\frac{\pi}{n}+o(n^{-3})>0,\end{aligned}$$

where $f(n)\in o(g(n))$ if $\frac{f(n)}{g(n)}\longrightarrow 0$ as $n\longrightarrow\infty$. This proves the second inequality.

Now we will prove that $4\cot\frac{\pi}{4k}<\csc\frac{\pi}{4(4k+1)}$. Using Lemma 2, we get

$$\begin{aligned} 4\cot\frac{\pi}{4k} &\leq 4\left(\frac{4k}{\pi}-\frac{\pi}{3(4k)}\right) \\ &= \frac{16k}{\pi}-\frac{\pi}{3k}. \end{aligned} \tag{22}$$

On the other hand, using $\csc x \geq \frac{1}{x}$, we obtain

$$\begin{aligned} \csc\frac{\pi}{4(4k+1)} &\geq \frac{4(4k+1)}{\pi} \\ &= \frac{16k}{\pi}+\frac{4}{\pi}. \end{aligned} \tag{23}$$

Using (22) and (23), we get the third inequality.

Since $\sin x$ is an increasing function in the interval $[0, \frac{\pi}{2}]$, we have $\sin\frac{\pi}{16k+8} < \sin\frac{\pi}{16k+4}$. Also we know that $\cos\frac{\pi}{16k+8} < 1$. This gives $2\sin\frac{\pi}{16k+8}\cos\frac{\pi}{16k+8} < 2\sin\frac{\pi}{16k+4}$. Using $\sin 2x = 2\sin x\cos x$, we get $\sin\frac{\pi}{8k+4} < 2\sin\frac{\pi}{4(4k+1)}$, that is, $\csc\frac{\pi}{4(4k+1)} < 2\csc\frac{\pi}{2(4k+2)}$. This proves the fourth inequality.

Next, we will prove that $2\csc\frac{\pi}{2(4k+2)} < \csc\frac{\pi}{4(4k+3)}$. Using (4), we have

$$\begin{aligned} 2\csc\frac{\pi}{2(4k+2)} &\leq 2\left(\frac{1}{\frac{\pi}{2(4k+2)}(1-\frac{\pi^2}{24(4k+2)^2})}\right) \\ &= \frac{4(4k+2)}{\pi}\left(1+\frac{\pi^2}{24(4k+2)^2-\pi^2}\right) \\ &= \frac{4(4k+2)}{\pi}+\frac{4(4k+2)\pi}{24(4k+2)^2-\pi^2}. \end{aligned}$$

Using Lemma 2, we get

$$\begin{aligned} \frac{4(4k+2)}{\pi}+\frac{4(4k+2)\pi}{24(4k+2)^2-\pi^2} &\leq \frac{4(4k+2)}{\pi}+\frac{4(6)\pi}{24(6)^2-\pi^2} \\ &\leq \frac{16k}{\pi}+2.63475. \end{aligned}$$

Thus

$$2\csc\frac{\pi}{2(4k+2)} \leq \frac{16k}{\pi} + 2.63475. \tag{24}$$

On the other hand, using (4), we obtain

$$\begin{aligned}\csc\frac{\pi}{4(4k+3)} &\geq \frac{4(4k+3)}{\pi}\\ &= \frac{16k}{\pi} + 3.81972.\end{aligned} \tag{25}$$

Using (24) and (25), we get the desired result.

Finally, we will show that $\csc\frac{\pi}{4(4k+3)} < 4\csc\frac{\pi}{4k+4}$. Since $\sin x$ is an increasing function in the interval $[0, \frac{\pi}{2}]$. This gives $\sin\frac{\pi}{16k+16} < \sin\frac{\pi}{16+12}$. As we know that $\cos\frac{\pi}{8k+8} < 1$ and $\cos\frac{\pi}{16k+16} < 1$, this gives $4\sin\frac{\pi}{16k+16}\cos\frac{\pi}{16k+16}\cos\frac{\pi}{8k+8} < 4\sin\frac{\pi}{16k+12}$. Using some of the basic trigonometric identities, we get $\sin\frac{\pi}{4k+4} < 4\sin\frac{\pi}{4(4k+3)}$, that is $\csc\frac{\pi}{4(4k+3)} < 4\csc\frac{\pi}{4k+4}$, and we are done. □

The next two theorems give extremal energy among n-vertex cycle-balanced and non cycle-balanced unicyclic sidigraphs, $n \geq 2$.

Theorem 3 (Khan et al. [9])**.** *The unicyclic n-vertex digraphs which contain a directed cycle C_2 have minimal total energy among all n-vertex unicyclic digraphs. The directed cycle C_n has the maximal total energy among all n-vertex unicyclic digraphs.*

Theorem 4. *Among n-vertex unicyclic sidigraphs with negative cycle, minimal total energy is attained in a sidigraph which contains $\boldsymbol{C}_2$. Moreover, among all non cycle-balanced unicyclic sidigraphs on n vertices, the cycle $\boldsymbol{C}_n$ has the largest total energy.*

The next theorem gives a few characteristics of positive and negative directed cycles.

Theorem 5. *Total energy of positive and negative directed cycles satisfies the following:*

(*i*) $E_t(C_n) = E_t(\boldsymbol{C}_n)$ *if and only if* $n \equiv 1(\mathrm{mod}2)$ *or* $n \equiv 2(\mathrm{mod}4)$.

(ii) $E_t(C_n) > E_t(\boldsymbol{C}_n)$ *if and only if* $n \equiv 4(\mathrm{mod} 8)$.

(iii) $E_t(C_n) < E_t(\boldsymbol{C}_n)$ *if and only if* $n \equiv 0(\mathrm{mod} 8)$.

Proof. Proof follows from Lemma 3. □

In the following theorem, we compare the total energy of positive directed cycles with their energy and iota energy.

Theorem 6. *Let* $n \geq 2$ *be an integer. Then* $E_t(C_n) > E(C_n)$ *and* $E_t(C_n) > E_c(C_n)$.

Proof. Proof follows from (7), (8) and Theorem 2.4 [10]. □

The next corollary is an immediate consequence of Theorem 6.

Corollary 1. *Let* S *be an* n*-vertex unicylic sidigraph with unique directed cycle* C_m, $2 \leq m \leq n$. *Then* $E_t(S) > E(S)$ *and* $E_t(S) > E_c(S)$.

In the next theorem, we give comparison between energy, iota energy and the total energy of negative directed cycles.

Theorem 7. *Let* $n \geq 2$ *be an integer. Then* $E_t(\boldsymbol{C}_n) > E(\boldsymbol{C}_n)$ *and* $E_t(\boldsymbol{C}_n) > E_c(\boldsymbol{C}_n)$.

Proof. Using (11), (19) and Theorem 2.11 [5], we can easily get the desired result. □

An immediate consequence of Theorem 7 is the following corollary.

Corollary 2. *Let* S *be an* n*-vertex unicylic sidigraph with unique directed cycle* $\boldsymbol{C}_m$, $2 \leq m \leq n$. *Then* $E_t(S) > E(S)$ *and* $E_t(S) > E_c(S)$.

4 Complex adjacency matrix and increasing property

In this section, we introduce complex adjacency matrix for sidigraphs in order to define total energy of sidigraph as the sum of absolute values of real parts of its eigenvalues. Let $S = (D, \alpha)$, where $D = (\mathcal{V}, \mathcal{A})$ is the underlying digraph of S and $\alpha : \mathcal{A} \longrightarrow \{-1, 1\}$ is the signing function. We define the complex adjacency matrix $A_t(S) = b_{jk}$ of S by:

$$b_{jk} = \begin{cases} (1-i)\,\alpha(v_j, v_k) & \text{if } v_j v_k \in \mathcal{A}, \\ 0 & \text{otherwise.} \end{cases}$$

The characteristic polynomial of the sidigraph S with respect to complex adjacency matrix $A_t(S)$ is $\phi_S(\lambda) = \det(\lambda I_n - A_t(S))$. The zeros of $\phi_S(\lambda)$ are called the A_t-eigenvalues of S. The spectrum of $A_t(S)$ is denoted by $\mathrm{Spec_t(S)}$. If $A(S)$ is the adjacency matrix of S, then note that $A_t(S) = (1-i)A(S)$. Therefore, $\mathrm{Spec_t(S)} = (1-i)\ \mathrm{Spec(S)}$. Thus, $\mathrm{Re}(\lambda) + \mathrm{Im}(\lambda) = \mathrm{Re}((1-i)\lambda)$, where λ is the eigenvalue of S. Let $\lambda_1, \ldots, \lambda_n$ be the A_t-eigenvalues of a sidigraph S. Then, the total energy of S can also be defined as:

$$E_t(S) = \sum_{k=1}^{n} |\mathrm{Re}(\lambda_\mathrm{k})|,$$

where $\mathrm{Re}(\lambda_k)$ denote the real part of the A_t-eigenvalue λ_k. The coefficient theorem for sidigraphs with complex adjacency matrix $A_t(S)$ is given by:

Theorem 8. *Let S be an n-vertex sidigraph with characteristic polynomial*

$$\Phi_S^t(\lambda) = \lambda^n + \sum_{k=1}^{n} a_k(S)(1-i)^k \lambda^{n-k},$$

then

$$a_k(S) = \sum_{L \in \mathcal{L}_k} (-1)^{p(L)} \prod_{Z \in c(L)} s(Z),$$

for all $k = 1, 2, \ldots, n$, where $\mathcal{L}_k$ is the set of all linear subsidigraphs L of S of order k, $p(L)$ denotes the number of components of L. Moreover $c(L)$ and $s(Z)$ represent the set of all cycles of L and the sign of cycle Z, respectively.

Let S be an n-vertex sidigraph and let $S_1, \ldots, S_r$ be its strong components. Then, from the proof of Theorem 2, the characteristic polynomial of S is given by:

$$\phi_S(\lambda) = \phi_{S_1}(\lambda) \; \ldots \; \phi_{S_r}(\lambda). \tag{26}$$

The following lemma is about the characteristic polynomial of sidigraph S such that all of its cycles are vertex-disjoint.

Lemma 4. *Let S be an n-vertex sidigraph containing cycles $C_1, \ldots, C_r$ of lengths m_1, m_2, $\ldots$, m_r, respectively. Assume that C_1, $\ldots, C_r$ are pairwise vertex-disjoint. Then*

$$\phi_S(\lambda) = \lambda^{n-m} \, \phi_{C_1}(\lambda) \, . \, \ldots \, \phi_{C_r}(\lambda),$$

where $m = \sum\limits_{i=1}^{r} m_i$.

Proof. Proof follows from equation (26). □

The concept of energy of polynomials is introduced by Mateljevic et al. [11] in 2010. The authors defined the energy of a complex polynomial $P(\lambda)$ of degree n as

$$E(P(\lambda)) = \sum_{k=1}^{n} sgn(\mathrm{Re}(\lambda_k))\lambda_k,$$

where $\lambda_1, \ldots, \lambda_n$ are the zeros of $P(\lambda)$. In general, the zeros of the complex polynomial do not satisfy the conjugate pairing property. Therefore, the energy of $P(\lambda)$ is not a real number. Moreover, by complex conjugate root theorem, the roots of the polynomial with real coefficients satisfy conjugate pairing property. Thus

$$E(P(\lambda)) = \sum_{k=1}^{n} sgn(\mathrm{Re}(\lambda_k))\lambda_k = \sum_{k=1}^{n} |\mathrm{Re}(\lambda_k)|. \tag{27}$$

Throughout the paper, we denote by $p.v \int\limits_{-\infty}^{\infty} F(\lambda)\, d\lambda$, the principal value of an integral $\int\limits_{-\infty}^{\infty} F(\lambda)\, d\lambda$. The following theorem is well known.

Theorem 9 (Mateljevic et al. [11]). *Let $\phi(\lambda)$ be a monic real polynomial of degree n and $\lambda_1, \ldots, \lambda_n$ be its zeros. Then*

$$\sum_{k=1}^{n} |\mathrm{Re}(\lambda_k)| = \sum_{k=1}^{n} sgn(\mathrm{Re}(\lambda_k))\lambda_k = \frac{1}{\pi}\, p.v \int\limits_{-\infty}^{\infty} \left(n - \frac{i\lambda\phi'(i\lambda)}{\phi(i\lambda)} \right) d\lambda,$$

where $\mathrm{Re}(\lambda_k)$ *is the real part of* λ_k.

If S is an n-vertex sidigraph which has cycles of length $h \equiv 0(\bmod\ 4)$, then all coefficients of $\Phi_S^t(\lambda)$ in Theorem 8 are real. Thus, in this case we can represent the total energy of sidigraphs in integral form.

Theorem 10. *Let S be an n-vertex sidigraph, which has cycles of length $h \equiv 0(\bmod\ 4)$ and $\Phi_S^t(\lambda)$ be the characteristic polynomial of the complex adjacency matrix $A_t(S)$. Then*

$$\sum_{k=1}^{n} |\mathrm{Re}(\lambda_k)| = \frac{1}{\pi}\ p.v \int\limits_{-\infty}^{\infty} \left(n - \frac{i\lambda\Phi'^t_S(i\lambda)}{\Phi_S^t(i\lambda)} \right) d\lambda,$$

where $\lambda_1, \ldots, \lambda_n$ are A_t-eigenvalues of S and $\mathrm{Re}(\lambda_k)$ *is the real part of* λ_k.

The following corollary is an immediate consequence of Theorem 10.

Corollary 3. *Let ϕ be a monic polynomial of degree n. Let $\lambda_1, \ldots, \lambda_n$ be its roots. Then*

$$\sum_{k=1}^{n} |\mathrm{Re}(\lambda_k)| = \frac{1}{\pi}\ p.v \int_{-\infty}^{\infty} \log|\gamma(t)| \frac{dt}{t^2},$$

where $\gamma(t) = t^n\phi(\frac{i}{t})$ and $\mathrm{Re}(\lambda_k)$ *is the real part of* λ_k.

Let $S_{n,h}$ be the set of sidigraphs with n vertices such that each sidigraph in $S_{n,h}$ has signed directed cycles of length h. Now we will study the increasing property of total energy of sidigraphs in $S_{n,h}$. The following theorem gives the characteristic polynomial of the sidigraphs in $S_{n,h}$.

Theorem 11 (Pirzada and Bhat [13]). *If $S \in S_{n,h}$, then $\Phi_S(\lambda) = \lambda^n + \sum_{k=1}^{\lfloor \frac{n}{h} \rfloor} (-1)^k c^*(S,kh)\lambda^{n-kh}$, $c^*(S,kh)$ is the number of positive linear subsidigraphs of order kh − number of negative linear subsidigraphs of order kh, $k = 1, \ldots, \lfloor \frac{n}{h} \rfloor$.*

Let $S^1_{n,h} = \left\{ S \in S_{n,h} \mid c^*(S,kh) \geq 0, k = 1, \ldots, \lfloor \frac{n}{h} \rfloor \right\}$. Pirzada and Bhat [13] define the quasi-order relation over $S^1_{n,h}$ as follows:

Let $S_1, S_2 \in S^1_{n,h}$. If for all $k = 1, \ldots, \lfloor \frac{n}{h} \rfloor$, $c^*(S_1,kh) \leq c^*(S_2,kh)$, then $S_1 \preceq S_2$ and if there exists k such that $c^*(S_1,kh) < c^*(S_2,kh)$, then $S_1 \prec S_2$. Clearly it is symmetric and transitive relation over $S^1_{n,h}$.

The next theorem is an analogue of Theorem 11.

Theorem 12. *If $S \in S^1_{n,h}$, then $\Phi^t_S(\lambda) = \lambda^n + \sum_{k=1}^{\lfloor \frac{n}{h} \rfloor} (-1)^k (1 - i)^k c^*(S,kh)\lambda^{n-kh}$, $c^*(S,kh)$ is the number of positive linear subsidigraphs of order kh − number of negative linear subsidigraphs of order kh, $k = 1, \ldots, \lfloor \frac{n}{h} \rfloor$.*

Farooq et al. [5] define a new subfamily $S^2_{n,h}$ of $S_{n,h}$, which consists of those sidigraphs $S \in S_{n,h}$ which have negative cycles of length h. The following theorem gives the characteristic polynomial of the sidigraphs in $S^2_{n,h}$.

Theorem 13 (Farooq et al. [5]). *Let $S \in S^2_{n,h}$. Then the characteristic polynomial of S according to adjacency matrix $A(S)$ is given by $\Phi_S(\lambda) = \lambda^n + \sum_{k=1}^{\lfloor \frac{n}{h} \rfloor} c(S,kh)\lambda^{n-kh}$, where $c(S,kh)$ is the number of linear subsidigraphs of S containing k negative cycles of order h, $k = 1, \ldots, \lfloor \frac{n}{h} \rfloor$.*

Farooq et al. [5] define a quasi-order relation over $S^2_{n,h}$ as follows: Let $S_1, S_2 \in S^2_{n,h}$. If for all $k = 1, \ldots, \lfloor \frac{n}{h} \rfloor$, $c(S_1, kh) \leq c(S_2, kh)$, then $S_1 \preceq S_2$ and if there exists k such that $c(S_1, kh) < c(S_2, kh)$, then $S_1 \prec S_2$. This relation is symmetric and transitive relation over $S^2_{n,h}$.

The next theorem is an analogue of Theorem 13.

Theorem 14. *Let $S \in S^2_{n,h}$. Then the characteristic polynomial of S according to adjacency matrix $A_t(S)$ is given by $\Phi^t_S(\lambda) = \lambda^n + \sum_{k=1}^{\lfloor \frac{n}{h} \rfloor} (1 - i)^k\, c(S, kh)\lambda^{n-kh}$, where $c(S, kh)$ is the number of linear subsidigraphs of S containing k negative cycles of order h, $k = 1, \ldots, \lfloor \frac{n}{h} \rfloor$.*

The following theorems give the increasing property of total energy over $S^1_{n,h}$ when $h \equiv 4 (\mathrm{mod}\ 8)$ and over $S^2_{n,h}$ when $h \equiv 0 (\mathrm{mod}\ 8)$. The proofs are similar to the proof of Theorem 5.3 [12]. However, for the sake of self-containment, we only include the proof of Theorem 16.

Theorem 15. *Let h be an integer of the form $h \equiv 4(\mathrm{mod}\ 8)$. Then total energy of a sidigraph $S \in S^1_{n,h}$ increases with respect to the quasi-order relation $\preceq$ defined over $S^1_{n,h}$. That is, if $S_1 \preceq S_2$, then $E_t(S_1) \leq E_t(S_2)$, where $S_1, S_2 \in S^1_{n,h}$.*

Theorem 16. *Let h be an integer of the form $h \equiv 0(\mathrm{mod}\ 8)$. Then total energy of a sidigraph $S \in S^2_{n,h}$ increases with respect to the quasi-order relation $\preceq$ defined over $S^1_{n,h}$. That is, if $S_1 \preceq S_2$, then $E_t(S_1) \leq E_t(S_2)$, where $S_1, S_2 \in S^2_{n,h}$.*

Proof. Let $S \in S^2_{n,h}$ be an n-vertex sidigraph. Since $h \equiv 0(\mathrm{mod}\ 8)$, there is a positive integer l such that $h = 8l$. Then by Theorem 13, the characteristic polynomial of $A_t(S)$ is given by

$$\Phi^t_S(\lambda) = \lambda^n + \sum_{k=1}^{\lfloor \frac{n}{h} \rfloor} (1 - i)^{kh} c(S, kh)\lambda^{n-kh}. \tag{28}$$

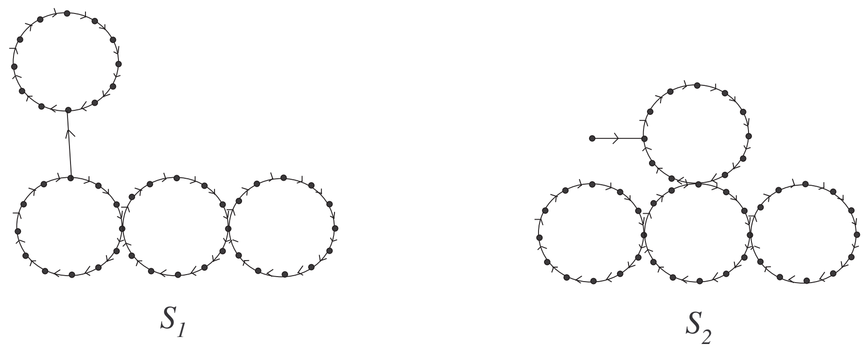

Figure 2. $S_1, S_2 \in S^2_{46,12}$

This gives

$$\begin{aligned}
\Phi^t_S(\frac{i}{\lambda}) &= \left(\frac{i}{\lambda}\right)^n + \sum_{k=1}^{\lfloor \frac{n}{h} \rfloor}(1-i)^{kh} c(S,kh)\left(\frac{i}{\lambda}\right)^{n-kh} \\
&= \left(\frac{i}{\lambda}\right)^n \left(1 + \sum_{k=1}^{\lfloor \frac{n}{h} \rfloor}(-2i)^{4kl} c(S,kh)\lambda^{kh}\right) \\
&= \left(\frac{i}{\lambda}\right)^n \left(1 + \sum_{k=1}^{\lfloor \frac{n}{h} \rfloor}2^{4kl} c(S,kh)\lambda^{kh}\right).
\end{aligned}$$

By applying Corollary 3, we get

$$\begin{aligned}
E_t(S) &= \frac{1}{\pi}\ p.v \int_{-\infty}^{\infty} \frac{1}{\lambda^2} \log \left| \lambda^n \frac{i^n}{\lambda^n} \left(1 + \sum_{k=1}^{\lfloor \frac{n}{h} \rfloor} 2^{4kl} c(S,kh)\lambda^{kh}\right)\right| d\lambda \\
&= \frac{1}{\pi}\ p.v \int_{-\infty}^{\infty} \frac{1}{\lambda^2} \log \left(1 + \sum_{k=1}^{\lfloor \frac{n}{h} \rfloor} 2^{4kl} c(S,kh)\lambda^{kh}\right) d\lambda.
\end{aligned}$$

The last equality is obtained by using $\frac{1}{\pi} p.v. \int_{-\infty}^{\infty} \log(i^n)\frac{d\lambda}{\lambda^2} = 0$. It is easy to see from the above total energy expression that total energy increases with respect to quasi-order relation $\preceq$ over $S^2_{n,h}$. □

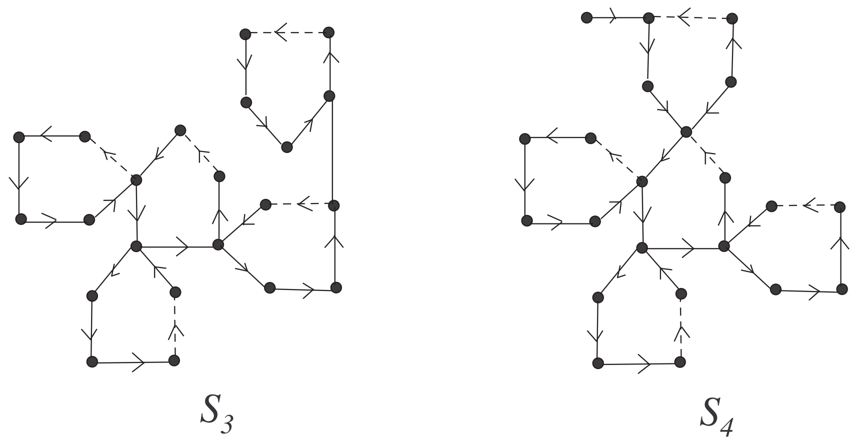

Figure 3. $S_3, S_4 \in S^2_{22,5}$

Example 3. *Consider the sidigraphs* S_1 *and* S_2 *as shown in Figure* 2*, where* $h \equiv 4(\mathrm{mod}\, 8)$ *and each cycle has negative sign. By Theorem 13, the characteristics polynomials of* S_1 *and* S_2 *are, respectively, given by*

$$\begin{aligned} \phi_{S_1} &= \lambda^{46} + 4\lambda^{34} + 4\lambda^{22} + \lambda^{10}, \\ \phi_{S_2} &= \lambda^{46} + 4\lambda^{34} + 3\lambda^{22} + \lambda^{10}. \end{aligned}$$

Here $c(S_1, kh) \geq c(S_2, kh)$*,* $k = 1, 2, 3$*. However,* $E_t(S_1) = 31.7354$ *and* $E_t(S_2) = 32.0260$*. Clearly,* $E_t(S_2) \geq E_t(S_1)$*. Thus, the total energy may not increase with respect to the quasi-order relation over the set* $S^2_{n,h}$ *when* $h \equiv 4(\mathrm{mod}\ 8)$*.*

Example 4. *Consider the sidigraphs* S_3 *and* S_4 *as shown in Figure* 3*, where* $h \equiv 1(\mathrm{mod} 2)$*. By Theorem 13, the characteristics polynomials of* S_3 *and* S_4 *are, respectively, given by*

$$\begin{aligned} \phi_{S_3} &= \lambda^{22} + 5\lambda^{17} + 7\lambda^{12} + 4\lambda^{7} + \lambda^{2}, \\ \phi_{S_4} &= \lambda^{22} + 5\lambda^{17} + 6\lambda^{12} + 4\lambda^{7} + \lambda^{2}. \end{aligned}$$

Here $c(S_3, kh) \geq c(S_4, kh)$*,* $k = 1, 2, 3, 4$*. However,* $E_t(S_3) = 18.1843$ *and* $E_t(S_4) = 18.2420$*. Clearly,* $E_t(S_4) \geq E_t(S_3)$*. Thus, the total energy may not increase with respect to the quasi-order relation over the set* $S^2_{n,h}$ *when* $h \equiv 1 (\mathrm{mod}\ 2)$*.*

For counter examples of increasing property of total energy in $S^1_{n,h}$ when $h \equiv 0 (\mathrm{mod}\ 8)$ or $h \equiv 1 (\mathrm{mod}\ 2)$, see Example 5.4 and 5.5 [9].

5 Upper bound for total energy of sidigraphs

In this section, we obtain upper bound for the total energy of sidigraphs. An alternating sequence of vertices and directed arcs is called a directed walk. Let $c^+(2)$ and $c^-(2)$, respectively, denote the number of positive and negative closed directed walks of length 2. Let K_2^+(respectively, K_2^-) be the sidigraph whose sign of an arc is $+1$(respectively, -1). The following Lemma is used to find upper bound.

Lemma 5 (Pirzada and Bhat [13])**.** *Let* S *be a sidigraph having* n *vertices and* a *arcs and let* $\lambda_1, \ldots, \lambda_n$ *be its eigenvalues. Then*

(1). $\sum_{k=1}^n (\mathrm{Re}(\lambda_k))^2 - \sum_{k=1}^n (\mathrm{Im}(\lambda_k))^2 = c^+(2) - c^-(2)$,

(2). $\sum_{k=1}^n (\mathrm{Re}(\lambda_k))^2 + \sum_{k=1}^n (\mathrm{Im}(\lambda_k))^2 \leq a$.

It is easy to see from the proof of Lemma 5 that

$$\sum_{k=1}^{n} \mathrm{Re}(\lambda_k)\,\mathrm{Im}(\lambda_k) = 0. \tag{29}$$

Let S be any sidigraph. We denote by $q \oplus S$, the direct sum of q copies of S. Now we will give upper bound for the total energy of sidigraphs. We remark that the idea of the proof is taken from Theorem 2.3 [14].

Theorem 17. *Let* S *be an* n*-vertex sidigraph and* a *be the total number of arcs of* S*. Let* $\lambda_1, \ldots, \lambda_n$ *be the eigenvalues of* S*. Then* $E_t(S) \leq \sqrt{na}$*. Moreover, if* S *is balanced, the equality holds if* $S = \frac{n}{2} \oplus \overleftrightarrow{K}_2^+$ *or*

$S = \frac{n}{2} \oplus \overleftrightarrow{K}_2^-$ *or* $S = \frac{n}{4} \oplus C_4$. *If* S *is unbalanced, the equality holds if* $S = \frac{n}{2} \oplus \boldsymbol{C}_2$.

Proof. Using (29) and part 2 of Lemma 5 and applying the Cauchy-Schwarz inequality to vectors $(|\mathrm{Re}(\lambda_1)+\mathrm{Im}(\lambda_1)|, \ldots, |\mathrm{Re}(\lambda_n)+\mathrm{Im}(\lambda_n)|)$ and $(1, 1, \ldots, 1) \in \mathbb{R}^n$, we have

$$
\begin{aligned}
E_t(S) &= \sum_{k=1}^{n} |\mathrm{Re}(\lambda_\mathrm{k}) + \mathrm{Im}(\lambda_\mathrm{k})| \\
&\leq \sqrt{n}\sqrt{\sum_{k=1}^{n} |\mathrm{Re}(\lambda_\mathrm{k}) + \mathrm{Im}(\lambda_\mathrm{k})|^2} \\
&= \sqrt{n}\sqrt{\sum_{k=1}^{n} (\mathrm{Re}(\lambda_k))^2 + \sum_{k=1}^{n} (\mathrm{Im}(\lambda_k))^2} \leq \sqrt{na}.
\end{aligned}
$$

For the second part, we consider two cases: (1). S is balanced, (2). S is unbalanced.

(1). Note that the eigenvalues of sidigraphs $\frac{n}{2} \oplus \overleftrightarrow{K}_2^+$ and $\frac{n}{2} \oplus \overleftrightarrow{K}_2^-$ are ± 1 and ± 1, respectively, each repeated $\frac{n}{2}$ times and the eigenvalues of sidigraph $\frac{n}{4} \oplus C_4$ are $\{\pm 1, \pm i\}$, each repeated $\frac{n}{4}$ times. Thus, $E_t(\frac{n}{2} \oplus \overleftrightarrow{K}_2^+) = \frac{n}{2}(|1| + |-1|) = n$, $E_t(\frac{n}{2} \oplus \overleftrightarrow{K}_2^-) = \frac{n}{2}(|1| + |-1|) = n$ and $E_t(\frac{n}{4} \oplus C_4) = \frac{n}{4}(2\,(\,|1| + |-1|\,)) = n$. On the other hand, clearly, the number of vertices and number of arcs in all these three sidigraphs are n and n. Therefore, for these sidigraphs, $E_t(S) = \sqrt{nn} = n$.

(2). The sidigraph $S = \frac{n}{2} \oplus \boldsymbol{C}_2$ has n vertices and n arcs. Thus, $E_t(S) = \sqrt{nn} = n$. However, from total energy formula (3), we have $E_t(\frac{n}{2} \oplus \boldsymbol{C}_2) = \frac{n}{2}(|1| + |-1|) = n$, since the eigenvalues of a sidigraph $\frac{n}{2} \oplus \boldsymbol{C}_2$ are $\pm i$, each repeated $\frac{n}{2}$ times. This completes the proof. ☐

Theorem 18. *Let* S *be a sidigraph on* n *vertices and* a *arcs and let* $S_1, \ldots, S_m$ *be its strong components with* $n_i > 1$ *vertices and* a_i *arcs, where* $i = 1, 2, \ldots, m$, *respectively. Then* $E_t(S) \leq a$. *Moreover, if* S *is balanced, the equality holds if* $S = \frac{a}{2} \oplus \overleftrightarrow{K}_2^+$ *plus some isolated vertices or* $S = \frac{a}{2} \oplus \overleftrightarrow{K}_2^-$ *plus some isolated vertices or* $S = \frac{a}{4} \oplus C_4$ *plus some* *isolated vertices, and if S is unbalanced, the equality holds if $S = \frac{a}{2} \oplus \boldsymbol{C}_2$ plus some isolated vertices.*

Proof. We assume that S is a strongly connected sidigraph with n vertices and a arcs. Then $n \leq a$. By Theorem 17, we have

$$E_t(S) \leq \sqrt{na} \leq a.$$

As $S_1 \ldots, S_m$ are strong components of S with n_i vertices and a_i arcs, where $i = 1, 2, \ldots, m$, therefore $\sum_{i=1}^{m} = n$ and $\sum_{i=1}^{m} a_i \leq a$. By Theorem 2, we have

$$E_t(S) = \sum_{i=1}^{r} E_t(S_i) \leq \sum_{i=1}^{r} \leq a.$$

The proof of the second part is similar to the proof of the second part of Theorem 17 and is, thus, omitted. □

6 T-Equienergetic sidigraphs

Two sidigraphs with the same spectrum are said to be cospectral, otherwise non-cospectral. Two T-equienergetic sidigraphs of the same order are the sidigraphs which have the same total energy. Two isomorphic sidigraphs are always cospectral and, thus, are T-equienergetic. In this section, we are interested to construct a few classes of non-cospectral T-equienergetic sidigraphs.

Example 5. *Consider the sidigraphs S and H as shown in Figure 2. The dotted lines represent negative arcs and solid ones represent positive arcs. The characteristic polynomials of S and H are, respectively, given by*

$$\begin{aligned} \phi_S(\lambda) &= (\lambda^6 - 1) \\ \phi_H(\lambda) &= (\lambda^6 + 1). \end{aligned}$$

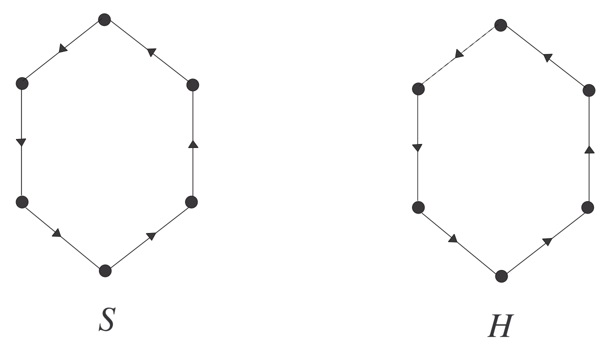

Figure 4. Equienergetic sidigraphs

Thus, the spectrums of S and H are, respectively, given by

$$\text{spec(S)} = \Big\{ \pm 1, \frac{1 \pm \sqrt{3}i}{2}, \frac{-1 \pm \sqrt{3}i}{2} \Big\}, \quad (30)$$

$$\text{spec(H)} = \Big\{ \pm i, \frac{\sqrt{3} \pm i}{2}, \frac{-\sqrt{3} \pm i}{2} \Big\}. \quad (31)$$

From (30) *and* (31)*, S and H are non-cospectral T-equienergetic sidigraphs*

Theorem 19. *Let $S_1 = (2 \oplus C_2) \times S$, $S_2 = (2 \oplus \boldsymbol{C}_2) \times S$ and $S_3 = C_4 \times S$, where S is an n-vertex sidigraph. Then $E_t(S_1) = E_t(S_3)$ and $E_t(S_2) = E_t(S_3)$. Moreover, S_1, S_3 and S_2, S_3 are non-cospectral sidgraphs. Similarly, S_1 and S_2 are non-cospectral T-equienergetic sidigraphs.*

Proof. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of S. Then spectrums of S_1, S_2 and S_3 are, respectively, given by

$$\begin{aligned}
\text{spec(S}_1) &= \{\lambda_1 \pm 1, \lambda_1 \pm 1, \ldots, \lambda_n \pm 1, \lambda_n \pm 1\}, \\
\text{spec(S}_2) &= \{\lambda_1 \pm i, \lambda_1 \pm i, \ldots, \lambda_n \pm i, \lambda_n \pm i\}, \\
\text{spec(S}_3) &= \{\lambda_1 \pm 1, \lambda_1 \pm i, \ldots, \lambda_n \pm 1, \lambda_n \pm i\}.
\end{aligned}$$

It is clear that S_1, S_3 and S_2, S_3 are non-cospectral T-equienergetic sidigraphs. Also S_1 and S_2 are non-cospectral T-equienergetic sidigraphs. □

Example 6. *For each odd* n*,* C_n *and* $\boldsymbol{C}_n$ *are non-cospectral T-equienergetic sidigraphs, because* $\mathrm{spec}(\mathrm{C_n}) = -\mathrm{spec}(\boldsymbol{C}_\mathrm{n})$ *and* $1 \in \mathrm{spec}(\mathrm{C_n})$ *but* $1 \notin \mathrm{spec}(\boldsymbol{C}_\mathrm{n})$*.*

Using Example 6, we have the following corollary of Theorem 19.

Corollary 4. *If* $n \equiv 1(\mathrm{mod}\ 2)$*, then* $(2 \oplus C_2) \times C_n$ *and* $(2 \oplus C_2) \times \boldsymbol{C}_n$ *are non-cospectral T-equienergetic sidgraphs. Similarly,* $(2 \oplus \boldsymbol{C}_2) \times C_n$ *and* $(2 \oplus \boldsymbol{C}_2) \times \boldsymbol{C}_n$ *are non-cospectral T-equienergetic sidigraphs.*

The following lemma will be useful to find pair of non-cospectral T-equienergetic sidigraphs.

Lemma 6. *Let* $n \geq 6$ *be a positive integer. Then*

(1). $E_t(C_n) = 2E_t(\boldsymbol{C}_{\frac{n}{2}})$ *if and only if* $n \equiv 2(\mathrm{mod}\ 4)$*.*

(2). $E_t(C_n) = 4E_t(\boldsymbol{C}_{\frac{n}{4}})$ *if and only if* $n \equiv 4(\mathrm{mod}\ 8)$*.*

(3). $E_t(C_n) = E_t(\boldsymbol{C}_n)$ *if and ony if* $n \equiv 2(\mathrm{mod}\ 4)$ *or* $n \equiv 1(\mathrm{mod}\ 2)$*.*

Proof. Proof is similar to the proof of Lemma 5.2 [10]. □

The proofs of the next Theorems are similar to the proof of Theorem 5.3 [10] and are, thus, omitted.

Theorem 20. *Let* S_1 *and* S_2 *be an* n*-vertex sidigraphs,* $n \geq 6$*, with* k *vertex-disjoint positive directed cycles and* k *vertex-disjoint negative directed cycles of lengths* $m_1, \ldots, m_k$*, where* $m_j \equiv 1(\mathrm{mod}\ 2)$ *or* $m_j \equiv 2(\mathrm{mod}\ 4)$*,* $j = 1, 2, \ldots, k$*. Then* $E_t(S_1) = E_t(S_2)$*. Moreover,* S_1 *and* S_2 *are non-cospectral.*

Theorem 21. *Let* S_1 *and* S_2 *be two* n*-vertex sidigraphs,* $n \geq 6$*, with* k *vertex-disjoint positive directed cycles and* $2k$ *vertex-disjoint negative directed cycles, respectively, of lengths* $m_1, \ldots, m_k$ *and* $\frac{m_1}{2}, \frac{m_1}{2}, \ldots, \frac{m_k}{2}, \frac{m_k}{2}$*, respectively, where* $m_j \equiv 2(\mathrm{mod}\ 4)$*,* $j = 1, 2, \ldots, k$*. Then* $E_t(S_1) = E_t(S_2)$*. Moreover,* S_1 *and* S_2 *are non-cospectral.*

Theorem 22. *Let S_1 and S_2 be two n-vertex sidigraphs, $n \geq 6$, with k vertex-disjoint positive directed cycles and 4k vertex-disjoint negative directed cycles, respectively, of lengths $m_1, \ldots, m_k$ and $\frac{m_1}{4}, \frac{m_1}{4}, \frac{m_1}{4}, \frac{m_1}{4}, \ldots, \frac{m_k}{4}, \frac{m_k}{4}, \frac{m_k}{4}, \frac{m_k}{4}$, respectively, where $m_j \equiv 4 (\mathrm{mod}\ 8)$, $j = 1, 2, \ldots, k$. Then $E_t(S_1) = E_t(S_2)$. Moreover, S_1 and S_2 are non-cospectral.*

The next two theorems give a class of non-cospectral T-equienergetic digraphs.

Theorem 23. (Khan et al. [9]) *Let S_1 and S_2 be two n-vertex sidigraphs, $n \geq 6$, with k and 2k vertex-disjoint directed cycles, respectively, of lengths $m_1, \ldots, m_k$ and $\frac{m_1}{2}, \frac{m_1}{2}, \ldots, \frac{m_k}{2}, \frac{m_k}{2}$, respectively, where $m_j \equiv 2 (\mathrm{mod}\ 4)$, $j = 1, 2, \ldots, k$. Then S_1 and S_2 are non-cospectral T-equienergetic digraphs.*

Theorem 24. (Khan et al. [9]) *Let S_1 and S_2 be two n-vertex sidigraphs, $n \geq 6$, with k and 2k vertex-disjoint directed cycles, respectively, of lengths $m_1, \ldots, m_k$ and $\frac{m_1}{2}, \frac{m_1}{2}, \ldots, \frac{m_k}{2}, \frac{m_k}{2}$, respectively, where $m_j \equiv 4 (\mathrm{mod}\ 8)$, $j = 1, 2, \ldots, k$. Then S_1 and S_2 are non-cospectral T-equienergetic digraphs.*

7 Conclusion

In this paper, we extend the concept of total energy of digraphs to sidigraphs and introduced complex adjacency matrix of sidigraphs. We calculated the total energy formulas for negative directed cycles and show that the total energy of negative directed cycles increases monotonically. We discuss the increasing property of total energy over the set $S^1_{n,h}$ and $S^2_{n,h}$ with respect to the quasi-order relation when $h \equiv 4 (\mathrm{mod}\ 8)$ and $h \equiv 0 (\mathrm{mod}\ 8)$, respectively. We find upper bound for total energy of sidigraphs. Finally, we find non-cospectral T-equienergetic sidigraphs. However, it is unclear whether the total energy of sidigraphs increases or not, with respect to the quasi-order relation over $S^1_{n,h}$ and $S^2_{n,h}$ when $h \equiv 2 (\bmod\ 4)$. We leave this problem for future work.

References

[1] B. D. Acharya, "Spectral criterion for the cycle balanced in networks," *J. Graph Theor.*, vol. 4, pp. 1–11, 1980.

[2] M. A. Bhatt, "Energy of weighted digraphs," *Discrete Applied Math.*, vol. 223, pp. 1–14, 2017.

[3] M. A. Bhat and S. Pirzada, "Unicyclic signed graphs with minimal energy," *Discrete Applied Math.*, vol. 226, pp 32–39, 2017. DOI: 10.1016/j.dam.2017.03.015.

[4] R. Farooq, M. Khan, and F. Ahmad, "Extremal iota energy of bicyclic digraphs," *Appl. Math. Comput.*, vol. 303, pp. 24–33, 2017.

[5] R. Farooq, M. Khan, and S. Chand, "On iota energy of signed digraphs," *Linear Multilinear A.*, vol. 67, no. 4, pp.705–724, 2019.

[6] K. A. Germina, K. S. Hameed, and T. Zaslavsky, "On products and line graphs of signed graphs, their eigenvalues and energy," *Linear Algebra Appl.*, vol. 435, pp. 2432–2450, 2010.

[7] I. Gutman, "The energy of a graph," *Ber. Math. statist. Sekt. Forschungszentrum. Graz.*, vol. 103, pp. 1–22, 1978.

[8] M. Khan, R. Farooq, and A. A. Siddiqui, "On the extremal energy of bicyclic digraphs," *J. Math. Inequal.*, vol. 9, no. 3, pp. 799–810, 2015.

[9] M. Khan and S. Hafeez, Total energy of digraphs, submitted.

[10] M. Khan, R. Farooq, and J. Rada, "Complex adjacency matrix and energy of digraphs," *Linear Multilinear A.*, vol. 65, no. 11, pp. 2170–2186, 2017.

[11] M. Mateljevic, V. Bozin, and I. Gutman, "Energy of a polynomial and the Coulson integral formula," *J. Math. Chem.*, vol. 48, pp. 1062–1068, 2010.

[12] I. Peña and J. Rada, "Energy of digraphs," *Linear Multilinear Algebra.*, vol. 56, no. 5, pp. 565–579, 2008.

[13] S. Pirzada and M. A. Bhat, "Energy of signed digraphs," *Discrete Applied Math.*, vol. 169, pp. 195–205, 2014.

[14] J. Rada, "The McClelland inequality for the energy of digraphs," *Linear Algebra Appl.*, vol. 430, pp. 800–804, 2009.

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