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A sharp upper bound on the independent 2-rainbow domination in graphs with minimum degree at least two

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Abstract

An independent 2-rainbow dominating function (I2-RDF) on a graph G is a function f from the vertex set $V(G)$ to the set of all subsets of the set $\{1,2\}$ such that $\{x \in V \mid f(x) \neq \emptyset\}$ is an independent set of G and for any vertex $v \in V(G)$ with $f(v) = \emptyset$ we have $\bigcup_{u\in N(v)} f(u) = \{1,2\}$. The *weight* of an I2-RDF f is the value $\omega(f) = \sum_{v\in V} |f(v)|$, and the independent 2-rainbow domination number $i_{r2}(G)$ is the minimum weight of an I2-RDF on G. In this paper, we prove that if G is a graph of order $n \geq 3$ with minimum degree at least two such that the set of vertices of degree at least 3 is independent, then $i_{r2}(G) \leq \frac{4n}{5}$.

Keywords: independent k-rainbow dominating function, independent k-rainbow domination number.

MSC 2010: 05C69.

1 Introduction

In this paper, G is a simple graph with vertex set $V(G)$ and edge set $E(G)$ (briefly V and E). For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in V(G) \mid uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. Similarly, the *open neighborhood* of a set $S \subseteq V$ is the set $N(S) = \cup_{v\in S} N(v)$, and the *closed neighborhood*

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of S is the set $N[S] = N(S) \cup S$. The *degree* of a vertex $v \in V$ is $\deg_G(v) = \deg(v) = |N(v)|$ and the *minimum degree* of a graph G is denoted by $\delta = \delta(G)$.

For a positive integer k, a *k-rainbow dominating function* (k-RDF) of a graph G is a function f from $V(G)$ to the set of all subsets of the set $\{1, 2, \ldots, k\}$ such that for every vertex $v \in V(G)$ with $f(v) = \emptyset$ the condition $\bigcup_{u \in N(v)} f(u) = \{1, 2, \ldots, k\}$ is fulfilled. The *weight* of a k-RDF f is the value $\omega(f) = \sum_{v \in V} |f(v)|$. The *$k$-rainbow domination number* of a graph G, denoted by $\gamma_{rk}(G)$, is the minimum weight of a k-RDF of G. Note that $\gamma_{r1}(G)$ is the classical domination number $\gamma(G)$. The k-rainbow domination number was introduced by Brešar, Henning, and Rall [4] and has been studied by several authors (see for example [1], [5], [6], [8], [12], [13], [15], [16]).

If we additionally require that the set of vertices $x \in V(G)$ with $f(x) \neq \emptyset$ induces an independent set of G, then the situation is very different. Let $k \geq 1$ be an integer, and let G be a graph. A k-RDF f of G is an *independent k-rainbow dominating function* (Ik-RDF) if $\{x \in V(G) \mid f(x) \neq \emptyset\}$ is an independent set of G. The minimum weight $w(f)$ of an Ik-RDF f of G is called the *independent k-rainbow domination number* of G, and is denoted by $i_{rk}(G)$. Clearly, $\gamma_{rk}(G) \leq i_{rk}(G)$ holds for every graph G. Moreover, independent k-rainbow domination can be seen as a generalization of independent domination, since the number $i_{r1}(G)$ is precisely the *independent domination number* $i(G)$ of G which has been widely studied (see a survey [10]). The independent rainbow domination number was studied in, for example, [3], [7], [9], [14]. Recently, Shao et al. [14] have shown that the independent k-rainbow domination problem is NP-complete. Therefore, it is natural to look for good bounds on the independent k-rainbow domination number of graphs, especially for the case $k = 2$ which is the most studied. Our main contribution in this paper is the following upper bound on the independent 2-rainbow domination number for a class of graphs with minimum degree at least two.

Theorem 1. Let G be a connected graph of order n with $\delta(G) \geq 2$ such that the set of vertices with degree at least three is an independent set. Then $i_{r2}(G) \leq \frac{4}{5}n$. This bound is sharp for the cycle C_5.

We recall that for trees, Amjadi et al. [2] showed that if T is a tree of order $n \geq 3$, then $i_{r2} \leq \frac{3n}{4}$. This result has recently been extended to connected bipartite graphs by Fujita et al. [9] who also gave other results on $i_{rk}(G)$ when $k \geq 3$. Moreover, it has been noticed in [7], that for any graph G, $i_{r2}(G) \leq 2i(G)$. Therefore, the $\frac{4}{5}n$ upper bound on the independent 2-rainbow domination number holds for any connected graph G of order n with $i(G) \leq \frac{2}{5}n$.

2 Proof of Theorem 1

For the proof of Theorem 1, we need three preparatory results.

Proposition 1. For $n \geq 5$, the path P_n has an I2-RDF f that assigns $\emptyset$ to the end-vertices of P_n and $\omega(f) \leq \frac{4n}{5}$.

Proof. Let $P_n = v_1v_2\dots v_n$ and define $f : V(P_n) \to \mathcal{P}(\{1,2\})$ as follows. If $n \equiv 0 \pmod 3$, then $f(v_{3i+2}) = \{1,2\}$ for $0 \leq i \leq \frac{n}{3} - 1$, and $f(x) = \emptyset$ otherwise; if $n \equiv 2 \pmod 3$, then $f(v_{n-1}) = \{1,2\}$, $f(v_{3i+2}) = \{1,2\}$ for $0 \leq i \leq \frac{n-5}{3}$, and $f(x) = \emptyset$ otherwise, and if $n \equiv 1 \pmod 3$, then $f(v_2) = \{1,2\}, f(v_4) = \{1\}$, $f(v_{3i+6}) = \{1,2\}$ for $0 \leq i \leq \frac{n-4}{3}$, and $f(x) = \emptyset$ otherwise. Clearly, $\omega(f) \leq \frac{4n}{5}$ and f is an I2-RDF of P_n assigning $\emptyset$ to the end-vertices of P_n. □

For integers $r \geq 3$ and $s \geq 1$, let $C_{r,s}$ be the connected graph obtained from a cycle $C_r = (u_1u_2\dots u_r)$ and a path $P_s = v_1v_2\dots v_s$ by adding the edge u_1v_1.

Proposition 2. For integers $r \geq 3$ and $s \geq 1$ with $r+s \geq 4$, the graph $C_{r,s}$ has *an I2-RDF* f that assigns $\emptyset$ to v_s and $\omega(f) \leq \frac{4(r+s)}{5}$.

Proof. Define $f : V(C_{r,s}) \to \mathcal{P}(\{1,2\})$ by:

$f(u_{3i+1}) = f(v_{3j+2}) = \{1,2\}$ for $0 \leq i \leq \frac{r-3}{3}$, $0 \leq j \leq \frac{s-3}{3}$, and $f(x) = \emptyset$ otherwise, when $r \equiv 0 \pmod 3$ and $s \equiv 0 \pmod 3$,

$f(u_{3i+1}) = f(v_{3j+3}) = \{1,2\}$ for $0 \leq i \leq \frac{r-3}{3}$, $0 \leq j \leq \frac{s-4}{3}$, and $f(x) = \emptyset$ otherwise, when $r \equiv 0 \pmod 3$ and $s \equiv 1 \pmod 3$,

$f(v_s) = \{1,2\}$, $f(u_{3i+2}) = f(v_{3j+2}) = \{1,2\}$ for $0 \le i \le \frac{r-3}{3}$, $0 \le j \le \frac{s-5}{3}$, and $f(x) = \emptyset$ otherwise, when $r \equiv 0 \pmod 3$ and $s \equiv 2 \pmod 3$,

$f(u_1) = \{1\}$, $f(u_{3i+3}) = f(v_{3j+2}) = \{1,2\}$ for $0 \le i \le \frac{r-4}{3}$, $0 \le j \le \frac{s-3}{3}$, and $f(x) = \emptyset$ otherwise, when $r \equiv 1 \pmod 3$ and $s \equiv 0 \pmod 3$,

$f(u_1) = \{1,2\}$, $f(u_{3i+3}) = f(v_{3j+3}) = \{1,2\}$ for $0 \le i \le \frac{r-4}{3}$, $0 \le j \le \frac{s-4}{3}$, and $f(x) = \emptyset$ otherwise, when $r \equiv 1 \pmod 3$ and $s \equiv 1 \pmod 3$,

$f(u_{3i+3}) = f(v_{3j+1}) = \{1,2\}$ for $0 \le i \le \frac{r-4}{3}$, $0 \le j \le \frac{s-2}{3}$, and $f(x) = \emptyset$ otherwise, when $r \equiv 1 \pmod 3$ and $s \equiv 2 \pmod 3$,

$f(u_{3i+1}) = f(v_{3j+2}) = \{1,2\}$ for $0 \le i \le \frac{r-2}{3}$, $0 \le j \le \frac{s-3}{3}$, and $f(x) = \emptyset$ otherwise, when $r \equiv 2 \pmod 3$ and $s \equiv 0 \pmod 3$,

$f(u_{3i+1}) = f(v_{3j+3}) = \{1,2\}$ for $0 \le i \le \frac{r-2}{3}$, $0 \le j \le \frac{s-4}{3}$, and $f(x) = \emptyset$ otherwise, when $r \equiv 2 \pmod 3$ and $s \equiv 1 \pmod 3$,

$f(u_r) = \{1\}$, $f(u_{3i+3}) = f(v_{3j+1}) = \{1,2\}$ for $0 \le i \le \frac{r-5}{3}$, $0 \le j \le \frac{s-2}{3}$, and $f(x) = \emptyset$ otherwise, when $r \equiv 2 \pmod 3$ and $s \equiv 2 \pmod 3$.

In either case, f is an I2-RDF of $C_{r,s}$ of weight at most $\frac{4(r+s)}{5}$ with the desired property. □

Let $\mathcal{F}$ be the family of all simple graphs obtained from some connected multigraph H without loops with $\delta(H) \ge 3$ by subdividing every edge of H at least once and at most four times. By definition, the smallest graph of $\mathcal{F}$ has order at least 5. Also, we note that if $G \in \mathcal{F}$, then the set $\{x \in V(G) \mid \deg(x) \ge 3\}$ is an independent set of G.

Proposition 3. If $G \in \mathcal{F}$, then G has an *I2-RDF* f that assigns a non-empty set to every vertex of degree at least 3 and $\omega(f) \le \frac{4n(G)}{5}$.

Proof. Let $G \in \mathcal{F}$ be a graph of order n. We proceed by induction on n. Clearly, the result is immediate for $n = 5$. Let $n \ge 6$ and assume that the result holds for all graphs in $\mathcal{F}$ of order less than n. Let $G \in \mathcal{F}$ be a graph of order n. Set $A = \{x \in V(G) \mid \deg(x) \ge 3\}$ and $B = V(G) - A$. Since $G \in \mathcal{F}$, A is independent. In the sequel, we will call an induced path P of G an *A-ear path* if $V(P) \subset B$ and P is connected to A by either its unique vertex (if $|V(P)| = 1$) or by each of its two end-vertices. For each $i \in \{1,2,3,4\}$, let Q_i be the set of all A-ear paths P of G of order i and let $\mathcal{Q} = \bigcup_{i=1}^{4} Q_i$. Clearly, $B = \bigcup_{P\in\mathcal{Q}} V(P)$. Moreover, for each A-ear path P, let $X_P = \{u \in A \mid u$ is adjacent to a vertex of $P\}$. Hence, $A = \bigcup_{P\in\mathcal{Q}} X_P$ and since $G \in \mathcal{F}$, we have $|X_P| = 2$ for each $P \in \mathcal{Q}$. Therefore, $|A| \geq 2$.

Assume first that $|A| = 2$ and let $A = \{u, v\}$. Note that $n = |A|+m_1+2m_2+3m_3+4m_4$ and $m_1+m_2+m_3+m_4 \geq 3$, where $m_i = |Q_i|$ for $i \in \{1,2,3,4\}$. Let $Q_4 = \{v_1^i v_2^i v_3^i v_4^i \mid 1 \leq i \leq m_4\}$ if $Q_4 \neq \emptyset$, $Q_3 = \{m_1^j m_2^j m_3^j \mid 1 \leq j \leq m_3\}$ if $Q_3 \neq \emptyset$, $Q_2 = \{w_1^k w_2^k \mid 1 \leq k \leq m_2\}$ if $Q_2 \neq \emptyset$ and $Q_1 = \{z_1^l \mid 1 \leq l \leq m_1\}$ if $Q_1 \neq \emptyset$. Suppose that $uv_1^i, um_1^j, uw_1^k, uz_1^l, v_4^i v, m_3^j v, w_2^k v, z_1^l v \in E(G)$ for each i, j, k, l. Define $g : V(G) \to \mathcal{P}(\{1,2\})$ by $g(u) = g(v) = \{1,2\}$, $g(v_2^i) = \{1,2\}$ for $1 \leq i \leq m_4$ and $g(m_2^j) = \{2\}$ for $1 \leq j \leq m_3$, and $g(x) = \emptyset$ otherwise. Obviously, g is an I2-RDF of G that assigns a non-empty set to every vertex in A. In addition, we have

$$i_{r2}(G) \leq 4 + 2m_4 + m_3 \leq \frac{4(2 + 4m_4 + 3m_3 + 2m_2 + m_1)}{5} = \frac{4n}{5}.$$

Next, we assume that $|A| \geq 3$. Suppose first, there are two vertices $u, v \in A$ such that $\deg(u)$, $\deg(v) \geq 4$ and there is an A-ear path $P = v_1 \ldots v_k$ with $k \geq 3$ connecting u and v. Let $G' = G - V(P)$. Since $G' \in \mathcal{F}$, by the induction hypothesis G' has an I2-RDF f such that $|f(u)|, |f(v)| \geq 1$ and $\omega(f) \leq \frac{4(n-k)}{5}$. Assume, without loss of generality, that $1 \in f(u)$. Then f can be extended to an I2-RDF g of G as follows: assign $\{1,2\}$ to v_{3i+2} for $0 \leq i \leq \frac{k}{3} - 1$ and $\emptyset$ to other vertices, when $k \equiv 0 \pmod 3$; assign $\{2\}$ to v_2, $\{1,2\}$ to v_{3i+1} for $1 \leq i \leq \frac{k-2}{3}$ and $\emptyset$ to other vertices, when $k \equiv 2 \pmod 3$; assign $\{1,2\}$ to u, v_{3i} for $1 \leq i \leq \frac{k-1}{3}$ and $\emptyset$ to other vertices, when $k \equiv 1$ (mod 3). Clearly, g is an I2-RDF of G of weight at most $\frac{4n(G)}{5}$ with the desired property. Hence, we can assume that there is no two vertices of degree at least four connected by A-ear path P of order at least three. Consider the following cases.

Case 1. $Q_4 \neq \emptyset$.
Let $P_1 = x_1^1 x_2^1 x_3^1 x_4^1 \in Q_4$ and let $ux_1^1, x_4^1 v_1 \in E(G)$, where $u, v_1 \in A$.

By assumption, u or v_1 has degree three, say $\deg(u) = 3$. Consider also the following situations.

(I) u is adjacent to two A-ear paths in Q_4.
Let $P_2 = x_1^2x_2^2x_3^2x_4^2 \in Q_4 - \{P_1\}$ such that ux_1^2, $v_2x_4^2 \in E(G)$. Let G' be the graph obtained from G by removing vertices $\{x_1^1, x_2^1, x_3^1, x_1^2, x_2^2\}$ and adding edges ux_4^1, ux_3^2. Clearly, $G' \in \mathcal{F}$, and by the induction hypothesis, there exists an I2-RDF f of G' of weight at most $\frac{4(n-5)}{5}$ assigning a non-empty set to every vertex of degree at least 3. It follows that $f(x_3^2) = \emptyset$ and $f(x_4^2) = \emptyset$, and, thus, $f(u) = f(v_2) = \{1, 2\}$ (to 2-rainbow dominate x_3^2, x_4^2). Define $g : V(G) \to \mathcal{P}(\{1, 2\})$ by $g(x_3^1) = g(x_3^2) = \{1, 2\}$, $g(x_1^1) = g(x_2^1) = g(x_1^2) = g(x_2^2) = \emptyset$, and $g(x) = f(x)$ otherwise. Clearly, g is an I2-RDF of G that assigns a non-empty set to all vertices of A and has weight $\omega(g) = \omega(f) + 4 \leq \frac{4(n-5)}{5} + 4 = \frac{4n}{5}$.

According to the previous situation, we may assume that P_1 is the unique A-ear path in Q_4 adjacent to u.

(II) u is adjacent to an A-ear path in Q_2.
Let $P_2 = x_1^2x_2^2$ be an A-ear path in Q_2 such that $ux_1^2, v_2x_2^2 \in E(G)$. Let G' be the graph obtained from G by removing vertices x_1^1, x_2^1, x_3^1 and adding the edge ux_4^1. Clearly, $G' \in \mathcal{F}$, and by the induction hypothesis, there exists an I2-RDF f of G' of weight at most $\frac{4(n-3)}{5}$ assigning a non-empty set to every vertex of degree at least 3. Likewise to situation (I), one can see that $f(u) = f(v_2) = \{1, 2\}$. Now define $g : V(G) \to \mathcal{P}(\{1, 2\})$ by $g(x_3^1) = \{1, 2\}$, $g(x_1^1) = g(x_2^1) = \emptyset$, and $g(x) = f(x)$ otherwise. Clearly, g is an I2-RDF of G that assigns a non-empty set to every vertex of A and has weight $\omega(g) = \omega(f) + 3 \leq \frac{4(n-3)}{5} + 2 < \frac{4n}{5}$.

(III) u is adjacent to an A-ear path in Q_3.
Let $P_2 = x_1^2x_2^2x_3^2$ be an A-ear path in Q_3 such that ux_1^2, $v_2x_3^2 \in E(G)$. Let G' be the graph obtained from G by removing $x_1^1, x_2^1, x_3^1, x_1^2$ and adding edges ux_4^1, ux_2^2. Since $G' \in \mathcal{F}$, by the induction hypothesis there is an I2-RDF f of G' of weight at most $\frac{4(n-4)}{5}$ that assigns non-empty sets to every vertex of degree at least 3. As above, $f(u) = f(v_2) = \{1, 2\}$. Define $g : V(G) \to \mathcal{P}(\{1, 2\})$ by $g(x_1^2) = \{1\}$, $g(x_3^1) = \{1, 2\}$, $g(x_1^1) = g(x_2^1) = g(x_1^2) = \emptyset$, and $g(x) = f(x)$ otherwise. Clearly, g is an

I2-RDF of G of weight $\omega(g) = \omega(f) + 3 \leq \frac{4(n-4)}{5} + 3 < \frac{4n}{5}$ having the desired property.

(IV) u is adjacent to two A-ear paths in Q_1.
Let $P_2 = x_1^2$ and $P_3 = x_1^3$ be A-ear paths in $\mathcal{Q}_1$ such that $ux_1^2, ux_1^3, v_2x_1^2, v_3x_1^3 \in E(G)$, where $v_2, v_3 \in A$.

- $v_1 \notin \{v_2, v_3\}$.
Let G' be the graph obtained from G by removing $x_1^1, x_2^1, x_3^1, x_4^1, u$ and adding the edges $v_1x_1^2$ and $v_1x_1^3$. Since $G' \in \mathcal{F}$, by the induction hypothesis there is an I2-RDF f of G' of weight at most $\frac{4(n-5)}{5}$ assigning non-empty sets to every vertex of degree at least 3. Since $|f(v_2)|, |f(v_3)| \geq 1$, we must have $f(x_1^2) = f(x_1^3) = \emptyset$. Now the function $g : V(G) \to \mathcal{P}(\{1,2\})$ defined by $g(u) = g(x_3^1) = \{1,2\}$, $g(x_1^1) = g(x_2^1) = g(x_4^1) = \emptyset$, and $g(x) = f(x)$ otherwise, is an I2-IRDF of G with the desired property and weight $\omega(g) = \omega(f) + 4 \leq \frac{4(n-5)}{5} + 4 \leq \frac{4n}{5}$.

- $v_1 \in \{v_2, v_3\}$.
Without loss of generality, assume that $v_1 = v_2$. Suppose first that $v_1 \neq v_3$ and let G' be the graph obtained from G by removing x_1^1, x_2^1, x_1^3, u and adding the edges $v_3x_1^2$, $v_3x_3^1$. Clearly, $G' \in \mathcal{F}$ and, thus, by the induction hypothesis, there is an I2-RDF f of G' of weight at most $\omega(f) \leq \frac{4(n-4)}{5}$ that assigns a non-empty set to every vertex of degree at least 3. As above, one can easily see that $f(v_1) = f(v_3) = \{1,2\}$ and, thus, $f(x_3^1) = f(x_4^1) = f(x_1^2) = \emptyset$. Now define the function $g : V(G) \to \mathcal{P}(\{1,2\})$ by $g(x_2^1) = \{1,2\}$, $g(x_1^1) = g(x_1^3) = \emptyset$, $g(u) = \{1\}$, and $g(x) = f(x)$ otherwise. Clearly, g is an I2-RDF of G with the desired property and weight $\omega(g) = \omega(f) + 3 \leq \frac{4(n-4)}{5} + 3 < \frac{4n}{5}$.

  Now assume that $v_1 = v_2 = v_3$. Since $|A| \geq 3$ and G is connected, we have $\deg(v_1) \geq 4$. Let $w \in A - \{u, v_1\}$ and let G' be the graph obtained from G by removing x_1^1, x_2^1, u, x_1^3 and adding the edges wx_3^1, wx_1^2. Since $G' \in \mathcal{F}$, by the induction hypothesis, there is an I2-RDF f of G' of weight at most $\omega(f) \leq \frac{4(n-4)}{5}$ such that f assigns a non-empty set to every vertex of degree at least

3. As above, we must have $f(v_1) = \{1,2\}$. Now the function $g : V(G) \to \mathcal{P}(\{1,2\})$ defined by $f(x_2^1) = \{1,2\}$, $g(u) = \{1\}$, $g(x_1^1) = g(x_1^3) = \emptyset$, and $g(x) = f(x)$ otherwise, is an I2-RDF of G such that g assigns a non-empty set to every vertex in A and $\omega(g) = \omega(f) + 3 \leq \frac{4(n-4)}{5} + 3 < \frac{4n}{5}$.

Seeing Case 1, we can assume from now on that $Q_4 = \emptyset$.

Case 2. $Q_3 \neq \emptyset$.

Let $P = x_1^1 x_2^1 x_3^1 \in Q_3$ and let $ux_1^1, x_3^1 v_1 \in E(G)$, where $u, v_1 \in A$. By assumption, we may assume, without loss generality, that $\deg(u) = 3$. Consider the following situations.

(V) u is adjacent to three A-ear-paths in Q_3.

Let $P_2 = x_1^2 x_2^2 x_3^2$ and $P_3 = x_1^3 x_2^3 x_3^3$ be two A-ear-paths in $Q_3 - \{P\}$ such that ux_1^3, ux_1^2, $v_2 x_3^2, v_3 x_3^3 \in E(G)$. Let G' be the graph obtained from G by removing $x_1^1, x_2^1, x_1^2, x_2^2, x_1^3, x_2^3$ and by adding edges ux_3^1, ux_3^2, ux_3^3. Then $G' \in \mathcal{F}$, and by the induction hypothesis, there is an I2-RDF f of G' of weight at most $\frac{4(n-6)}{5}$ that assigns non-empty sets to vertices of degree at least 3. Without loss of generality, assume that $1 \in f(u)$. Define $g : V(G) \to \mathcal{P}(\{1,2\})$ by $g(u) = \{1,2\}$, $g(x_1^i) = \emptyset$ for $i \in \{1,2,3\}$, $g(x_2^i) = \{1\}$ if $f(v_i) = \{1,2\}$, $g(x_2^i) = \{1,2\} - f(v_i)$ if $|f(v_i)| = 1$ for $i \in \{1,2,3\}$, and $g(x) = f(x)$ otherwise. Clearly, g is an I2-RDF of G with the desired property and weight $\omega(g) = \omega(f) + 4 \leq \frac{4(n-6)}{5} + 4 < \frac{4n}{5}$.

(VI) u is adjacent to an A-ear path in Q_2.

Let $P_2 = x_1^2 x_2^2 \in Q_2$ such that $ux_1^2, v_2 x_1^2 \in E(G)$. Let G' be the graph obtained from G by removing x_1^1, x_2^1 and adding the edge ux_3^1. Then $G' \in \mathcal{F}$ and by the induction hypothesis, there is an I2-RDF f of G' of weight at most $\frac{4(n-2)}{5}$ such that $f(x) \neq \emptyset$ for every $x \in A$. Since $f(x_1^2) = f(x_2^2) = \emptyset$, we deduce that $f(u) = \{1,2\}$. Define now the function $g : V(G) \to \mathcal{P}(\{1,2\})$ by $g(x_1^1) = \emptyset$, $g(x_2^1) = \{1\}$ if $f(v_1) = \{1,2\}$ or $g(x_2^1) = \{1,2\} - f(v_1)$ if $|f(v_1)| = 1$, and $g(x) = f(x)$ otherwise. Clearly, g is an I2-RDF of G of weight $\omega(g) = \omega(f) + 1 \leq \frac{4(n-2)}{5} + 1 < \frac{2n}{3}$ and such that $g(x) \neq \emptyset$ for every $x \in A$.

(VII) u is adjacent to two A-ear paths in Q_3 and to an A-ear path in Q_1**.**
Let $P_2 = x_1^2x_2^2x_3^2 \in Q_3$ and $P_3 = x_1^3 \in Q_1$ such that $ux_1^3, ux_1^2, v_2x_3^2, v_3x_1^3 \in E(G)$. Let G' be the graph obtained from G by removing $x_1^1, x_2^1, x_1^2, x_2^2$ and adding edges ux_3^1, ux_3^2. Clearly, $G' \in \mathcal{F}$ and by the induction hypothesis, there is an I2-RDF f of G' of weight at most $\frac{4(n-4)}{5}$ such that $f(x) \neq \emptyset$ for every $x \in A$. Define the function $g : V(G) \to \mathcal{P}(\{1,2\})$ by $g(u) = \{1,2\}$, $g(x_1^i) = \emptyset$ for $i \in \{1,2\}$, $g(x_2^i) = \{1\}$ if $f(v_i) = \{1,2\}$, $g(x_2^i) = \{1,2\} - f(v_i)$ if $|f(v_i)| = 1$, and $g(x) = f(x)$ otherwise. Note that $|g(u)| - |f(u)| \leq 1$. Clearly, g is an I2-RDF of G of weight $\omega(g) = \omega(f) + 3 \leq \frac{4(n-4)}{5} + 3 < \frac{4n}{5}$ such that $g(x) \neq \emptyset$ for every $x \in A$.

(VIII) u is adjacent to an A-ear-path in Q_3 and to two A-ear paths in Q_1.
Let $P_2 = x_1^2$ and $P_3 = x_1^3$ be A-ear paths in Q_1 such that $ux_1^2, ux_1^3 \in E(G)$. Suppose $v_2x_1^2, v_3x_1^3 \in E(G)$, where $v_2, v_3 \in A$.

- $v_1 \notin \{v_2, v_3\}$.
  Let G' be the graph obtained from G by removing x_1^1, x_2^1, x_3^1, u and adding the edges $v_1x_1^2$ and $v_2x_1^3$. Clearly, $G' \in \mathcal{F}$ and by the induction hypothesis, there is an I2-RDF f of G' of weight at most $\omega(f) \leq \frac{4(n-4)}{5}$ such that $f(x) \neq \emptyset$ for every $x \in A - \{u\}$. Without loss of generality, we assume that $1 \in f(v_1)$ and define the function g on $V(G)$ by $g(u) = \{1,2\}$, $g(x_1^1) = g(x_3^1) = \emptyset$, $g(x_2^1) = \{2\}$, and $g(x) = f(x)$ otherwise. Clearly, g is an I2-RDF of G of weight $\omega(g) = \omega(f) + 3 \leq \frac{4(n-4)}{5} + 3 < \frac{2n}{3}$. In addition, $g(x) \neq \emptyset$ for every $x \in A$.

- $v_1 \in \{v_2, v_3\}$ and $v_1 = v_2 = v_3$.
  Since $|A| \geq 3$, we have $\deg(v_1) \geq 4$. First, let there exist a path v_3zv_4 in G, where $v_4 \in A$ and $z \notin \{x_1^3, x_1^2\}$. Since $\deg(v_4) \geq 3$, we deduce that $A - \{u, v_3, v_4\} \neq \emptyset$. Let $w \in A - \{u, v_3, v_4\}$. Assume that $\deg(v_1) \geq 5$, and let G' be the graph resulting from the deletion of vertices u, x_1^1, x_1^2, x_1^3 and the addition of the edge wx_2^1. Then $G' \in \mathcal{F}$ and by the induction hypothesis, there is an

I2-RDF f of G' of weight at most $\frac{4(n-4)}{5}$ such that $f(x) \neq \emptyset$ for every $x \in A - \{u\}$. We also have $f(v_3) = f(w) = \{1, 2\}$. But then the function g defined on $V(G)$ by $f(u) = \{1, 2\}$, $f(x_2^1) = \{1\}$, $f(x_1^2) = f(x_1^3) = \emptyset$, and $g(x) = f(x)$ otherwise is an I2-RDF of G of weight $\omega(f) + 3 \leq \frac{4(n-4)}{5} + 3 < \frac{2n}{3}$ and such that $g(x) \neq \emptyset$ for every $x \in A$. Now assume that $\deg(v_1) = 4$, and let G' be the graph obtained from G by removing $u, x_1^1, x_2^1, x_3^1, x_1^2, x_1^3$ and by adding the edge wv_3. Note that v_3 has degree two in G' and, thus, belongs to an A'-ear path joining v_4 and w. Since $G' \in \mathcal{F}$, by the induction hypothesis, there is an I2-RDF f of G' of weight at most $\frac{4(n-6)}{5}$ such that $f(x) \neq \emptyset$ for every $x \in A - \{u, v_3\}$. Clearly, $f(v_4) = f(w) = \{1, 2\}$ and $f(z) = f(v_3) = \emptyset$. Now f can be extended to an 2-IRDF of G with the desired property by assigning $\{1\}$ to u, $\{2\}$ to v_3, $\{1, 2\}$ to x_2^1 and $\emptyset$ to $x_1^1, x_3^1, x_1^2, x_1^3$ and, thus, $\omega(g) = \omega(f) + 4 < \frac{4n}{5}$.

Now let there exist a path v_3zyv_4 in G, where $v_4 \in A$ and $z \notin \{x_3^1\}$. As above, we have $A - \{u, v_3, v_4\} \neq \emptyset$, and so pick a vertex $w \in A - \{u, v_3, v_4\}$. If $\deg(v_1) \geq 5$, then the result follows, as above, by considering the same graph G' obtained from the removal of vertices u, x_1^1, x_1^2, x_1^3 and the addition of the edge wx_2^1. Hence, we assume that $\deg(v_1) = 4$. Then delete vertices $u, x_1^1, x_2^1, x_3^1, v_3, x_1^2, x_1^3$ and add the edge zw, and let G' be the resulting graph. Clearly, $G' \in \mathcal{F}$ and by the induction hypothesis, there is an I2-RDF f of G' of weight at most $\frac{4(n-7)}{5}$ such that $f(x) \neq \emptyset$ for every $x \in A - \{u, v_3\}$. Since $f(v_4) = f(w) = \{1, 2\}$ and $f(z) = f(y) = \emptyset$, function f can be extended to an I2-RDF of G by assigning $\{1, 2\}$ to u, v_3, $\{1\}$ to x_2^1 and $\emptyset$ to other vertices. It follows that $\omega(g) = \omega(f) + 5 < \frac{4n}{5}$ and $g(x) \neq \emptyset$ for every $x \in A$.

- $v_1 \in \{v_2, v_3\}$ and $|\{v_1, v_2, v_3\}| = 2$.
Suppose, without loss of generality, that $v_1 = v_2$ and $v_1 \neq v_3$. Let G' be the graph obtained from G by removing x_1^1, x_3^1, u and by adding the edges $v_3x_1^2, v_3x_2^1$. Then $G' \in \mathcal{F}$ and by the induction hypothesis, there is an I2-RDF f of G' of weight at most $\frac{4(n-3)}{5}$ such that $g(x) \neq \emptyset$ for every $x \in A - \{u\}$. As above, we

must have $f(v_1) = f(v_3) = \{1, 2\}$ and so $f(x_2^1) = f(x_3^1) = \emptyset$. In this case, the function g defined by $g(u) = \{1\}$, $= g(x_2^1) = \{2\}$, $g(x_1^1) = g(x_1^3) = \emptyset$, and $g(x) = f(x)$ otherwise, is an I2-RDF of G of weight $\omega(g) = \omega(f) + 2 \le \frac{4(n-3)}{5} + 2 < \frac{4n}{5}$. Moreover, $g(x) \neq \emptyset$ for every $x \in A - \{u\}$.

Considering Case 2, we may assume that $Q_3 = \emptyset$.

Case 3. $Q_2 \neq \emptyset$.
Let $P_1 = x_1^1 x_2^1 \in Q_2$ with $ux_1^1, x_2^1 v_1 \in E(G)$. Without loss of generality, we assume that $\deg(u) \le \deg(v_1)$. Consider the following situations.

(IX) $\deg(u) \ge 4$ and u is adjacent to at least two A-ear paths in Q_2.
By assumption we have $\deg(v_1) \ge 4$. Let $P_2 = x_1^2 x_2^2$ be a second A-ear path in Q_2 such that $ux_1^2 \in E(G)$ and let $v_2 x_2^2 \in E(G)$ for some $v_2 \in A$. Remove vertices x_1^1, x_2^1 and let G' be the resulting graph. Then $G' \in \mathcal{F}$ and by the induction hypothesis, there is an I2-RD-function f of G' of weight at most $\frac{4(n-2)}{5}$ such that $f(x) \neq \emptyset$ for every $x \in A$. Clearly, $f(u) = \{1, 2\}$ and $|f(v_1)| \ge 1$. Define the function g by $g(x_1^1) = g(x_2^1) = \emptyset$, $g(v_1) = \{1, 2\}$, and $g(x) = f(x)$ otherwise. Then g is an 2-IRDF of G of weight $\omega(f) + 1 \le \frac{4(n-2)}{5} + 1 < \frac{4n}{5}$ having the property that $g(x) \neq \emptyset$ for every $x \in A$.

(X) $\deg(u) = 3$ and u is adjacent to at least two A-ear paths in Q_2.
Let $P_2 = x_1^2 x_2^2 \in Q_2$ be an A-ear path in G such that $ux_1^2 \in E(G)$ and $v_2 x_2^2 \in E(G)$, where $v_2 \in A$.

- $v_1 = v_2$.
  Let G' be the graph obtained from G by removing x_1^1 and adding the edge ux_2^1. Then $G' \in \mathcal{F}$ and by the induction hypothesis, there is an I2-RD-function f of G' of weight $\frac{4(n-1)}{5}$ with the desired property, in particular $f(u) = f(v_1) = \{1, 2\}$. In this case, f can be extended to an I2-IRDF of G by assigning $\emptyset$ to x_1^1 such that f satisfies the conditions.

- $v_1 \neq v_2$.
  Since $\deg(u) = 3$, let uPv_3 be a path in G such that $P \in (Q_1 \cup Q_2) - \{P_1, P_2\}$. Let $z \in V(P)$ be the vertex adjacent to u. We may assume that $v_1 \notin \{v_2, v_3\}$. Let G' be the resulting graph after removing vertices x_1^1, x_2^1, u and adding edges $v_1x_1^2$ and v_1z. Then $G' \in \mathcal{F}$ and by the induction hypothesis, there exists an I2-RDF f of G' satisfying our conditions. Since $f(v_1) = \{1, 2\}$, we can define the function g on $V(G)$ by $g(u) = \{1, 2\}$, $g(x_1^1) = g(x_1^2) = \emptyset$, and $g(x) = f(x)$ otherwise. Then g is an I2-RDF of G of weight $\omega(f) + 2 \leq \frac{4(n-3)}{5} + 2 < \frac{4n}{5}$ such that $g(x) \neq \emptyset$ for every $x \in A$.

(XI) The other neighbors of u belong to ear-paths in Q_1.
Considering the above cases and subcases, we may assume that $\mathcal{Q} = Q_1 \cup Q_2$ and that each vertex in A is adjacent to at most one A-ear path in Q_2. In that case, since $G \in \mathcal{F}$, it is obtained from connected multigraph H without loops with $\delta(H) \geq 3$ by subdividing any edge at most twice so that the set of edges of H subdivided twice is independent (in H). Hence, let $u_1v_1, \ldots, u_kv_k$ be the edges of H subdivided twice and let A'' be the set of all vertices in H for which all edges that are incident are subdivided once. Therefore, we have $|V(H)| = 2k + |A''|$ and $|E(H)| = \frac{1}{2}\sum_{v \in V(H)} \deg(v) \geq \frac{3}{2}|V(H)| = 3k + \frac{3}{2}|A''|$ (because $\delta(H) \geq 3$, k edges of H are subdivided twice and the remaining edges are subdivided once). Hence, the order of G is

$$n = |V(H)| + |E(H)| + k \geq 6k + \frac{5}{2}|A''|.$$

It is easy to see that the function g defined on $V(G)$ by $g(x) = \{1, 2\}$ for $x \in V(H)$, and $g(x) = \emptyset$ otherwise, is an I2-RDF of G that assigns non-empty sets to vertices of A and $\omega(g) = 2|V(H)| = 4k + 2|A''| < \frac{4(6k + \frac{5}{2}|A''|)}{5} \leq \frac{4n}{5}$. This completes the proof. □

Now, we can proceed to the proof of Theorem 1.

Proof of Theorem 1. We use an induction on the order n. If $n \leq 5$, then, clearly, G is connected having at most two vertices of degree at least three. More precisely, $G \in \{C_3, C_4, C_5\}$ or G is either obtained from two cycles C_3 sharing the same vertex or G is the complete bipartite graph $K_{2,3}$. In this case, it can be easily checked that $i_{r2}(C_n) \leq \frac{4}{5}n$, establishing the base case. Let $n \geq 6$, and assume that the result holds for all graphs G' of order less than n with minimum degree at least two such that the set of vertices with degree at least three is independent. Let G be a graph of order n such that $\delta(G) \geq 2$ and the set of vertices with degree at least three is independent. We can assume that G is connected for otherwise the result follows by applying the induction hypothesis on each component of G.

If $\Delta(G) = 2$, then $G = C_n$. Since for the cycle C_n, $i(C_n) = \lceil n/3 \rceil$, we obtain that $i_{r2}(C_n) \leq 2\lceil n/3 \rceil$ and, clearly, $2\lceil n/3 \rceil \leq \frac{4}{5}n$ for all $n \geq 8$. Since $i_{r2}(C_7) = 5 < \frac{4}{5}n$, we deduce that $i_{r2}(C_n) \leq \frac{4}{5}n$. Hence, assume that $\Delta(G) \geq 3$, and let $A = \{v \in V(G) \mid \deg(v) \geq 3\}$ and $B = V(G) - A$. Consider the A-ear paths and keep the same notations as defined in the proof of Proposition 3. Note that $A = \bigcup_{P \in \mathcal{Q}} X_P$, $V(G) = A \cup \bigcup_{P \in \mathcal{Q}} V(P)$ and $1 \leq |X_P| \leq 2$ for each $P \in Q$. Assume first that there exists an A-ear path P such that $\delta(G - V(P)) = 1$. Since G is simple, this means that $|V(P)| \geq 2$ and some vertex of G of degree three is adjacent to the end-vertices of P. Thus, $|X_P| = 1$. In that case, let $X_P = \{a\}$ and $N_G(a) - V(P) = \{b\}$. Clearly, $b \in B$ (since A is independent) and, thus, there is a unique A-ear path P' in which b is an end-vertex of P'. Let c be the other end-vertex of P' (possibly $b = c$). Let G' be the graph resulting from the deletion of vertex a and all vertices of P and P'. Then $\delta(G') \geq 2$ and by the induction hypothesis, $i_{r2}(G') \leq \frac{4|V(G')|}{5}$. On the other hand, since $G'' = G[V(P) \cup V(P') \cup \{a\}]$ is isomorphic to $C_{|V(P)|+1, |V(P')|}$, by Proposition 2, G'' has an I2-RDF g such that $\omega(g) \leq \frac{4n(G'')}{5}$ and $g(c) = \emptyset$. Now, for any $i_{r2}(G')-$function, the function h defined on $V(G)$ by $h(x) = f(x)$ for all $x \in V(G')$ and $h(x) = g(x)$ for all $x \in V(G'')$ is an I2-RDF of G. Therefore,

$$
\begin{aligned}
i_{r2}(G) &\leq i_{r2}(G') + i_{r2}(G'') \\
&\leq \frac{4|V(G')|}{5} + \frac{4|V(P) \cup V(P') \cup \{a\}|}{5} = \frac{4n}{5}.
\end{aligned}
$$

In the next, we can assume that $\delta(G - V(P)) \geq 2$ for each A-ear path $P \in \mathcal{Q}$. It follows that $|X_P| = 2$ for each A-ear path $P \in \mathcal{Q}$. Assume that $\mathcal{Q}-(Q_1 \cup Q_2 \cup Q_3 \cup Q_4) \neq \emptyset$, and let $P \in \mathcal{Q}-(Q_1 \cup Q_2 \cup Q_3 \cup Q_4)$. By Proposition 1, P has an I2-RDF g such that $\omega(g) \leq \frac{4|V(P)|}{5}$ and g assigns $\emptyset$ to the end-vertices of the path P. Now, let G' be the graph obtained from G' by removing all vertices of P. By the induction hypothesis, we have $i_{r2}(G') \leq \frac{4|V(G')|}{5}$. Clearly, for every $i_{r2}(G')$-function f, the function h defined on $V(G)$ by $h(x) = f(x)$ for all $x \in V(G')$ and $h(x) = g(x)$ for all $x \in V(P)$, is an I2-RDF of G and, thus, $i_{r2}(G) \leq i_{r2}(G') + i_{r2}(P) \leq \frac{4}{5}n$. Assume now that $\mathcal{Q} = Q_1 \cup Q_2 \cup Q_3 \cup Q_4$. Then $G \in \mathcal{F}$ and the result follows from Proposition 3. $\square$

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