Computer Science Journal of Moldova, vol.**28**, no.2(83), 2020

On α-spectral theory of a directed k-uniform hypergraph

Gholam-Hasan Shirdel, Ameneh Mortezaee,
Effat Golpar-Raboky

Abstract

In this paper, we study a k-uniform directed hypergraph in general form and introduce its adjacency tensor, Laplacian tensor and signless Laplacian tensor. For the k-uniform directed hypergraph $\mathcal{H}$ and $0 \leq \alpha < 1$ the convex linear combination of $\mathcal{D}$ and $\mathcal{A}$ has been defined as $\mathcal{A}_\alpha = \alpha\mathcal{D} + (1-\alpha)\mathcal{A}$, where $\mathcal{D}$ and $\mathcal{A}$ are the degree tensor and the adjacency tensor of $\mathcal{H}$, respectively. We propose some spectral properties of $\mathcal{A}_\alpha$. We also introduce power directed hypergraph and cored directed hypergraph and investigate their α-spectral properties.

Keywords: Directed Hypergraph, Adjacency tensor, Laplacian tensor, Signless Laplacian tensor, Eigenvalue, α-spectral theory, Odd-bipartite Hypergraph.

MSC 2010: 05C65, 15A18.

1 Introduction

Directed hypergraphs are deeply used as a successful data structure in modeling the problems arising in computer science [3] and operations research, and in recent years have found applications in data mining, clustering, association rules [13], image processing [1] and optical network communications [8]. On the other hand, spectral theory of hypergraphs gives useful and important information about them. In 2005 eigenvalues and eigenvectors of real tensor are defined [9], [14]. Qi [14] introduced the spectral theory of supersymmetric real tensor. In [15] the spectral theory of undirected hypergraphs was presented via eigenvalues and eigenvectors of the adjacency tensor, Laplacian tensor and signless Laplacian tensor. Recently a number of papers appeared in different aspects of spectral theory of hypergraphs.

On the other hand, Nikiforov in [11] proposed the spectral theory of the convex combination of the adjacency matrix and the degree matrix of a graph (see also [4],[12]) and then Lin et.al. [10] expanded it for the hypergraph. Let $\mathcal{H}$ be a hypergraph, $\mathcal{A}(\mathcal{H})$ and $\mathcal{D}(\mathcal{H})$ be the adjacency tensor and the degree tensor of $\mathcal{H}$, respectively. For $0 \leq \alpha < 1$, the convex linear combination, $\mathcal{A}_\alpha$, of $\mathcal{D}$ and $\mathcal{A}$ is defined by

$$\mathcal{A}_\alpha(\mathcal{H}) = \alpha\mathcal{D}(\mathcal{H}) + (1-\alpha)\mathcal{A}(\mathcal{H}) \qquad \forall \ 0 \leq \alpha < 1.$$

The spectral radius of $\mathcal{A}_\alpha(\mathcal{H})$ is called α-spectral radius of $\mathcal{H}$. Lin et.al. [10] gave an upper bound for the α-spectral radius in an n-vertex connected irregular k-uniform hypergraph $\mathcal{H}$ using number of vertices, maximum degree and diameter. Then Gue et.al. [5] studied α-spectral radius of uniform hypergraphs and also proposed some transformations that increase α-spectral radius and determine the unique hypergraphs with maximum α-spectral radius in some classes of uniform hypergraphs.

In spite of a lot of researches in spectral theory and α-spectral theory of undirected hypergraphs, there is almost a blank for (α-)spectral directed hypergraph theory. A special case of the k-uniform directed hypergraph, with one tail node and $k-1$ head nodes, and some its spectral properties were studied in [17]. In this paper, we present the α-spectral properties of the generalized directed hypergraphs and extend some classical results of undirected hypergraphs. We also introduce power directed hypergraphs and cored directed hypergraphs and propose some their α-spectral properties.

In Section 2, we discuss the needed fundamental results of tensors and introduce k-uniform directed hypergraphs in general form with their adjacency tensors, Laplacian tensors and signless Laplacian tensors. In Section 3, the H-eigenvalues of $\mathcal{A}_\alpha$ of k-uniform directed hypergraph are studied. We also introduce power directed hypergraphs and cored directed hypergraphs in Section 4. Finally, we conclude in Section 5.

2 Preliminaries

We first present some basic definitions of tensors. Then we introduce the general k-uniform directed hypergraph with its adjacency tensor, Laplacian tensor and signless Laplacian tensor.

2.1 Tensors and some related subjects

A real tensor $\mathcal{T} = (t_{i_1 \cdots i_k})$ of order k and dimension n, for integers k≥3 and n≥2, is a multi-dimensional array with entries $t_{i_1 \cdots i_k} \in \mathbb{R}$, for $i_j \in [n] := \{1, 2, \cdots, n\}$ and $j \in [k]$ (see [14]).

Definition 1. *[16]: Let $\mathcal{T}$ be a k order n dimension tensor and P and Q be $n \times n$ matrices. The tensor $\mathcal{S} = P\mathcal{T}Q^{k-1}$ is a k order n dimension tensor with the entries*

$$s_{i_1 \cdots i_k} = \sum_{j_1, \cdots, j_k = 1}^{n} t_{j_1 \cdots j_k} p_{i_1 j_1} q_{j_2 i_2} \cdots q_{j_k i_k}.$$

Let $x = (x_1, \cdots, x_n)^T \in \mathbb{C}^n$, we write x^k as a k order n dimension tensor with $(i_1, \cdots, i_k)$-th entry $x_{i_1} x_{i_2} \cdots x_{i_k}$. Then $\mathcal{T}x^{k-1}$ is an n dimensional vector whose i-th component is

$$(\mathcal{T}x^{k-1})_i = \sum_{i_2, \cdots, i_k = 1}^{n} t_{i i_2 \cdots i_k} x_{i_2} \cdots x_{i_k}.$$

The identity tensor of order k and dimension n, $\mathcal{I} = (i_{i_1 \cdots i_k})$, is defined as $i_{i_1 \cdots i_k} = 1$ iff $i_1 = \cdots = i_k \in [n]$, and zero otherwise.

Definition 2. *[14]: Let $\mathcal{T}$ be a nonzero k order n dimension tensor. Then $\lambda \in \mathbb{C}$ is called an eigenvalue of $\mathcal{T}$ if the polynomial system $(\lambda\mathcal{I} - \mathcal{T})x^{[k-1]} = 0$ has a nonzero solution $x \in \mathbb{C}^n$, where $x^{[k-1]} = (x_1^{k-1}, \cdots, x_n^{k-1})^T$. In this case x is called an eigenvector of $\mathcal{T}$ corresponding to λ and (λ, x) is called an eigenpair of $\mathcal{T}$.*

If $(\lambda, x) \in \mathbb{R} \times \mathbb{R}^n/\{0\}$, then λ is called an H-eigenvalue and x is called an H-eigenvector of $\mathcal{T}$ [14].

The set of all eigenvalues of $\mathcal{T}$, denoted by $Spec(\mathcal{T})$, is called the spectrum of $\mathcal{T}$. The H-spectrum of $\mathcal{T}$, denoted by $Hspec(\mathcal{T})$, is defined as follows:

$$Hspec(\mathcal{T}) = \{\lambda \in \mathbb{R} | \lambda \text{ is an H-eigenvalue of } \mathcal{T}\}.$$

2.2 K-uniform directed hypergraph

In this subsection we present some needed concepts and definitions of directed hypergraphs and then we introduce adjacency tensor of a k-uniform directed hypergraph in general form. The following definition of the k-uniform directed hypergraph was presented in [1].

Definition 3. *A k-uniform directed hypergraph $\mathcal{H}$ is a pair $= (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = [n]$ is a set of elements called vertices and $\mathcal{E} = \{\vec{e}_1, \cdots, \vec{e}_m\}$ is the set of arcs. Each $\vec{e}_i$, $(i = 1, \cdots, m)$, is considered as an ordered pair (e_i^+, e_i^-), where e_i^+, e_i^- are two nonempty subsets of $\mathcal{V}$ such that $e_i^+ \cap e_i^- = \phi$, $|e_i^+ \cup e_i^-| = k$. e_i^+ is called the tail of $\vec{e}_i$ and e_i^- is its head.*

Note that we assume that in the k-uniform directed hypergraph for any k vertices there exists at most one arc containing them.
The out-degree of a vertex $j \in \mathcal{V}$ is defined as $d_j^+ = |E_j^+|$, where $E_j^+ = \{\vec{e} \in \mathcal{E} | j \in e^+\}$ and the in-degree of a vertex $j \in \mathcal{V}$ is defined as $d_j^- = |E_j^-|$, where $E_j^- = \{\vec{e} \in \mathcal{E} | j \in e^-\}$. The degree of j is defined as $d_j = d_j^+ + d_j^-$. The hypergraph $\mathcal{H}$ is r-out-regular (or r-in-regular or r-regular, respectively) if for each $j \in \mathcal{V}$, $d_j^+ = r$ (or $d_j^- = r$ or $d_j = r$, respectively).

Let $i, j \in \mathcal{V}$ and $i \neq j$. Two vertices i and j are called weak-connected, if there is a sequence of arcs $\vec{e}_1, \cdots, \vec{e}_l$ such that $i \in e_1^+ \cup e_1^-$, $j \in e_l^+ \cup e_l^-$ and $(e_s^+ \cup e_s^-) \cap (e_{s+1}^+ \cup e_{s+1}^-) \neq \phi$ for all $s \in [l-1]$. Two vertices i and j are called strong-connected, denoted by $i \to j$, if there is a sequence of arcs $\vec{e}_1, \cdots, \vec{e}_l$ such that $i \in e_1^+$, $j \in e_l^-$ and $e_s^- \cap e_{s+1}^+ \neq \phi$ for all $s \in [l-1]$. A directed hypergraph $\mathcal{H}$ is called weak-connected, if every pair of different vertices of $\mathcal{H}$ is weakly-connected and $\mathcal{H}$ is called strong-connected, if $i \to j$ and $j \to i$ for all $i, j \in \mathcal{V}$ and $i \neq j$. A directed hypergraph is complete if $\mathcal{E}$ contains all possible arcs with different number of vertices in their tails.

Now we introduce adjacency tensor of a k-uniform directed hypergraph. In [17], authors discussed the case that each arc has only one tail and introduce the adjacency tensor, Laplacian tensor and signless Laplacian tensor. In this paper we consider general form of a k-uniform directed hypergraph and present the following definition of its adjacency tensor:

Definition 4. *The adjacency tensor of a k-uniform directed hypergraph $\mathcal{H}$ is the k order n dimension tensor $\mathcal{A} = (a_{i_1 \cdots i_k})$ whose entries are as follows:*

$$a_{i_1,\ldots,i_k} = \begin{cases} \frac{1}{(l_{\vec{e}}-1)!(k-l_{\vec{e}})!} & \textit{if} \, \exists \, \vec{e} = (e^+, e^-) \in \mathcal{E} \;\; s.t \;\; e^+ = \{i_1, \cdots, i_{l_{\vec{e}}}\}, \\ & \qquad\qquad\qquad\qquad\qquad e^- = \{i_{l_{\vec{e}}+1}, \cdots, i_k\} \\ 0 & \textit{otherwise.} \end{cases}$$

Similar to [17] the degree tensor $\mathcal{D}$ defined as the k order n dimension diagonal tensor whose diagonal element $d_{i \cdots i}$ is d_i^+, the out-degree of vertex i, for all $i \in [n]$. Also the Laplacian tensor of $\mathcal{H}$ is $\mathcal{L} = \mathcal{D} - \mathcal{A}$ and $\mathcal{Q} = \mathcal{D} + \mathcal{A}$ is the signless Laplacian tensor of $\mathcal{H}$.

As it has been said before, Lin in [3] defined the convex linear combinations of $\mathcal{A}_\alpha$ of $\mathcal{D}$ and $\mathcal{A}$ as $\mathcal{A}_\alpha = \alpha\mathcal{D} + (1-\alpha)\mathcal{A}$, where $0 \leq \alpha < 1$.

Now the following definition of an odd bipartite directed hypergraph is presented (just as in undirected hypergraph [6]).

Definition 5. *Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a k-uniform directed hypergraph. $\mathcal{H}$ is called an odd bipartite if k is even and there exists a partition of $\mathcal{V}$ so that $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$, $\mathcal{V}_1 \neq \phi$ and*

$$\forall \, \vec{e} = (e^+, e^-) \in \mathcal{E} \qquad |(e^+ \cup e^-) \cap \mathcal{V}_1| \quad \textit{is an odd number.}$$

3 H-Eigenvalues of $\mathcal{A}_\alpha$

Throughout this article, let $\mathcal{H} = (V, E)$ be a k-uniform hypergraph with n vertices, tensors $\mathcal{A}$,$\mathcal{D}$, $\mathcal{L} = \mathcal{D} - \mathcal{A}$ and $\mathcal{Q} = \mathcal{D} + \mathcal{A}$ are the adjacency tensor, the degree tensor, Laplacian tensor and signless Laplacian tensor of $\mathcal{H}$, respectively. For $0 \leq \alpha \leq 1$, let $\mathcal{A}_\alpha$ be defined as $\mathcal{A}_\alpha = \alpha\mathcal{D} + (1-\alpha)\mathcal{A}$.

The notation of weakly irreducible nonnegative tensors was introduced in [2].

Definition 6. *Let $\mathcal{T} = (t_{i_1 \cdots i_k})$ be a k order n dimension nonnegative tensor and $G(\mathcal{T}) = (V, E(\mathcal{T}))$ be a directed graph, where $V = [n]$ and a directed edge $(i, j) \in E(\mathcal{T})$ if there exists $\{i_2, \cdots, i_k\} \in [n]$ such that $j \in \{i_2, \cdots, i_k\}$ and $t_{ii_2 \ldots i_k} > 0$. Now $\mathcal{T}$ is called weakly irreducible if $G(\mathcal{T})$ is strongly connected.*

Let $\mathcal{H}$ be a k-uniform undirected hypergraph, then the adjacency of $\mathcal{H}$, $\mathcal{A}$ is weakly irreducible iff $\mathcal{H}$ is connected [2]. For k-uniform directed hypergraph $\mathcal{H}$ if each arc has only one tail, then $\mathcal{A}_\alpha$ is weakly irreducible iff $\mathcal{H}$ is strongly connected, i.e. the strongly connectivity of $\mathcal{H}$ is equivalent to strongly connectivity of $G(\mathcal{A}_\alpha)$. But we have just the sufficient condition in general:

Lemma 1. *Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a k-uniform directed hypergraph with adjacency tensor $\mathcal{A}$ and degree tensor $\mathcal{D}$. Then, $\mathcal{A}_\alpha = \alpha\mathcal{D} + (1 - \alpha)\mathcal{A}$ is weakly irreducible if $\mathcal{H}$ is strongly connected.*

Proof. Suppose that $\mathcal{H}$ is strongly connected. By Definition 6, we should show that $G(\mathcal{A}_\alpha)$ is strongly connected. Let $i, j \in V$ and $i \neq j$. Since $\mathcal{H}$ is strongly connected, there exists a sequence of vertices and arcs in $\mathcal{H}$ such that:

$$i = j_1 \quad \vec{e}_1 \quad j_2 \quad \vec{e}_2 \quad j_3 \quad \cdots \quad \vec{e}_{q-1} \quad j_q \quad \vec{e}_q \quad j_{q+1} = j,$$

where $j_2, \cdots, j_q \in \mathcal{V}$, $\vec{e}_1, \cdots, \vec{e}_q \in \mathcal{E}$ and $j_t \in e_t^+$, $j_{t+1} \in e_t^-$ for all $t = 1, \cdots, q$. On the other hand, $a_{e_t^+ e_t^-} > 0$ for $t = 1, \cdots, q$, since $\vec{e}_t = (e_t^+, e_t^-) \in \mathcal{E}$, then $a^{(\alpha)}_{e_t^+ e_t^-} > 0$. Hence $e_t = (j_t, j_{t+1})$ is a directed edge in $G(\mathcal{A}_\alpha)$, for all $t = 1, \cdots, q$. Therefore there exists a sequence of vertices and directed edges in $G(\mathcal{A}_\alpha)$:

$$i = j_1 \quad e_1 \quad j_2 \quad e_2 \quad j_3 \quad \cdots \quad e_{q-1} \quad j_q \quad e_q \quad j_{q+1} = j,$$

i.e. $i \to j$ in $G(\mathcal{A}_\alpha)$. Similarly it can be proved that $j \to i$ in $G(\mathcal{A}_\alpha)$. Thus $G(\mathcal{A}_\alpha)$ is strongly connected and then $\mathcal{A}_\alpha$ is weakly irreducible. $\square$

Now we study the H-eigenvalues of $\mathcal{A}_\alpha$. We have the following lemma:

Lemma 2. *Let $\mathcal{H}$ be a k-uniform directed hypergraph. Suppose that $X \in \mathbb{R}^n$, then we have:*

$$(\mathcal{A}_\alpha \mathbf{x}^{[k-1]})_i = \alpha d_i^+ + (1-\alpha) \sum_{\vec{e} \in E_i^+} \prod_{s \in (e^+ \cup e^-) \setminus \{i\}} x_s.$$

Proof.

$$\begin{aligned}
(\mathcal{A}_\alpha \mathbf{x}^{[k-1]})_i = & \sum_{i_2,\cdots,i_k=1}^{n} a^{(\alpha)}_{ii_2\cdots i_k} x_{i_2} \cdots x_{i_k} \\
= & \alpha \sum_{i_2,\cdots,i_k=1}^{n} d_{ii_2\cdots i_k} x_{i_2} \cdots x_{i_k} + (1-\alpha) \sum_{i_2,\cdots,i_k=1}^{n} a_{ii_2\cdots i_k} x_{i_2} \cdots x_{i_k} \\
= & \alpha d_i^+ x_i^{k-1} + (1-\alpha) \sum_{\substack{\vec{e}=(e^+,e^-)\in\mathcal{E} \\ i \in e^+,\ |e^+|=l_{\vec{e}}}} \frac{(l_{\vec{e}}-1)!(k-l_{\vec{e}})!}{(l_{\vec{e}}-1)!(k-l_{\vec{e}})!} \prod_{s \in (e^+ \cup e^-) \setminus \{i\}} x_s \\
= & \alpha d_i^+ x_i^{k-1} + (1-\alpha) \sum_{\vec{e} \in E_i^+} \prod_{s \in (e^+ \cup e^-) \setminus \{i\}} x_s.
\end{aligned}$$

□

Now we have the following theorems.

Theorem 1. *Let $\mathcal{H}$ be a k-uniform directed hypergraph with n vertices. Then each $(\alpha d_j^+, \mathbf{1}_j)$ is an H-eigenpair of $\mathcal{A}_\alpha$ for $j = 1, \cdots, n$.*

Proof. Let $i \in [n]$. By Lemma 2, if $i = j$, then we have:

$$(\mathcal{A}_\alpha \mathbf{1}_j^{[k-1]})_i = \alpha d_j^+ 1 + (1-\alpha) \sum_{\vec{e} \in E_i^+} 0 = \alpha d_j^+$$

and for $i \neq j$, we have:

$$(\mathcal{A}_\alpha \mathbf{1}_j^{[k-1]})_i = \alpha d_j^+ 0 + (1-\alpha) \sum_{\vec{e} \in E_i^+} 0 = 0.$$

By Definition 2, the result follows. □

Theorem 2. *Let $\mathcal{H}$ be a k-uniform directed hypergraph and $\mathcal{A}_\alpha = \alpha\mathcal{D} + (1-\alpha)\mathcal{A}$, where $\frac{1}{2} \leq \alpha < 1$. If λ is an H-eigenvalue of $\mathcal{A}_\alpha$, then we have:*

$$(2\alpha - 1)\delta^+ \leq \lambda \leq \Delta^+,$$

where δ^+ and Δ^+ are the minimum and maximum out-degree in $\mathcal{H}$, respectively.

Proof. Suppose that $\mathbf{x}$ is an H-eigenvector of $\mathcal{A}_\alpha$ associated with λ and $|x_i| = \max\{|x_1|, \cdots, |x_n|\}$. By Lemma 2, we have:

$$\begin{aligned}
&\lambda x_i^{k-1} = \alpha d_i^+ x_i^{k-1} + (1-\alpha) \sum_{\vec{e}\in E_i^+} \prod_{s\in(e^+\cup e^-)\setminus\{i\}} x_s \\
\Longrightarrow\ &(\lambda - \alpha d_i^+) x_i^{k-1} = (1-\alpha) \sum_{\vec{e}\in E_i^+} \prod_{s\in(e^+\cup e^-)\setminus\{i\}} x_s \\
\Longrightarrow\ &|\lambda - \alpha d_i^+||x_i|^{k-1} = (1-\alpha) \sum_{\vec{e}\in E_i^+} \prod_{s\in(e^+\cup e^-)\setminus\{i\}} |x_s| \leq (1-\alpha)\sum_{\vec{e}\in E_i^+} |x_i| \\
\Longrightarrow\ &|\lambda - \alpha d_i^+| \leq (1-\alpha) d_i^+ \\
\Longrightarrow\ &(2\alpha - 1) d_i^+ \leq \lambda \leq d_i^+ \\
\Longrightarrow\ &(2\alpha - 1)\delta^+ \leq \lambda \leq \Delta^+.
\end{aligned}$$

□

Lemma 3. *Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be an n-vertex k-uniform complete directed hypergraph and $i \in \mathcal{V}$ be an arbitrary vertex. Then $d_i = \binom{n-1}{k-1}$.*

Proof. Since $\mathcal{H}$ is complete, then $\mathcal{E}$ contains all possible arcs. Therefore vertex i has common arcs with any $k-1$ vertices that is $\binom{n-1}{k-1}$.Then $d_i = d_i^+ + d_i^- = \binom{n-1}{k-1}$. □

Theorem 3. *Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be an n-vertex k-uniform complete directed hypergraph. If $d_i^+ = \binom{n-1}{k-1}$ for each $i \in \mathcal{V}$, then the largest H-eigenvalue of tensor $\mathcal{A}_\alpha$, $\lambda(\mathcal{A}_\alpha)$, is $\binom{n-1}{k-1}$.*

Proof. We show that $\lambda = \binom{n-1}{k-1}$ with $\mathbf{x} = \mathbf{1}$ is an H-eigenpair of $\mathcal{A}_\alpha$. By Lemma 2, we have:

$$\begin{aligned}(\mathcal{A}_\alpha \mathbf{x}^{[k-1]})_i &= \alpha d_i^+ x_i^{k-1} + (1-\alpha) \sum_{\vec{e} \in E_i^+} \prod_{s \in (e^+ \cup e^-) \setminus \{i\}} x_s \\ &= \alpha \binom{n-1}{k-1} + (1-\alpha) \sum_{\vec{e} \in E_i^+} 1 = \alpha \binom{n-1}{k-1} + (1-\alpha) \binom{n-1}{k-1} \\ &= \binom{n-1}{k-1} = \lambda x_i^{k-1}.\end{aligned}$$

On the other hand, by Lemma 3, $\Delta^+ = \binom{n-1}{k-1}$, then the result follows from Theorem 2. $\square$

The next theorem characterizes the extreme weak-connected directed hypergraphs with respect to the upper bound of the largest $\mathcal{A}_\alpha$ H-eigenvalue.

Theorem 4. *Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a weak-connected k-uniform directed hypergraph. Then $\lambda(\mathcal{A}_\alpha) = \Delta^+$ if and only if $\mathcal{H}$ is out-regular.*

Proof. Suppose that $\mathcal{H}$ is out-regular. It is easy to see that $\lambda = \Delta^+$ with the H-eigenvector $\mathbf{x} = \mathbf{1}$ is an H-eigenvalue of $\mathcal{A}_\alpha$, then $\lambda(\mathcal{A}_\alpha) = \Delta^+$. On the other hand, assume that $\lambda(\mathcal{A}_\alpha) = \Delta^+$ and $x \in \mathbb{R}^n$ is its corresponding H-eigenvector. Let $|x_i| = max\ \{|x_j| \big| j \in [n]\}$. By Definition 2, we have:

$$\begin{aligned}&\Delta^+ x_i^{k-1} = \alpha d_i^+ x_i^{k-1} + (1-\alpha) \sum_{\vec{e} \in E_i^+} \prod_{s \in (e^+ \cup e^-) \setminus \{i\}} x_s \\ &\Rightarrow \Delta^+ |x_i^{k-1}| \leq \alpha d_i^+ |x_i^{k-1}| + (1-\alpha) \sum_{\vec{e} \in E_i^+} \prod_{s \in (e^+ \cup e^-) \setminus \{i\}} |x_s| \\ &\Rightarrow \Delta^+ = \alpha d_i^+ + (1-\alpha) \sum_{\vec{e} \in E_i^+} \prod_{s \in (e^+ \cup e^-) \setminus \{i\}} \frac{|x_s|}{|x_i|} \leq \alpha d_i^+ + (1-\alpha) \sum_{\vec{e} \in E_i^+} 1 = d_i^+ \\ &\Rightarrow \Delta^+ = d_i^+.\end{aligned}$$

and we must have $|x_i| = |x_j|$ for all $j \in e^+ \cup e^-$, where $\vec{e} = (e^+, e^-) \in E_i^+$. Applying the same augment for all such j, we have that $\Delta^+ = d_j^+$ and $|x_i| = |x_j| = |x_l|$ for all $l \in e^+ \cup e^-$, where $\vec{e} = (e^+, e^-) \in E_j^+$. Since $\mathcal{H}$ is weak-connected, we see that $d_j^+ = \Delta^+$ for all $j \in \mathcal{V}$, then $\mathcal{H}$ is out-regular. □

Suppose that x is an H-eigenvector of the $\mathcal{A}_\alpha$ of a k-uniform directed hypergraph corresponding to H-eigenvalue λ. The following theorem gives a sufficient condition for equality of some components of x.

Theorem 5. *Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a k-uniform directed hypergraph and $i, j \in \mathcal{V}$ such that $E_i^+ = E_j^+$. Then $d_i^+ = d_j^+ = d$. Now suppose that (λ, x) is an H-eigenpair of $\mathcal{A}_\alpha$, such that $\lambda \neq \alpha d$. Then $|x_i| = |x_j|$ and if k is odd, then $x_i = x_j$.*

Proof. Clearly, $d_i^+ = d_j^+ = d$ by the definition of E_i^+. Now suppose that (λ, x) is an H-eigenpair of $\mathcal{A}_\alpha$, such that $\lambda \neq \alpha d$. By Definition 2, we have:

$$\lambda x_i^{k-1} = \alpha d x_i^{k-1} + (1-\alpha) x_j \sum_{\vec{e} \in E_i^+} \prod_{s \in (e^+ \cup e^-) \setminus \{i,j\}} x_s$$

and

$$\lambda x_j^{k-1} = \alpha d x_j^{k-1} + (1-\alpha) x_i \sum_{\vec{e} \in E_j^+} \prod_{s \in (e^+ \cup e^-) \setminus \{i,j\}} x_s.$$

Hence,

$$(\lambda - \alpha d) x_i^k = (\lambda - \alpha d) x_j^k \quad \overset{\lambda \neq \alpha d}{\Longrightarrow} \quad x_i^k = x_j^k.$$

The conclusions follow from the last equality. □

4 Cored directed hypergraphs and Power directed hypergraphs

In this section we introduce two classes of k-uniform directed hypergraphs: 1. Cored directed hypergraphs and 2. Power directed hypergraphs. Hu, Qi and Shao in [7] introduced these two classes in undirected hypergraphs and investigated their spectral properties. We extend their definitions and analyze the α-spectral properties of power directed hypergraphs and cored directed hypergraphs.

4.1 Cored directed hypergraphs

We begin with the definition of Cored directed hypergraphs.

Definition 7. *Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a directed hypergraph. $\mathcal{H}$ is a cored directed hypergraph if there exists in each arc $\vec{e} = (e^+, e^-)$ a vertex $i \in e^+$ such that $d_i^+ = 1$ and $d_i^- = 0$. Such vertex is called core vertex and a vertex with out-degree greater than one is called intersection vertex.*

By Theorem 5, we have the following lemma:

Lemma 4. *Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a cored k-uniform directed hypergraph and $(\lambda, \mathbf{x})$ be an H-eigenpair of $\mathcal{A}_\alpha$ and $\lambda \neq \alpha$. If i and j are two core vertices in arc $\vec{e}$, then $x_i = x_j$ when k is odd and $|x_i| = |x_j|$ when k is even.*

In the following we study a special cored directed hypergraph.

Definition 8. *Let $\mathcal{S} = (\mathcal{V}, \mathcal{E})$ be a cored k-uniform directed hypergraph. We call it a directed squid if $\mathcal{V} = \{1, 1_1, 2_1, \cdots, k_1, \cdots, 1_{(k-1)}, 2_{(k-1)}, \cdots, k_{(k-1)}\}$ and the arc set $\mathcal{E} = \{\vec{e}_i \mid i = 0, \cdots, k-1\}$ in which*

$$\begin{aligned} &\vec{e}_0 = (\{1\}, \{1_1, 1_2 \cdots, 1_{(k-1)}\}), \\ &\vec{e}_i = (\{1_i\}, \{2_i, 3_i, \cdots, k_i\}), \qquad i = 1, \cdots, k-1. \end{aligned}$$

By the Definition 8, it's straightforward that $d_1^+ = d_{1_1}^+ = d_{1_2}^+ = \cdots, d_{1_{(k-1)}}^+ = 1$, and $d_i^+ = 0$ otherwise.

The following theorem determines $Hspec(\mathcal{A}_\alpha)$ of the directed squid $\mathcal{S}$.

Theorem 6. *Let $\mathcal{S} = (\mathcal{V}, \mathcal{E})$ be a k-uniform directed squid, then $Hspec(\mathcal{A}_\alpha) = \{0, \alpha\}$.*

Proof. It is easy to see that $(0, \mathbf{x})$ is an H-eigenpair of $\mathcal{A}_\alpha$, where

$$x_t = \begin{cases} 1 & t = 1, \\ 0 & t = 1_i \quad (\text{for } i = 1, \cdots, k-1), \\ 1 & t = 2_i \quad (\text{for } i = 1, \cdots, k-1), \\ 0 & t = j_i \quad (\text{for } i = 1, \cdots, k-1\ ,\ j = 3, \cdots, k). \end{cases}$$

Now let $\mathbf{x}$ be an H-eigenvector of $\mathcal{A}_\alpha$ corresponding to H-eigenvalue $\lambda \neq 0$. By Lemma 2, we have:

$$(\lambda - \alpha) x_1^{k-1} = (1 - \alpha) \prod_{i=1}^{k-1} x_{1_i}, \tag{1}$$

$$(\lambda - \alpha) x_{1_i}^{k-1} = (1 - \alpha) \prod_{j=2}^{k} x_{j_i}, \qquad i = 1, 2, \cdots, k-1, \tag{2}$$

$$\lambda x_{j_i}^{k-1} = 0, \qquad i = 1, 2, \cdots, k-1, j = 2, \cdots, k. \tag{3}$$

By (3), $x_{j_i} = 0$ for all i, j. By taking it in (2), we have $(\lambda - \alpha) x_{1_i}^{k-1} = 0$. Now three cases are considered:
(**i**) : $x_{1_i} \neq 0$ for $i = 1, 2, \cdots, k-1$, then $\lambda = \alpha$ and by (1), $\prod_{i=1}^{k-1} x_{1_i} = 0$ that is a contradiction.
(**ii**) : $x_{1_i} = 0$ for $i = 1, 2, \cdots, k-1$, then by 1, $(\lambda - \alpha) x_1^{k-1} = 0$. Thus $\lambda = \alpha$ and $x_1 \neq 0$.
(**iii**) : $x_{1_i} = 0$ and $x_{1_j} \neq 0$ for some $i, j = 1, 2, \cdots, k-1$. Then $\lambda = \alpha$ and $x_1 \in \mathbb{R}$.
Therefore, $\lambda = \alpha$ is the only nonzero H-eigenvalue of $\mathcal{A}_\alpha$. □

4.2 Power directed hypergraphs

Definition 9. *Let $G = (V, E)$ be a directed graph and $k \geq 3$. The k-th power of G, $\mathcal{G}^k = (\mathcal{V}, \mathcal{E})$ is defined as the k-uniform directed hypergraph with the set of arcs*

$$\mathcal{E} = \{\vec{e} = (e^+, e^-) \mid e \in E\},$$

where if $e = (i_1^e, i_2^e) \in E$, then $e^+ = \{i_1^e, i_{e,1}, i_{e,2}, \cdots, i_{e,k-2}\}$ and $e^- = \{i_2^e\}$, and the set of vertices $\mathcal{V} = \left(\bigcup_{e \in E} \{i_{e,1}, i_{e,2}, \cdots, i_{e,k-2}\}\right) \cup V$.

It is easy to see that each power directed hypergraph is a cored directed hypergraph, but on the contrary, it is not generally correct, for example, the directed squid which was studied in the previous subsection.

The next theorem gives some basic results about an ordinary arc in a power directed hypergraph.

Theorem 7. *Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a power k-uniform directed hypergraph and $\mathbf{x}$ be an H-eigenvector of $\mathcal{A}_\alpha$, corresponding to $\lambda \neq \alpha$. If $\vec{e} = (e^+, e^-) \in \mathcal{E}$ is an arbitrary arc with $e^+ = \{i_1^e, i_{e,1}, i_{e,2}, \cdots, i_{e,k-2}\}$ and $e^- = \{i_2^e\}$, then we have:*

(1) If $d^+_{i_1^e} > 1$, $d^+_{i_2^e} \geq 1$ and $x_{i_{e,1}} = \beta \neq 0$, then $x_{i_1^e} x_{i_2^e} = \frac{(\lambda-\alpha)\beta^2}{(1-\alpha)}$ when k is odd and $x_{i_1^e} x_{i_2^e} = \frac{(\lambda-\alpha)\beta^2}{(1-\alpha)}$ or $-\frac{(\lambda-\alpha)\beta^2}{(1-\alpha)}$ when k is even.

(2) If $d^+_{i_1^e} = 1$, $d^+_{i_2^e} \geq 1$ and $x_{i_{e,1}} = \beta \neq 0$, then $x_{i_2^e} = \frac{(\lambda-\alpha)\beta}{(1-\alpha)}$ when k is odd and $x_{i_2^e} = \frac{(\lambda-\alpha)\beta}{(1-\alpha)}$ or $-\frac{(\lambda-\alpha)\beta}{(1-\alpha)}$ when k is even.

(3) If $d^+_{i_2^e} = 0$, then $x_j = 0$ for $j \in \{i_{e,1}, i_{e,2}, \cdots, i_{e,k-2}, i_2^e\}$.

Proof. By Lemma 4, $x_{i_{e,j}} = \beta$ for $j = 2, \cdots, k-2$ when k is odd and $|x_{i_{e,j}}| = \beta$ for $j = 2, \cdots, k-2$ when k is even.
For (1), by Definition 2, we have:

$$(1-\alpha)\beta^{k-3} x_{i_1^e} x_{i_2^e} = (\lambda-\alpha)\beta^{k-1} \qquad \textit{if } k \textit{ is odd},$$

$$\left\{\begin{array}{rl} & (1-\alpha)\beta^{k-3} x_{i_1^e} x_{i_2^e} = (\lambda-\alpha)\beta^{k-1} \\ & \textit{or} \\ - & (1-\alpha)\beta^{k-3} x_{i_1^e} x_{i_2^e} = (\lambda-\alpha)\beta^{k-1} \end{array}\right. \qquad \textit{if } k \textit{ is even}.$$

The result follows from $\beta \neq 0$.

For (2), by Lemma 4, $x_{i_1^e} = \beta$ or $x_{i_1^e} = -\beta$. By Definition 2, we have:

$$(1-\alpha)\beta^{k-2}x_{i_2^e} = (\lambda-\alpha)\beta^{k-1} \qquad \textit{if } k \textit{ is odd,}$$

$$\begin{cases} \quad (1-\alpha)\beta^{k-2}x_{i_2^e} = (\lambda-\alpha)\beta^{k-1} \\ \quad \textit{or} & \textit{if } k \textit{ is even.} \\ - \quad (1-\alpha)\beta^{k-2}x_{i_2^e} = (\lambda-\alpha)\beta^{k-1} \end{cases}$$

The result follows from $\beta \neq 0$.
For (3), since $d^+_{i_2^e} = 0$, then $x_{i_2^e} = 0$. Thus by Definition 2 and Lemma 4, $x_j = 0$ for $j \in \{i_{e,1}, i_{e,2}, \cdots, i_{e,k-2}\}$. □

In the following we study a special power directed hypergraph which is a called directed hyperwheel.

Definition 10. *Let $\mathcal{W}_d = (\mathcal{V}, \mathcal{E})$ be a power k-uniform directed hypergraph. We call it a directed hyperwheel if $\mathcal{V} = V_0 \cup V_1 \cup \cdots V_d \cup \bar{V}_1 \cup \bar{V}_2 \cup \cdots \cup \bar{V}_d$ is a disjoint partition of $\mathcal{V}$ in which $V_0 = \{1\}$, $V_i = \{1_i, 2_i, \cdots, (k-1)_i\}$ and $\bar{V}_i = \{1^i, 2^i, \cdots, (k-2)^i\}$ for $i = 1, 2, \cdots, d$ and the arc set $\mathcal{E} = \{\vec{e}_i,\ \vec{a}_i |\ i = 1, \cdots, d\}$ in which*

$$\begin{aligned} &\vec{e}_i = (\{1, 1_i, \cdots, (k-2)_i\}, \{(k-1)_i\}), && i = 1, \cdots, d, \\ &\vec{a}_i = (\{(k-1)_i, 1^i, \cdots, (k-2)^i\}, \{(k-1)_{i+1}\}), && i = 1, \cdots, d-1, \\ &\vec{a}_d = (\{(k-1)_d, 1^d, \cdots, (k-2)^d\}, \{(k-1)_1\}). \end{aligned}$$

By Definition 10, it can be shown easily.

Lemma 5. *Let $\mathcal{W}_d = (\mathcal{V}, \mathcal{E})$ be a directed k-uniform hyperwheel, then $d_1^+ = d$, $d_j^+ = 1$ for $j \neq 1$ and $d^-_{(k-1)_i} = 2$ for $i = 1, \cdots, d$, $d_j^- = 0$ for $i = 1, \cdots, d$ and $j \neq (k-1)_i$.*

In the following theorems the H-spectrum of $\mathcal{A}_\alpha$ of $\mathcal{W}_d$ are determined.

Theorem 8. *Let $\mathcal{W}_d = (\mathcal{V}, \mathcal{E})$ be a directed k-uniform hyperwheel. Then $Hspec(\mathcal{A}_\alpha) = \{\alpha, \alpha d, 1\}$ when d and k are odd, and $Hspec(\mathcal{L}) = \{\alpha, \alpha d, 1, 2\alpha - 1\}$ otherwise.*

Proof. By Theorem 1 and Lemma 5, $\alpha, \alpha d \in Hspec(\mathcal{A}_\alpha)$. Now suppose that $\mathbf{x}$ is an H-eigenvector of $\mathcal{A}_\alpha$ corresponding to H-eigenvalue $\lambda \neq \alpha, \alpha d$. The proof is divided into two cases, which contain several subcases respectively:

1: k is odd.

By Lemma 4, we have:

$$\begin{aligned} &x_{1_i} = x_{2_i} = \cdots = x_{(k-2)_i} = \alpha_i, && i = 1, \cdots, d, \\ &x_{1^i} = x_{2^i} = \cdots = x_{(k-2)^i} = x_{(k-1)_i} = \beta_i, && i = 1, \cdots, d. \end{aligned}$$

Now by Definition 2, we have:

$$(\lambda - \alpha d)x_1^{k-1} = (1-\alpha)\sum_{i=1}^{d} \beta_i \alpha_i^{k-2}, \tag{4}$$

$$(\lambda - \alpha)\alpha_i^{k-1} = (1-\alpha)x_1\alpha_i^{k-3}\beta_i, \qquad i = 1, \cdots, d, \tag{5}$$

$$(\lambda - \alpha)\beta_i^{k-1} = (1-\alpha)\beta_i^{k-2}\beta_{i+1}, \qquad i = 1, \cdots, d-1, \tag{6}$$

$$(\lambda - \alpha)\beta_d^{k-1} = (1-\alpha)\beta_d^{k-2}\beta_1. \tag{7}$$

By (6) and (7), if $\beta_i = 0$ for some $i = 1, \cdots, d$, then all $\beta_i = 0$ and thus by (5) and (4), $\mathbf{x} = 0$ that is a contradiction. Therefore, $\beta_i \neq 0$ for $i = 1, \cdots, d$. Then by (6) and (7), $\frac{(\lambda-\alpha)}{(1-\alpha)} = \frac{\beta_{i+1}}{\beta_i} = \frac{\beta_1}{\beta_d}$ for $i = 1, \cdots, d-1$, then we have:

$$\begin{aligned} \beta_1 = \frac{(\lambda-\alpha)^d}{(1-\alpha)^d}\beta_1 \quad &\Longrightarrow \quad (\lambda-\alpha)^d = (1-\alpha)^d \\ &\Longrightarrow \quad \begin{cases} \lambda = 1, 2\alpha - 1 & \text{if } d \text{ is even} \\ \lambda = 1 & \text{if } d \text{ is odd} \end{cases} \end{aligned}$$

2: k is even.

By Lemma 4, we have:

$$\begin{aligned} &|x_{1_i}| = |x_{2_i}| = \cdots = |x_{(k-2)_i}|, && i = 1, \cdots, d, \\ &|x_{1^i}| = |x_{2^i}| = \cdots = |x_{(k-2)^i}| = |x_{(k-1)_i}|, && i = 1, \cdots, d. \end{aligned}$$

Now let $x_{1_i} = \alpha_i$ and $x_{(k-1)_i} = \beta_i$ for $i = 1, \cdots, d$. With a little modification in (4), (5), (6) and (7) and by similar argument in the

previous case, $\beta_i \neq 0$ for $i = 1, \cdots, d$. Now we consider two subcases:
(i) d is even. There are two cases:

- $\beta_d = \frac{(\lambda-\alpha)^{d-1}}{(1-\alpha)^{d-1}}\beta_1$, then we have:

$$\begin{aligned}
&\textit{if } \lambda < \alpha \;\Rightarrow\; \beta_1 \textit{ and } \beta_d \textit{ have different signs}\\
&\qquad\Rightarrow\; \beta_1 = \frac{(\lambda-\alpha)}{(1-\alpha)}\beta_d \;\Rightarrow\; \frac{(\lambda-\alpha)^d}{(1-\alpha)^d} = 1 \;\Rightarrow\; \lambda = 2\alpha - 1;\\
&\textit{if } \lambda > \alpha \;\Rightarrow\; \beta_1 \textit{ and } \beta_d \textit{ have the same sign}\\
&\qquad\Rightarrow\; \beta_1 = \frac{(\lambda-\alpha)}{(1-\alpha)}\beta_d \;\Rightarrow\; \frac{(\lambda-\alpha)^d}{(1-\alpha)^d} = 1 \;\Rightarrow\; \lambda = 1.
\end{aligned}$$

- $\beta_d = -\frac{(\lambda-\alpha)^{d-1}}{(1-\alpha)^{d-1}}\beta_1$, then we have:

$$\begin{aligned}
&\textit{if } \lambda > \alpha \;\Rightarrow\; \beta_1 \textit{ and } \beta_d \textit{ have different signs}\\
&\qquad\Rightarrow\; \beta_1 = -\frac{(\lambda-\alpha)}{(1-\alpha)}\beta_d \;\Rightarrow\; \frac{(\lambda-\alpha)^d}{(1-\alpha)^d} = 1 \;\Rightarrow\; \lambda = 1;\\
&\textit{if } \lambda < \alpha \;\Rightarrow\; \beta_1 \textit{ and } \beta_d \textit{ have the same sign}\\
&\qquad\Rightarrow\; \beta_1 = -\frac{(\lambda-\alpha)}{(1-\alpha)}\beta_d \;\Rightarrow\; \frac{(\lambda-\alpha)^d}{(1-\alpha)^d} = 1 \;\Rightarrow\; \lambda = 2\alpha - 1.
\end{aligned}$$

(ii) d is odd. There are two cases:

- $\beta_d = \frac{(\lambda-\alpha)^{d-1}}{(1-\alpha)^{d-1}}\beta_1$, then β_1 and β_d have the same sign and we have:

$$\begin{aligned}
&\textit{if } \lambda < \alpha \;\Rightarrow\; \beta_1 = -\frac{(\lambda-\alpha)}{(1-\alpha)}\beta_d \;\Rightarrow\; \frac{(\lambda-\alpha)^d}{(1-\alpha)^d} = -1 \Rightarrow \lambda = 2\alpha - 1;\\
&\textit{if } \lambda > \alpha \;\Rightarrow\; \beta_1 = \frac{(\lambda-\alpha)}{(1-\alpha)}\beta_d \;\Rightarrow\; \frac{(\lambda-\alpha)^d}{(1-\alpha)^d} = 1 \;\Rightarrow\; \lambda = 1.
\end{aligned}$$

- $\beta_d = -\frac{(\lambda-\alpha)^{d-1}}{(1-\alpha)^{d-1}}\beta_1$, then β_1 and β_d have different signs and we have:

$$\textit{if } \lambda < \alpha \;\Rightarrow\; \beta_1 = \frac{(\lambda-\alpha)}{(1-\alpha)}\beta_d \;\Rightarrow\; \frac{(\lambda-\alpha)^d}{(1-\alpha)^d} = -1 \;\Rightarrow\; \lambda = 2\alpha - 1;$$

$$\textit{if } \lambda > \alpha \;\Rightarrow\; \beta_1 = -\frac{(\lambda-\alpha)}{(1-\alpha)}\beta_d \;\Rightarrow\; \frac{(\lambda-\alpha)^d}{(1-\alpha)^d} = 1 \;\Rightarrow\; \lambda = 1.$$

$\square$

5 Conclusion

In this paper we consider a k-uniform directed hypergraph in general form and introduce its adjacency tensor, Laplacian tensor and signless Laplacian tensor. Then we propose theorems in spectral theory of the convex linear combination of $\mathcal{D}$ and $\mathcal{A}$ that has been defined as $\mathcal{A}_\alpha = \alpha\mathcal{D} + (1-\alpha)\mathcal{A}$, where $\mathcal{D}$ and $\mathcal{A}$ are the degree tensor and the adjacency tensor of $\mathcal{H}$, respectively. Cored directed hypergraphs and power directed hypergraphs are introduced, and some their α-spectral properties are presented.

References

[1] A. Ducournau and A. Bretto, "Random walks in directed hypergraphs and application to semisupervised image segmentation," *Comput. Vision Image Understanding*, vol. 120, pp. 91–102, 2014.

[2] S.Friedland, S.Gaubert, and L.Han, "Perron-Frobenius theorems for nonnegative multilinear forms and extension," *Linear Algebra Appl.*, vol. 438, pp. 738–749, 2013.

[3] G. Gallo, G. Longo, and S. Pallottino, "Directed hypergraphs and applications," *Discrete Appl. Math.*, vol. 42, pp. 177–201, 1993.

[4] H. Guo and B. Zhou, "On the α-spectral radius of graphs," [Online]. Available: arXiv:1805.03456.

[5] H. Guo and B.Zhou, "On the α-spectral radius of uniform hypergraphs," [Online]. Available: arxiv:1807.08112.

[6] S. Hu and L. Qi, "The eigenvectors associated with the zero eigenvalues of the Laplacian and signless Laplacian tensors of a uniform hypergraph", *Discrete Applied Mathematics*, vol. 169, pp. 140–151, 2014.

[7] S. Hu, L. Qi, and J. Shao, "Cored hypergraphs, power hypergraphs and their Laplacian eigenvalues," *Linear Algebra and Its Applications*, vol. 439, pp. 2980–2998, 2013.

[8] K. Li and L. Wang, "A polynomial time approximation scheme for embedding a directed hypergraph on a ring," *Inf. Process. Lett.*, vol. 97, pp. 203–207, 2006.

[9] L. Lim, "Singular values and eigenvalues of tensors: a variational approach," in *Proceedings of the IEEE International Workshop on Computational Advances in Multi Sensor Adaptive Processing, CAMSAP'05.*, vol. 1, 2005, pp. 129–132.

[10] H. Lin, H. Guo, and B. Zhou, "On the α-spectral radius of irregular uniform hypergraphs," *Linear Multilinear Algebra*, vol. 68, pp. 265–277, 2020.

[11] V. Nikiforov, " Merging the A- and Q-spectral theories," *Appl. Anal. Discrete Math.*, vol. 11, pp. 81–107, 2017.

[12] V. Nikiforov, G. Pastén, O. Rojo, and R. L. Soto, "On the α-spectral of trees," *Linear Algebra Appl.*, vol. 520, pp. 286–305, 2017.

[13] G. Pandey, S. Chawla, S. Poon, B. Arunasalam, and J. G. Davis, "Association Rules Network: Definition and Applications," *Statistical analysis and data mining*, vol. 1, pp. 260–279, 2009.

[14] L. Qi, "Eigenvalues of a real supersymmetric tensor," *J. Symbolic Comput.*, vol. 40, pp. 1302–1324, 2005.

[15] L. Qi, "H+-Eigenvalues of Laplacian and signless Lapaclian tensors," *Communications in Mathematical Sciences*, vol. 12, pp. 1045–1064, 2014.

[16] J. Y. Shao, "A general product of tensors with applications," *Linear Algebra Appl.*, vol. 439, pp. 2350–2366, 2013.

[17] J. Xie and L. Qi, "Spectral directed hypergraph theory via tensors," *Linear and Multilinear Algebra*, vol. 64, pp. 780–794, 2016.

Gholam Hasan Shirdel, Ameneh Mortezaee,
Effat Golpar-Raboky

Received October 14, 2017
Revised July 21, 2020
Accepted July 29, 2020

Gholam Hasan Shirdel
Department of Mathematics, University of Qom
Qom, I. R. Iran
E–mail: `g.h.shirdel@qom.ac.ir`

Ameneh Mortezaee
Department of Mathematics, University of Qom
Qom, I. R. Iran
E–mail: `a.mortezee@stu.qom.ac.ir`

Effat Golpar-Raboky
Department of Mathematics, University of Qom
Qom, I. R. Iran
E–mail: `q.raboky@qom.ac.ir`