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Chromatic Spectrum of K_s-WORM Colorings of K_n*

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Abstract

An *H-WORM* coloring of a simple graph G is the coloring of the vertices of G such that no copy of $H \subseteq G$ is monochrome or rainbow. In a recently published article by one of the authors [3], it was claimed that the number of r-partitions in a K_s-WORM coloring of K_n is $\zeta_r = \left\{ {n \atop r} \right\}$, where $\left\{ {n \atop r} \right\}$ denotes the Stirling number of the second kind, for all $3 \leq r \leq s < n$. We found that $\zeta_r = \left\{ {n \atop r} \right\}$ if and only if $\lceil \frac{n+3}{2} \rceil < s \leq n$ with $r < s$. Further investigations into ζ_2, given any K_3-WORM coloring of K_n, show its relation with the number of spanning trees of cacti and the Catalan numbers. Moreover, we extend the notion of H-WORM colorings to $(H_1; H_2)$-*mixed* colorings, where H_1 and H_2 are distinct subgraphs of G; these coloring constraints are closely related to those of mixed hypergraph colorings.

Keywords: Catalan numbers, Chromatic spectrum, Mixed hypergraph coloring, Stirling numbers.

MSC 2020: 05C15, 05C30, 05C35, 05C65.

1 Preliminaries

A *partition* $\mathscr{P}$ of a set A is a set of nonempty subsets of A such that each element of A is in exactly one subset of A. The elements of $\mathscr{P}$ are often called *blocks* and an r-*partition* is a partition with r number of *blocks*.

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A *coloring* of a set S is a mapping $c : S \to [r]$, where $[r] = \{1, 2, \ldots, r\}$ and an r-coloring of S is a coloring of the elements of S using exactly r colors. As such, a coloring $c(S)$ is a partition of the set S since all of the elements of S are assigned a color; elements that share the same color (monochrome subset) belong to the same block and elements with different colors (rainbow subset) belong to distinct blocks. A set $A \subseteq S$ is said to be *monochrome* if all of its elements share the same color and A is *rainbow* if all of its elements have different colors.

Let $G = (V, E)$ denote a simple graph and let H be a subgraph of G; we write $H \subseteq G$. An *H-WORM* (vertex) coloring of G is the coloring of the vertices of G such that no copy of $H \subseteq G$ is monochrome or rainbow. This coloring constraint was first introduced by W. Goddard, K. Wash and H. Xu [9], [10], and independently studied by Cs. Bujtás and Zs. Tuza [4], [6], [9], [10]. In [2], it was extended to the notion of $\mathscr{F}$-WORM colorings, where $\mathscr{F}$ represents a collection of distinct subgraphs of G instead of a single subgraph of G. Given any H-WORM coloring of G, the sequence $(\zeta_\alpha, \ldots, \zeta_\beta)$ is called a *chromatic spectrum*, where α and β are known as *lower* and *upper chromatic numbers*, respectively. Each chromatic spectral value, ζ_r with $\alpha \le r \le \beta$, counts the number of *proper* r-partitions which are the r distinct partitions that satisfy the coloring constraint. We note that the term *partition vector* was used in [2] to describe chromatic spectrum. The integer set $F = \{r : \zeta_r > 0\}$ with $\alpha \le r \le \beta$ is called a *feasible set* and it has been the subject of numerous research publications (see e.g., [5], [7], [8], [16], [22], [23]). In general, (see [6], for instance) it is NP-hard to determine α and it is NP-complete to decide whether or not a graph G admits a K_3-WORM coloring. Moreover, it is a far more difficult problem to find the chromatic spectral values ζ_r, since one must first determine the feasible set. Recently, the chromatic spectra of some 2-trees given any K_3-WORM coloring have been found [2], [3]. Specifically, in [2], the chromatic spectral values in any K_s-WORM coloring of K_n have been determined. Unfortunately, there was an oversight in the estimates of some of these spectral values. In this article, we provide a result (Theorem 2) that resolves the issue by classifying these spectral values in relation to the well-know Stirling number of the second kind. Further, given any K_3-

WORM coloring of K_n, we found that the lower spectral values, ζ_2, are closely related to the well-known Catalan numbers. Other relations between the lower spectral value and the number of spanning trees of some cacti are also shown. In the last section, we extend the notion of H-WORM colorings to $(H_1;H_2)$-mixed colorings. In particular, some related results between $(K_r;K_s)$-mixed colorings of K_n and complete (r,s)-uniform mixed hypergraph colorings are established.

2 K_s-WORM Colorings

The *Stirling number of the second kind* (see for e.g., [17]) which we denote by $\left\{ {n \atop r} \right\}$, counts the r-partitions of a set of order n. Clearly $\left\{ {n \atop 1} \right\} = 1 = \left\{ {n \atop n} \right\}$ for $n \geq 1$. It is often computed with the identity $\left\{ {n \atop r} \right\} = \dfrac{1}{r!}\sum_{j=0}^{r}(-1)^n\binom{r}{j}(r-j)^n$. There are several other well-known combinatorial identities and generating functions on the Stirling numbers of the second kind that can be found in [11]. Recently ([3]), the Stirling numbers of the second kind appeared in the chromatic spectrum of some K_s-WORM colorings of K_n. However, many of the proposed spectral values turned out to be significantly less than the Stirling numbers of the second kind. The main result of this section outlines the spectral values that are equal to the Stirling numbers of the second kind and those that are not.

We begin with a restatement of the original work in [3] concerning the spectral values.

Theorem 1. *[[3], **Theorem 2.1**] The partition vector in a K_s-WORM coloring of K_n is $(\zeta_{\lfloor \frac{n}{s-1} \rfloor}, \ldots, \zeta_{s-1})$, where $\zeta_r = \left\{ {n \atop r} \right\}$ for all $3 \leq s < n$.*

Here is one simple counterexample to Theorem 1.

Let $n = 4$ and $s = 3$. Clearly, $3 \leq s < n$. Suppose there is a K_3-WORM coloring of K_4 and let $V = \{a,b,c,d\}$, where $V = V(K_4)$. The total number of 2-partitions (with no constraint) of V is $\left\{ {4 \atop 2} \right\} = 7$. Now, consider ζ_2, the number of 2-partitions.

Claim. $\zeta_2 < \left\{ {4 \atop 2} \right\}$ (when $r = 2$).

Proof. It suffices to show that there is a 2-partition that fails to satisfy the K_3-WORM coloring condition of K_4: no monochrome K_3 and no rainbow K_3. Suppose $V = \{a, b, c, d\}$. Clearly, $\{\{a, b, c\}, \{d\}\}$ is a 2-partition of V that contains a subset of size 3, giving a monochrome K_3. Thus, it is not a proper 2-partition of V. So, the number of all proper 2-partitions of V must be smaller than $\left\{ {4 \atop 2} \right\}$, the number of all 2-partitions of V. $\square$

Later, in Corollary 6, we show that $\zeta_2 = \frac{4(4-1)}{4} = 3$, in which case the 2-partitions of V that satisfy the K_3-WORM coloring condition are $\{\{a, b\}, \{c, d\}\}$, $\{\{a, c\}, \{b, d\}\}$ and $\{\{a, d\}, \{b, c\}\}$. The four remaining 2-partitions that fail to satisfy the K_3-WORM coloring condition are: $\{\{a\}, \{b, c, d\}\}$, $\{\{b\}, \{a, c, d\}\}$, $\{\{c\}, \{a, b, d\}\}$, $\{\{d\}, \{a, b, c\}\}$.

Lemma 1. *There is a K_s-WORM vertex coloring of K_n if and only if $n \le (s-1)^2$ for all $3 \le s \le n$.*

Proof. Suppose there is a K_s-WORM coloring of K_n. There is an r-partition of $[n]$ such that no block contains s or more elements from $[n]$. It follows that $n \le r(s-1)$. Moreover, $r \le s-1$, or else some subgraph $K_s \subseteq K_n$ is rainbow. Conversely, suppose that there is a K_s-WORM vertex coloring of K_n and $n \ge (s-1)^2+1$. Let $A = [n]$. Partition the elements of A into $(s-1)$-blocks, each containing $(s-1)$ elements. Any remaining element, since there is at least one, say $x \in A$, must be added to the elements of one of blocks, giving a monochrome s-block, or else, at least one extra block is needed for x, giving a rainbow s-set. Hence, $n \le (s-1)^2$.

$\square$

Remark 1.

A result similar to Lemma 1 had been first proved by Goddard, Wash, and Xu in [9]. Also, from the previous two results, it is clear that not every K_n admits a K_s-WORM coloring for some $s < n$. For example, there is no K_2-WORM coloring of K_5 even though there is a

2-partition of K_5. Moreover, it is easy to see that an $(s-1)$-partition of $[n]$ is a K_s-WORM coloring of K_n if every block is of size $s-1$ or less. Thus, it is clear that an r-partition of $[n]$ is a K_s-WORM coloring of K_n if and only if $r \le s-1$ and every block is of size $s-1$ or less. Such partition will be said to be *representative* of a K_s-WORM coloring of K_n.

The next result is simply a restatement of Lemma 1.

Corollary 1. *Suppose $K_s \subseteq K_n$. There is a K_s-WORM vertex coloring of K_n if and only if $\lceil \sqrt{n} \rceil < s \le n$ for all $n \ge 3$.*

Corollary 2. *The feasible set in a K_s-WORM vertex coloring of K_n is $F = \{\lceil \frac{n}{s-1} \rceil, \dots, s-1\}$, for all $s > \lceil \sqrt{n} \rceil$.*

Proof. Suppose there is a K_s-WORM vertex coloring of K_n, in which case $\lceil \sqrt{n} \rceil < s \le n$. It follows that each block is of size at most $s-1$, giving at least $\lceil \frac{n}{s-1} \rceil$ distinct blocks for all $n \ge 3$. □

Here, we correct Theorem 1 with Theorem 2 which identifies the spectral values that are actually equal to the Stirling numbers of the second kind, given any K_s-WORM coloring of K_n, $n \ge 3$.

Theorem 2. *Suppose ζ_i denotes a spectral value in a K_s-WORM coloring of K_n. Then $\zeta_i \le \left\{ {n \atop i} \right\}$ with equality if and only if $n \le s+i-2$, for $2 \le i < s \le n$.*

Proof. Suppose there is a K_s-WORM coloring of K_n. It is clear that $\zeta_r \le \left\{ {n \atop r} \right\}$, $r < s \le n$ since every K_s-WORM coloring of K_n is a partition of $[n]$. Consider all r-partitions of $[n]$, for some $r \le s-1$. If all partitions have only blocks of size less than s, in which case they are representatives of a K_s-WORM coloring of K_n, then there are exactly $\left\{ {n \atop r} \right\}$ such partitions.

Now, consider any r-partition of $[n]$, with $r \le s-1$. If $r < s-1$, we simultaneously remove elements from some existing blocks to form new blocks of the partition until we attain the maximum number of allowable blocks, when $r = s-1$; this is always possible since $r < s \le n$. Let $\mathscr{P} = \dot{\bigcup}_{j=1}^{r} A_j$ denote the newly formed r-partition of $[n]$, with $r = s-1$, such that $|A_i| \leq |A_k|$, for $1 \leq i < k \leq r$. $\mathscr{P}$ is representative of a K_s-WORM coloring of K_n and without loss of generality, we can assume that there is a block $A_r \in \mathscr{P}$ such that $|A_r| = s-1$; otherwise, the result follows easily from the next argument.

Suppose that $|A_t| \geq 2$, for some t, with $1 \leq t \leq r-1$. Let $x_t \in A_t$ and define $A'_t = A_t - \{x_t\}$, and $A'_r = A_r \cup \{x_t\}$. Let $\mathscr{P}' = \dot{\bigcup}_{j=1}^{r-2} A_j \cup A'_t \cup A'_r$ is an r-partition. This implies $|A'_r| = s$ in which case $\mathscr{P}'$ is not representative of a K_s-WORM coloring of K_n. Thus, given some $|A_j| = s-1$, every r partitions of $[n]$ are representives of K_s-WORM colorings of K_n if and only if $|A_1| = \ldots = |A_{r-1}| = 1$, for each $A_j \in \mathscr{P}$, $1 \leq j \leq r-1$. Hence, it must be that $n \leq s+r-2$ whenever the number of r-partitions in a K_s-WORM coloring of K_n is $\left\{ {n \atop r} \right\}$, giving the result for all $2 \leq r < s \leq n$. □

The next result is a special case of Theorem 2 when $n = s$.

Corollary 3. *Given a K_n-WORM coloring of K_n, the chromatic spectrum is $(\zeta_2, \ldots, \zeta_{n-1})$, where $\zeta_i = \left\{ {n \atop i} \right\}$ with $2 \leq i \leq n-1$.*

Proof. By definition, every r-partition of $[n]$ represents a coloring of K_n. Further, with $2 \leq r \leq n-1$, no r-partition of $[n]$ contains a block of size n, by the pigeonhole principle. Clearly, if either $r = 1$ or $r = n$, then $[n]$ becomes monochrome or rainbow, respectively. □

Corollary 4. *Given any K_s-WORM coloring of K_n, if $s \geq \lceil \frac{n+3}{2} \rceil$, then the chromatic spectrum is $(\zeta_2, \ldots, \zeta_{s-1})$, where $\zeta_i = \left\{ {n \atop i} \right\}$ with $2 \leq i \leq s-1$.*

Proof. The result follows from Theorem 2 when $n \leq s+r-2$ and the fact that $r \leq s-1$.

□

The next result follows from Corollary 2 and Corollary 4, for all $n \geq 4$.

Corollary 5. *Given any K_s-WORM coloring of K_n, if $\lceil\sqrt{n}\rceil < s < \lceil\frac{n+3}{2}\rceil$, then the chromatic spectrum is $(\zeta_{\lceil\frac{n}{s-1}\rceil},\dots,\zeta_{s-1})$, where $\zeta_i < \left\{ {n \atop i} \right\}$ for all $\lceil\frac{n}{s-1}\rceil \le i \le s-1$.*

In the case when $n = 3$ it is clear that we obtain $\zeta_2 = \left\{ {3 \atop 2} \right\}$, showing that the bounds on s in the previous two results are tight.

In Figure 1, we summarize the results on the feasible sets, the chromatic spectral values given any K_s-WORM coloring of K_n, $n \ge 3$.

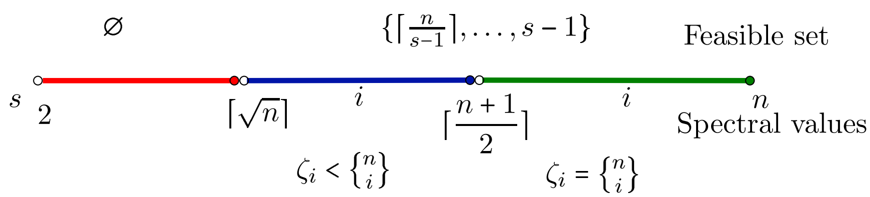

Figure 1. K_s-WORM coloring of K_n, for $3 \le s \le n$

In the next result we give the lower spectral values, ζ_2, when $\zeta_2 < \left\{ {n \atop 2} \right\}$; these values are related to the Catalan numbers as shown in a remark after the result. We note here that, for $r \ge 3$, the exact values of ζ_r when $s < \lceil\frac{n+3}{2}\rceil$ remain to be found.

Corollary 6. *Suppose there is a K_s-WORM coloring of K_n. If $s = \lceil\frac{n}{2}\rceil + 1$, then the chromatic spectrum is $(\zeta_2,\dots,\zeta_{s-1})$, where*

$$\zeta_2 = \begin{cases} \binom{n}{\frac{n}{2}}/2, & \text{if } n \text{ is even} \\ \binom{n}{\lceil\frac{n}{2}\rceil}, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Suppose there is a K_s-WORM coloring of K_n. Because $\lceil\frac{n}{2}\rceil+1 < \lceil\frac{n+3}{2}\rceil$ for all $n \ge 4$, it is clear from Corollary 5 that $0 < \zeta_2 < \left\{ {n \atop 2} \right\}$. Now, consider a 2-partition of $[n]$ that is representative of a K_s-WORM coloring. Since $s = \lceil \frac{n}{2} \rceil + 1$, the blocks of the partition must be of sizes $\lceil \frac{n}{2} \rceil$ and $\lfloor \frac{n}{2} \rfloor$. When n is even, the result follows by symmetry, when considering all $\frac{n}{2}$-subsets of $[n]$ to form one block of the partition, leaving the remaining (half) elements of $[n]$ for the other block. On the other hand, when n is odd, each $\lceil \frac{n}{2} \rceil$ subset of $[n]$ paired with the remaining $\lfloor \frac{n}{2} \rfloor$ elements of $[n]$ form distinct 2-partitions, giving the result.

□

Remark 2.

The k^{th} *Catalan number* is known to be given by $C_k = \frac{1}{k+1}\binom{2k}{k}$; besides having numerous combinatorial meanings (see for e.g., [19]), C_k also counts the number of triangulations of a convex $(k+2)$-gon. Further, from Corollary 6 we establish the following relation between 2-partitions in a K_{s+1}-WORM coloring of K_{2s}, and the k^{th} Catalan number. Namely,

$$\zeta_2 = \frac{(s+1)}{2} C_s.$$

Thus, the number of 2-partitions of a K_{s+1}-WORM coloring of K_{2s} is $\dfrac{(2s)!}{2(s!)^2}$, $s \geq 2$. In the case when $s = 2$ we obtain $\zeta_2 = 3$, a value that is supported by both Corollary 6 and the remark following the counterexample given at the beginning of this section.

Suppose $\tau(G)$ denotes the number of spanning trees of a graph G. It is clear that $\tau(G) = 1$ for any acyclic graph, and when $G = C^m$, which we denote a cycle on m vertices, $\tau(G) = m$. Further, it is well-known (see for e.g., [1]) that if $G = K_{m,n}$, a complete bipartite graph, then $\tau(G) = m^{n-1}n^{m-1}$. Here, we show a close relation between K_3-WORM colorings and spanning trees. We recall that the *girth* of a graph is the length of its smallest cycle and a *cactus* is a simple connected graph in which every pair of cycles share at most one vertex. Figure 2 shows some cacti with girth 3.

Proposition 1. *Suppose G is a cactus with k cycles of length $m_1, m_2, \ldots, m_k$. Then the number of its spanning trees is $\tau(G) = \prod_{i=1}^{k} m_i$, for each $m_i \geq 3$.*

Proof. If $G = C^{m_1}$, then $\tau(G) = m_1$, each spanning tree is obtained by deleting exactly one edge from C^{m_1}. Because every two cycles share exactly one vertex, the argument follows by induction on $k \geq 1$. ☐

Suppose G denotes a simple graph and $\mathscr{F} = \{F_1, \ldots, F_k\}$ is a collection of distinct subgraphs $F_i \subseteq G$, $1 \leq i \leq k$. An *$\mathscr{F}$-WORM* coloring of G is the coloring of the vertices of G such that no copy of $F_i \subseteq G$ is monochrome or rainbow. This notion was first introduced in [2] as a generalization of F-WORM colorings. For the next two results, given a cactus G, we denote by $\mathscr{C} = \{C^1, C^2, \ldots, C^k\}$ the collection of all distinct cycles $C^i \subseteq G$.

Proposition 2. *Let G denote a cactus of order n. If there is a $\mathscr{C}$-WORM coloring of G, then the feasible set is $F = \{2, \ldots, n-k\}$, with $1 \leq i \leq k$.*

Proof. Suppose there is a $\mathscr{C}$-WORM coloring of G. It follows that no C^i is rainbow or monochrome for each $1 \leq i \leq k$. We give the colorings or partitions that produce the infimum and the supremum of F. Color the vertices of G in such a way that, for each of the k cycles, exactly two vertices are monochrome while all other vertices of G are kept rainbow. This gives the supremum. On the other hand, select a pair of vertices from each of the k cycles. Color all such pairs with a single color, and any other (remaining) vertices of G with another color. This gives the infimum of G. In both cases it is easy to verify that each coloring is representative of a proper $\mathscr{C}$-WORM coloring. ☐

Proposition 3. *Suppose G denotes a cactus with k cycles, each of length $m_1, m_2, \ldots, m_k$, $m_i \geq 3$. The number of 2-partitions in a $\mathscr{C}$-WORM coloring of G is at least $\prod_{i=1}^{k} m_i$, the number of its spanning trees.*

Proof. In every spanning tree of a graph G, exactly one edge $e_i \in E(C^{m_i})$, for each $1 \leq i \leq k$, is removed to create an acyclic connected graph. Let A_1 be the block whose elements are the endpoints of e_i, $1 \leq i \leq k$, and let A_2 be the block that contains any remaining vertices of G. Clearly no $C^{m_j} \subseteq G$ is monochrome or rainbow, and the collection $\{A_1, A_2\}$ is a 2-partition of G. Hence the number of 2-partitions $\zeta_2(G) \geq \tau(G)$, and the result follows from Proposition 1.

□

Proposition 4. *Suppose G is a cactus with k cycles each of length 3. If ζ_2 counts the 2-partitions of G in a K_3-WORM coloring, then $\zeta_2 \geq 3^k$ with equality if and only if G is bridgeless.*

Proof. Suppose G is a bridgeless cactus with k cycles each of length 3 and consider a K_3-WORM coloring. It follows from Proposition 3 that $\zeta_2 \geq 3^k$. Further, a 2-coloring of each $K_3 \subseteq G$ is a 2-coloring of G since no other vertex of G lies outside some K_3. This implies that $\zeta_2 \leq 3^k$. Hence the equality.

□

Remark 3.

Proposition 4 points out that a K_3-WORM coloring of G is equivalent to a graph operation that yields spanning trees provided G is bridgeless (see Figure $2(c)$)–In every 2-coloring of $K_3 \subseteq G$, exactly one pair of vertices share the same color, in which case the edge incident to those vertices can be considered “deleted”–Because no edge is a bridge, the resulting graph G' remains connected. Also since G is a cactus, no edge is shared by two or more cycles, and the removal of an edge does not induce any (larger) cycle as a subgraph. Therefore the resulting graph G' contains no cycle and yet, it includes all vertices of G, giving a spanning tree.

Figure 2 helps illustrate this previous remark. Both Figures $2(a)$ and $2(b)$ show some K_3-WORM 2-colorings that do not represent spanning trees; in both cases $\zeta_2 \geq 3^3$. Further, in Figure $2(c)$, G is bridgeless and $\zeta_2 = 3^4$.

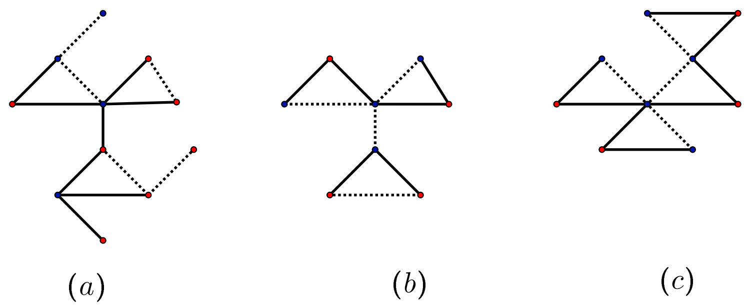

Figure 2. Spanning trees and 2-colorings of some cacti

Remark 4.

It is shown in [2] that when $G = \theta(1,2,\ldots,2)$, an $(n-1)$-bridge, then $\zeta_2(G) = 2^{n-2}+1$. Further when $G = F_n$, a Fan on $n \geq 3$ vertices, $\zeta_2 = \dfrac{1}{\sqrt{5}}\Big[\dfrac{\beta-2}{\beta}\alpha^n - \dfrac{\alpha-2}{\alpha}\beta^n\Big]$, a shifted Fibonacci number, with $\alpha = \dfrac{1+\sqrt{5}}{2}$ and $\beta = \dfrac{1-\sqrt{5}}{2}$.

Although, given any K_3-WORM coloring of G, $\tau(G) = \zeta_2(G)$ when G is a bridgeless cactus of girth 3, it is also true that, for any acyclic graph G, $1 = \tau(G) < \zeta_2(G)$. With this observation, it is tempting to claim that, for any graph G with girth 3, the number of its spanning trees $\tau(G) \leq \zeta_2(G)$, the number of its 2-partitions in an K_3-WORM coloring. However, the next proposition shows that it is not always the case. In particular, $\zeta_2(G) < \tau(G)$ when $G = \theta(1,2,\ldots,2)$.

Proposition 5. *If $G = \theta(1,2,\ldots,2)$, an $(n-1)$-bridge on $n \geq 3$ vertices, then $\tau(G) = n2^{n-3}$.*

Proof. Consider the path $\{u,v\} \subset E(G)$ where every other vertices of G are adjacent to both u and v. In every spanning tree of G, either (i) the edge uv is deleted or (ii) uv is kept, in which case, for each K_3 which necessarily includes uv, exactly one of the edges incident to either u or v is deleted. From case (i), it follows that the resulting graph is $K_{n-2,2}$

where u and v are the vertices of part size 2 and $\tau(K_{n-2,2})$ counts the number of its spanning trees. In case (ii), this is equivalent to a 2-partition of G, given a K_3-WORM coloring. However, $\zeta_2(G)$ includes the case when $c(u) = c(v)$, and there is exactly one such partition which we remove since it falls under case (i). Together, we have

$$\begin{aligned} \tau(G) &= \tau(K_{n-2,2}) + \zeta_2(G) - 1 \qquad (1) \\ &= (n-2)2^{n-3} + 2^{n-2} \\ &= n2^{n-3}, \end{aligned}$$

giving the result for all $n \geq 3$. □

It is clear from equation 1 that when $n = 3$, the equality $\zeta_2(G) = 3 = \tau(G)$ holds since $\tau(K_{1,2}) = 1$ and for all $n \geq 4$, the inequality $\zeta_2(G) < \tau(G)$ holds.

Example 1.

Given Proposition 5, the case when $n = 4$, i.e., when $G = \theta(2,1,1)$, is illustrated by Figure 3. We show all spanning trees of G (after the arrows), and $\tau(G) = 8$. The spanning trees in case (ii), when $c(u) = c(v)$, are not representative of K_3-WORM colorings; each pair of deleted edges would yield a corresponding monochrome K_3. Also, $\zeta_2(G) = 5$ which are obtained from case (i), when $c(u) \neq c(v)$, and one additional graph from case (ii) which has exactly one edge deleted; this produces a C_4, which is trivially counted as a 2-partition in a K_3-WORM coloring of G.

3 $(K_r; K_s)$-mixed coloring

A *hypergraph* $\mathscr{H}$ is an ordered pair $(\mathscr{V}, \mathscr{E})$, where $\mathscr{V}$ is a finite set of vertices with *order* $|\mathscr{V}| = n$ and $\mathscr{E}$ is a collection of nonempty subsets of $\mathscr{V}$, called *hyperedges*. When $\mathscr{E}$ is a collection of all nonempty s-subsets of $\mathscr{V}$, then $\mathscr{H}$ is called a *complete s-uniform* hypergraph.

Hypergraphs are extensively used in machine learning techniques, data mining and information retrieval tasks such as clustering and classification. See for instance [12]–[14].

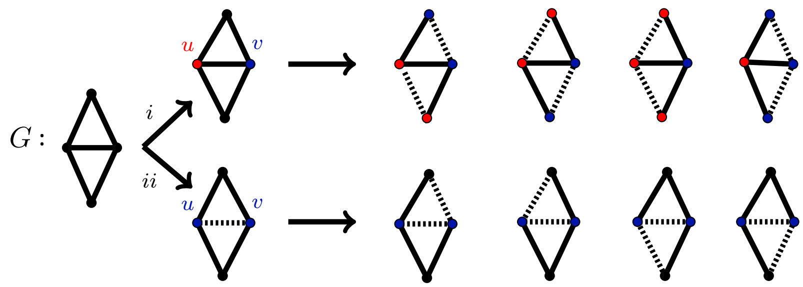

Figure 3. K_3-WORM 2-colorings and the spanning trees of $G = \theta(2,1,1)$

Given any vertex coloring of $\mathscr{H}$, if no hyperedge $e \in \mathscr{E}$ is monochrome, $\mathscr{H} = (\mathscr{V}, \mathscr{D})$ is called a *$\mathscr{D}$-hypergraph* which is the classic hypergraph vertex coloring. When no hyperedge $e \in \mathscr{E}$ is rainbow, $\mathscr{H} = (\mathscr{V}, \mathscr{C})$ is called a *cohypergraph*. In the event no hyperedge $e \in \mathscr{E}$ is monochrome or rainbow, $\mathscr{H} = (\mathscr{V}, \mathscr{B})$ is called a *bihypergraph*, where $\mathscr{B}$ is a nonempty intersection of $\mathscr{C}$ and $\mathscr{D}$ or when $\mathscr{C} = \mathscr{D}$. As a generalization of these different coloring constraints on the vertices of $\mathscr{V}$, it is customary to define a *mixed hypergraph* $\mathscr{H} = (\mathscr{V}, \mathscr{C}, \mathscr{D})$ as a triple such that $\mathscr{C}$ and $\mathscr{D}$ are (not necessarily distinct) subsets of $\mathscr{E}$. Mixed hypergraph colorings were first introduced by Voloshin (see for e.g., [8], [20]–[23]). They are often used to encode partitioning constraints and also to construct cyber security models [15], [18].

A mixed hypergraph $\mathscr{H}$ is said to be *uncolorable* if its feasible set $F = \varnothing$, in which case $\mathscr{H}$ admits no proper coloring. It is obvious that when $|e| \leq 2$ for some $e \in \mathscr{B}$, $\mathscr{H}$ is uncolorable, so we assume $|e| \geq 3$ for every $e \in \mathscr{B}$. Various classes of uncolorable mixed hypergraphs have been extensively discussed, including *complete (r,s)-uniform mixed hypergraphs* [7], [20], [23] which are complete uniform mixed hypergraphs such that, for every hyperedges $d \in \mathscr{D}$ and $c \in \mathscr{C}$, $|d| = r$ and $|c| = s$.

Remark 5.

With the concept of mixed hypergraph colorings, it is natural to define an $(H_1; H_2)$-*mixed (vertex) coloring* of a graph G as the coloring of the vertices of G such that no $H_1 \subseteq G$ is rainbow and no $H_2 \subseteq G$ is monochrome. In particular when $H_1 = H_2 = H$, an $(H; H)$-mixed coloring of G is an H-*WORM* coloring. Thus, by definition, a $(K_r; K_s)$-mixed coloring of K_n is equivalent to a proper coloring of a complete (r, s)-uniform mixed hypergraphs, and when $r = s$, a K_s-WORM coloring is a complete s-uniform bihypergraph. Within these contexts, we later state equivalent results without offering any additional proof. Further, it is clear that, if $\mathscr{P} = \dot{\bigcup}_{j=1}^{k} A_j$ is a k-partition of $[n]$, then $\mathscr{P}$ is *representative* of a $(K_r; K_s)$-mixed coloring of K_n if and only if $k < r$ and $|A_j| < s$.

Proposition 6. *There exists a set of k positive integers $t_1, \ldots, t_k$ such that $\sum_{\substack{i=1 \\ t_i < s}}^{r-1} t_i = n$ if and only if $n \le (s-1)(r-1)$, $2 \le r \le s \le n$.*

Proof. Take $r-1$ integers, say t_i's, such that each $t_i \le s-1$. They add up to at most $(s-1)(r-1)$, giving the result. □

Lemma 2. *There is a (K_r, K_s)-mixed coloring of K_n if and only if $n \le (s-1)(r-1)$, $2 \le r \le s \le n$.*

Proof. Let $\mathscr{P} = \dot{\bigcup}_{j=1}^{k} A_j$ denote an k-partition of $[n]$ such that $|A_i| < s$. For all $k < r$, let $|A_i| = t_i$ with $3 \le t_i < s$ and the result follows from Proposition 6. □

Lemma 3. *A complete (r, s)-uniform mixed hypergraph of order n is colorable if and only if $n \le (s-1)(r-1)$, $2 \le r \le s \le n$.*

We note here that Zs. Tuza and V. Voloshin ([20], Theorem 8) were first to prove the negation of the statement in Lemma 3, and Lemma 1 is a special case when $r = s$. We state the equivalent statement of this special case in the next Lemma.

Lemma 4. *A complete s-uniform bihypergraph of order $n \geq 3$ is colorable if and only if $n \leq (s-1)^2$, for $3 \leq s \leq n$.*

Corollary 7. *Suppose C_s denotes the s^{th} Catalan number for all $s \geq 2$. The number of 2-colorings of a complete $(s+1)$-uniform bihypergraph of order $2s$, from a list of $r \geq 2$ colors is $\binom{r}{2}(s+1)C_s$.*

Proof. Following Remark 2, and the fact that there are exactly $r(r-1)$ ways of colorings the elements of each 2-partition, we have a total of $\dfrac{r(r-1)(s+1)}{2}C_s$ such colorings. □

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