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Total k-rainbow domination subdivision number in graphs

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Abstract

A total k-rainbow dominating function (TkRDF) of G is a function f from the vertex set $V(G)$ to the set of all subsets of the set $\{1,\ldots,k\}$ such that (i) for any vertex $v \in V(G)$ with $f(v) = \emptyset$ the condition $\bigcup_{u\in N(v)} f(u) = \{1,\ldots,k\}$ is fulfilled, where $N(v)$ is the open neighborhood of v, and (ii) the subgraph of G induced by $\{v \in V(G) \mid f(v) \neq \emptyset\}$ has no isolated vertex. The total k-rainbow domination number, $\gamma_{trk}(G)$, is the minimum weight of a TkRDF on G. The total k-rainbow domination subdivision number $\mathrm{sd}_{\gamma_{trk}}(G)$ is the minimum number of edges that must be subdivided (each edge in G can be subdivided at most once) in order to increase the total k-rainbow domination number. In this paper, we initiate the study of total k-rainbow domination subdivision number in graphs and we present sharp bounds for $\mathrm{sd}_{\gamma_{trk}}(G)$. In addition, we determine the total 2-rainbow domination subdivision number of complete bipartite graphs and show that the total 2-rainbow domination subdivision number can be arbitrary large.

Keywords: total k-rainbow domination, total k-rainbow domination subdivision number, k-rainbow domination.
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1 Introduction

In this paper, G is a simple graph with vertex set $V(G)$ and edge set $E(G)$ (briefly V and E). For every vertex $v \in V$, the *open neighborhood*

$N_G(v) = N(v)$ is the set $\{u \in V \mid uv \in E\}$ and the *closed neighborhood* of v is the set $N_G[v] = N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $\deg_G(v) = \deg(v) = |N(v)|$. The *minimum degree* and the *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. A *leaf* is a vertex of degree one, a *support vertex* is a vertex adjacent to a leaf and a strong support vertex is a vertex adjacent to at least two leaves. Let L_v denote the set of leaves adjacent to the vertex v.

A subset S of vertices of G is a *dominating (total dominating) set* if $N[S] = V$ $(N(S) = V)$. The *domination (total domination) number* $\gamma(G)$ $(\gamma_t(G))$ is the minimum cardinality of a (total) dominating set of G. A (total) dominating set with cardinality $\gamma(G)$ $(\gamma_t(G))$ is called a $\gamma(G)$-set ($\gamma_t(G)$-set). The domination and its variations have been attracted considerable attention and surveyed in three books [20], [21]. Velammal [26] defined the *domination subdivision number* $sd_\gamma(G)$ to be the minimum number of edges that must be subdivided (where each edge in G can be subdivided at most once) in order to increase the domination number. The domination subdivision number has been studied by several authors (see for instance [7], [18]). Similar concepts related to connected domination were studied in [17], to total domination in [16], [19], [22], to Roman domination in [5], [6], [8], to rainbow domination in [12], [15], to weakly convex domination in [13] and to convex domination in [14].

Let k be a positive integer, and let $[k] := \{1, 2, \ldots, k\}$. A function $f : V(G) \to 2^{[k]}$ is a *k-rainbow dominating function* (kRDF) of G if for each vertex $v \in V(G)$ with $f(v) = \emptyset$, the condition $\bigcup_{u \in N(v)} f(u) = [k]$ is fulfilled. The *weight* of a kRDF f on G is $\omega(f) = \sum_{v \in V(G)} |f(v)|$. The *$k$-rainbow domination number* of G, $\gamma_{rk}(G)$, is the minimum weight of a kRDF on G. The k-rainbow domination number was introduced by Brešar, Henning and Rall [9] and has been studied by several authors [6], [10], [11], [23]–[25].

A k-rainbow dominating function f on G, is called a *total k-rainbow dominating function* (TkRDF) if the subgraph of G induced by the set $\{v \in V(G) \mid f(v) \neq \emptyset\}$ has no isolated vertex. The *total k-rainbow domination number* of G, $\gamma_{trk}(G)$, is the minimum weight of a $Tk RDF$

of G. A TkRDF f of G with weight $\gamma_{trk}(G)$ is called a $\gamma_{trk}(G)$-function. Note that $\gamma_{tr1}(G)$ is equal to the classical total domination number, denoted by $\gamma_t(G)$. The total k-rainbow domination has been studied in [1]–[3].

The *total k-rainbow domination subdivision number* $\mathrm{sd}_{\gamma_{trk}}(G)$ of a graph G is the minimum number of edges that must be subdivided (where each edge in G can be subdivided at most once) in order to increase the total k-rainbow domination number of G. (An edge $uv \in E(G)$ is subdivided if the edge uv is deleted, but a new vertex x is added, along with two new edges ux and vx. The vertex x is called a subdivision vertex). Observation 1 below shows that the total k-rainbow domination number of graphs cannot decrease when an edge of graph is subdivided.

The purpose of this paper is to initiate the study of the total k-rainbow domination subdivision number in graphs. We first present some sharp bounds on $\mathrm{sd}_{\gamma_{trk}}(G)$, and then determine the total 2-rainbow domination subdivision number of complete bipartite graphs. In addition, we show that the total 2-rainbow domination subdivision number can be arbitrary large. Although it may not be immediately obvious that the total k-rainbow domination subdivision number is defined for all graphs without isolated vertices, we will show this shortly (see Corollary 4).

We make use of the following results in this paper.

Proposition A. *[2] For any graph G of order n without isolated vertices*

$$min\{n, k, \gamma_{rk}(G), \gamma_t(G)\} \leq \gamma_{trk}(G) \leq k\gamma_t(G).$$

Proposition B. *[2] Let $k \geq 2$ be an integer, and let G be a graph of order $n \geq k$. Then $\gamma_{trk}(G) = k$ if and only if $n = k$ and there exists a set $A = \{v_1, v_2, \ldots, v_t\} \subseteq V(G)$ with $2 \leq t \leq k$ such that the induced subgraph $G[A]$ has no isolated vertex and $V(G) - A \subseteq N(v_i)$ for each $1 \leq i \leq t$.*

Proposition C. *[2]* For $n \geq 3$, $\gamma_{tr2}(C_n) = \lceil \frac{2n}{3} \rceil$.

Corollary 1. For $n \geq 3$,

$$\mathrm{sd}_{\gamma_{tr2}}(C_n) = \begin{cases} 1 & \text{if} \quad n \equiv 0, 1 \pmod 3 \\ 2 & \text{if} \quad n \equiv 2 \pmod 3. \end{cases}$$

Proposition D. *[2]* For $n \geq 2$, $\gamma_{tr2}(P_n) = \lceil \frac{2n+2}{3} \rceil$.

Corollary 2. For $n \geq 2$,

$$\mathrm{sd}_{\gamma_{tr2}}(P_n) = \begin{cases} 1 & \text{if} \quad n \equiv 0, 2 \pmod 3 \\ 2 & \text{if} \quad n \equiv 1 \pmod 3. \end{cases}$$

Proposition E. *[2]* If $k \geq 3$ and $n \geq 3$, then $\gamma_{trk}(P_n) = \gamma_{trk}(C_n) = n$.

Corollary 3. *If $k \geq 3$ and $n \geq 3$, then* $\mathrm{sd}_{\gamma_{trk}}(P_n) = \mathrm{sd}_{\gamma_{trk}}(C_n) = 1$.

Observation 1. Let G be a graph and $u \in V(G)$ be a support vertex with a leaf neighbor v. If f is a $\gamma_{trk}(G)$-function, then $|f(u)| + |f(v)| \geq 2$.

2 Bounds and exact values

In this section we present basic results on the total k-rainbow domination subdivision number in graphs. Our first result shows that the total k-rainbow domination number of a graph can not be decreased by subdividing an edge.

Proposition 1. Let G be a simple connected graph of order $n \geq 3$ and $e = uv \in E(G)$. If G' is obtained from G by subdividing the edge $e = uv$ with vertex x, then $\gamma_{trk}(G') \geq \gamma_{trk}(G)$.

Proof. Let f be a $\gamma_{trk}(G')$-function. If $f(x) = \emptyset$, then $f|_{V(G)}$ is a TkRDF of G. Let $f(x) \neq \emptyset$. Since f is a TkRDF of G', we have $\max\{|f(u)|, |f(v)|\} \geq 1$. Suppose without loss of generality that $|f(v)| \geq 1$. Define $g : V(G) \to 2^{[k]}$ by $g(u) = f(u) \cup f(x)$, and $g(x) = f(x)$ otherwise. Obviously g is a TkRDF of G with weight $\gamma_{trk}(G')$ and so $\gamma_{trk}(G') \geq \gamma_{trk}(G)$. $\square$

Theorem 1. *Let G be a graph and $u \in V(G)$ be a vertex with degree at least two. Then* $\mathrm{sd}_{\gamma_{trk}}(G) \leq \deg(u)$.

Proof. Let $N(u) = \{u_1, u_2, \ldots, u_r\}$ and let G' be the graph obtained from G by subdividing the edges $uu_1, \ldots, uu_r$ with subdivision vertices $x_1, \ldots, x_r$, respectively. Suppose f is a $\gamma_{trk}(G)$-function. If $f(u) = \{1, 2, \ldots, k\}$, then in order that u to be totally rainbow dominated, we may assume that $|f(x_1)| \geq 1$, and the function $g : V(G) \to 2^{[k]}$ defined by $g(u) = \{1\}$, $g(u_i) = f(u_i) \cup f(x_i)$ for $1 \leq i \leq r$, and $g(x) = f(x)$ otherwise, is a TkRDF of G of weight less than $\gamma_{trk}(G')$. If $1 \leq |f(u)| \leq k-1$, then in order that u to be totally rainbow dominated, we can assume that $|f(x_1)| \geq 1$, and the function $g : V(G) \to 2^{[k]}$ defined by $g(u) = f(x_1)$, $g(u_i) = f(u_i) \cup f(x_i)$ for $2 \leq i \leq r$, and $g(x) = f(x)$ otherwise, is a TkRDF of G of weight less than $\gamma_{trk}(G')$. Henceforth we assume that $f(u) = \emptyset$. In order that u to be rainbow dominated, we must have $\sum_{i=1}^{r} |f(x_i)| \geq k$, and in order that x_i to be totally rainbow dominated, we must have $|f(u_i)| \geq 1$ for each i. Then the function $g : V(G) \to 2^{[k]}$ defined by $g(u) = \{1\}$, and $g(x) = f(x)$ otherwise, is a TkRDF of G of weight less than $\gamma_{trk}(G')$, and this implies that $\mathrm{sd}_{\gamma_{trk}}(G) \leq \deg(u)$. □

The following results are immediate consequences of Theorem 1.

Corollary 4. If $k \geq 2$ is an integer and G is a connected graph of order $n \geq 2$, then

$$\mathrm{sd}_{\gamma_{trk}}(G) \leq \Delta(G).$$

Furthermore, this bound is sharp for C_n when $n \equiv 2 \pmod 3$ and $k = 2$.

Corollary 4 shows that the total k-rainbow domination subdivision number is well-defined for all non-trivial graphs when $k \geq 2$.

Corollary 5. If $k \geq 2$ is an integer and G is a connected graph with $\delta(G) \geq 2$, then

$$\mathrm{sd}_{\gamma_{trk}}(G) \leq \delta(G).$$

This bound is sharp for C_n when $n \equiv 2 \pmod 3$ and $k = 2$.

Corollary 6. If G is a graph and u, v are two adjacent vertices each of degree at least two, then $\mathrm{sd}_{\gamma_{trk}}(G) \leq \deg(u)+\deg(v)-|N(u)\cap N(v)|-1$.

Corollary 7. If G is a connected graph of $n \geq 3$ and $v \in V(G)$ is a support vertex, then $\mathrm{sd}_{\gamma_{trk}}(G) \leq \deg(v)$.

Ahangar et al. [2] proved that for any connected graph G of order $n \geq 3$, $\gamma_{trk}(G) \leq n - \delta(G) + 2$. Using this bound and Theorem 1 we obtain the next result.

Corollary 8. For any connected graph G with $\delta(G) \geq 2$,

$$\mathrm{sd}_{\gamma_{trk}}(G) \leq n - \gamma_{trk}(G) + 2.$$

Moreover, Ahangar et al. [2] showed that for any connected graph G of order $n \geq 3$, $\gamma_{trk}(G) \geq \lceil \frac{kn}{\Delta(G)+k-1} \rceil$. Applying this lower bound and Corollary 8, the next result follows.

Corollary 9. If G is a connected graph with $n \geq 3$, then $\mathrm{sd}_{\gamma_{trk}}(G) \leq n - \left\lceil \frac{kn}{\Delta(G)+k-1} \right\rceil + 2$.

Now we provide some sufficient conditions to have small total k-rainbow domination number.

Proposition 2. If $k \geq 2$ is an integer and G contains a strong support vertex, then

$$\mathrm{sd}_{\gamma_{trk}}(G) = 1.$$

Proof. Let v be a strong support vertex of G and let $v_1, v_2 \in L_v$. Assume that G' is the graph obtained from G by subdividing the edge vv_1 with vertex x. Suppose f is a $\gamma_{trk}(G')$-function. By Observation 1, we have $|f(x_1)| + |f(v_1)| \geq 2$, $|f(v)| + |f(v_2)| \geq 2$ and $|f(v)| \geq 1$. Define $g : V(G) \to 2^{[k]}$ by $g(v_1) = \{1\}$, and $g(x) = f(x)$ otherwise. Clearly, g is a TkRDF of G with weight smaller than $w(f)$, implying that $\mathrm{sd}_{\gamma_{trk}}(G) = 1$. □

Proposition 3. Let $n > k \geq 2$ be integers and G a simple graph with $\gamma_{trk}(G) = k$. Then $\mathrm{sd}_{\gamma_{trk}}(G) = 1$.

Proof. Suppose $e = uv$ is an edge of G and G' is the graph obtained from G by subdividing the edge uv with subdivision vertex x. We show that $\gamma_{trk}(G') > \gamma_{trk}(G)$. Suppose, to the contrary, that $\gamma_{trk}(G') = \gamma_{trk}(G) = k$. By Theorem B, there exists a set $A = \{v_1, v_2, \ldots, v_t\} \subseteq V(G')$ with $2 \leq t \leq k$ such that the induced subgraph $G'[A]$ has no isolated vertex and $V(G') - A \subseteq N(v_i)$ for each $1 \leq i \leq t$. It follows that $x \in V(G) \setminus A$ and $A = \{u, v\}$. Since $G[A]$ has no isolated vertex, there must exist another edge $e' = uv$ in G which leads to a contradiction because G is simple. Thus $\gamma_{trk}(G') > \gamma_{trk}(G)$ and so $\mathrm{sd}_{\gamma_{trk}}(G) = 1$. $\square$

Proposition 4. *Let $k \geq 2$ be an integer and G be a connected graph of order $n \geq k + 2$ with $\gamma_{trk}(G) = k + 1$. Then* $\mathrm{sd}_{\gamma_{trk}}(G) \leq 2$.

Proof. The result is immediate for $n = 4$. Assume that $n \geq 5$. If G is a star, then by Proposition 2 we have $\mathrm{sd}_{\gamma_{trk}}(G) = 1$. Assume that G is not a star, and let $M = \{u_1v_1, u_2v_2\}$ be a matching in G. Let G' be the graph obtained from G by subdividing the edges u_1v_1, u_2v_2 with vertices x, y, and let f be a $\gamma_{trk}(G')$-function. We show that $\gamma_{trk}(G') \geq k+2$. If $f(x) = f(y) = \emptyset$, then we must have $f(u_i) \cup f(v_i) = \{1, 2, \ldots, k\}$ for $i = 1, 2$, and this implies that $\gamma_{trk}(G') \geq 2k \geq k + 2$ as desired. Suppose without loss of generality that $|f(x)| \geq 1$. Then, in order that x to be totally dominated, we may assume that $|f(u_1)| \geq 1$. Now if $f(y) = \emptyset$, then we have $f(u_2) \cup f(v_2) = \{1, 2, \ldots, k\}$, implying that $\gamma_{trk}(G') \geq k + 2$. Suppose $|f(y)| \geq 1$. If $f(z) \neq \emptyset$ for each $z \in V(G) \setminus \{u_1, u_2, v_1, v_2\}$, then, clearly, $\gamma_{trk}(G') \geq k+2$; and if $f(z) = \emptyset$ for some $z \in V(G) \setminus \{u_1, u_2, v_1, v_2\}$, then we have $\cup_{u \in N(z)} f(u) = \{1, \ldots, k\}$, and this implies that $\gamma_{trk}(G') \geq k + 2$ because $x, y \notin N(z)$. Thus, $\mathrm{sd}_{\gamma_{trk}}(G) \leq 2$, and the proof is complete. $\square$

Lemma 1. *Let $k \geq 2$ be an integer and G be a connected graph containing a triangle uvw. If G' is obtained from G by subdividing the edges uv, vw, wu with vertices x_1, x_2, x_3, respectively, then for any $\gamma_{trk}(G')$-function,* $|f(u)| + |f(v)| + |f(w)| + \sum_{i=1}^{3} |f(x_i)| \geq \min\{5, k+2\}$.

Proof. Let f be a $\gamma_{trk}(G')$-function such that $f(u)$ is as large as possible. By the choice of f we may assume that $f(u) \neq \emptyset$. If $f(u) = \{1, 2, \ldots, k\}$, then in order that x_2 to be totally rainbow dominated, we must have $|f(w)|+|f(x_2)|+|f(v)| \geq 2$, and this leads to the result. Suppose $|f(u)| < k$. If $f(x_1), f(x_3) \neq \emptyset$, then in order that x_2 to be totally rainbow dominated, we must have $|f(w)| + |f(x_2)| + |f(v)| \geq 2$, implying that $|f(u)| + |f(v)| + |f(w)| + \sum_{i=1}^{3} |f(x_i)| \geq 5$ as desired. Without loss of generality, assume that $f(x_1) = \emptyset$. Then $|f(u)| + |f(v)| \geq k$. If $f(x_2), f(x_3) \neq \emptyset$, then obviously $|f(u)|+|f(v)|+|f(w)|+\sum_{i=1}^{3} |f(x_i)| \geq k+2$. Assume that $f(x_3) = \emptyset$ (the case $f(x_2) = \emptyset$ is similar). In order that x_3 to be rainbow dominated, we must have $|f(u)| + |f(w)| \geq k$. Note that $|f(u) \cap f(w)| \geq 1$. If $f(x_2) \neq \emptyset$, then, clearly, $|f(u)|+|f(v)|+|f(w)| + \sum_{i=1}^{3} |f(x_i)| \geq k + 2$, and if $f(x_2) = \emptyset$, then in order that x_2 to be rainbow dominated, we have $f(v) \cup f(w) = \{1, 2, \ldots, k\}$, and so $|f(u)| + |f(v)| + |f(w)| + \sum_{i=1}^{3} |f(x_i)| \geq k + 2$.

□

Proposition 5. *Let G be a simple connected graph of order at least three. If G has a vertex $v \in V(G)$ which is contained in a triangle vuw such that $N(u) \cup N(w) \subseteq N[v]$, then $\mathrm{sd}_{\gamma_{trk}}(G) \leq 3$.*

Proof. Let $N(v) = \{v_1 = u, v_2 = w, v_3, \ldots, v_{\deg(v)}\}$ and G' be obtained from G by subdividing the edges vu, vw, uw with vertices x_1, x_2, x_3, respectively. By Lemma 1, the inequality $|f(u)| + |f(v)| + |f(w)| + \sum_{i=1}^{3} |f(x_i)| \geq \min\{5, k + 2\}$ holds; and the function $g : V \to \mathcal{P}(\{1, 2, \ldots, k\})$ defined by $g(v) = \{1, 2, \ldots, k\}$, $g(u) = \{1\}$, $g(w) = \emptyset$, and $g(x) = f(x)$ otherwise, is a TkRDF of G of weight less than $\gamma_{trk}(G')$, implying that $\mathrm{sd}_{\gamma_{trk}}(G) \leq 3$. □

3 The special case $k = 2$

In this section we focus on the case $k = 2$.

3.1 An upper bound

Here we present an upper bound on $\mathrm{sd}_{\gamma_{tr2}}(G)$.

Theorem 2. For any connected graph G of order $n \geq 3$ with $\delta(G) = 1$,

$$\mathrm{sd}_{\gamma_{tr2}}(G) \leq \min\{\gamma_{tr2}(G) - 1, \alpha'(G) + 1\}$$

where $\alpha'(G)$ is the matching number of G. This bound is sharp for complete graphs.

Proof. The result is immediate for $\gamma_{tr2}(G) = 2$ or 3 by Propositions 3 and 4. Assume that $\gamma_{tr2}(G) \geq 4$. Let $u \in V$ be vertex of degree one, $uv \in E(G)$ and $N(v) = \{u = v_1, v_2, \ldots, v_k\}$. By the proof of Theorem 1, subdividing all edges adjacent to u increases the total 2-rainbow domination number. First, we prove that $\mathrm{sd}_{\gamma_{tr2}}(G) \leq \gamma_{tr2}(G)-1$. Now let S be a largest subset of $N(v)$ containing u, such that subdividing the edges uv_i for $v_i \in S$ does not increase the total 2-rainbow domination number. If $|S| \leq 2$, then $\mathrm{sd}_{\gamma_{tr2}}(G) \leq 3 \leq \gamma_{tr2}(G) - 1$. Let $|S| \geq 3$ and assume without loss of generality that $S = \{v_1, \ldots, v_r\}$. Let G' be the graph obtained from G by subdividing the edges $vv_1, \ldots, vv_r$ with vertices $x_1, \ldots, x_r$. By the choice of S, we have $\gamma_{tr2}(G) = \gamma_{tr2}(G')$. Let f be a $\gamma_{tr2}(G')$-function. Clearly, $|f(v_1)| + |f(x_1)| \geq 2$. If $|f(v)| \geq 1$, then the function $g : V(G) \to \mathcal{P}(\{1, 2\})$ defined by $g(v_1) = \{1\}$, $g(v_i) = f(v_i) \cup f(x_i)$ for $1 \leq i \leq r$, and $g(x) = f(x)$ otherwise, is a T2RDF of G of weight less than $\gamma_{tr2}(G)$, which is a contradiction. Hence we assume that $f(v) = \emptyset$. In order that x_i to be totally rainbow dominated, we must have $|f(x_i)| + |f(v_i)| \geq 2$ for each $1 \leq i \leq r$. Then we have $\gamma_{tr2}(G) = \gamma_{tr2}(G') \geq 2s > s + 1 \geq \mathrm{sd}_{\gamma_{tr2}}(G)$. Thus $\mathrm{sd}_{\gamma_{tr2}}(G) \leq \gamma_{tr2}(G) - 1$.

Next we show that $\mathrm{sd}_{\gamma_{tr2}}(G) \leq \alpha'(G)+1$. If $\mathrm{sd}_{\gamma_{tr2}}(G) \leq 2$, then the result is immediate. Suppose $\mathrm{sd}_{\gamma_{tr2}}(G) \geq 3$. By Corollary 7, we may have $\deg(v) \geq \alpha' + 1$. Let S be a smallest subset of $N(v)$ containing u, such that subdividing the edges uv_i for $v_i \in S$ increases the total 2-rainbow domination number. We may assume without loss of generality that $S = \{vv_1, \ldots, vv_r\}$. By assumption we have $r \geq 3$. Let G' be the graph obtained from G by subdividing the edges $vv_1, vv_2, \ldots, vv_{r-1}$ with vertices $x_1, x_2, \ldots, x_{r-1}$, respectively. Then $\gamma_{tr2}(G) = \gamma_{tr2}(G')$. Let f be $\gamma_{tr2}(G)$-function. By Observation 1, we have $|f(v_1)|+|f(x_1)| \geq 2$. As above we may assume that $f(v) = \emptyset$. If $|f(v_1)| + |f(x_1)| \geq 3$, then the function $g : V(G) \to \mathcal{P}(\{1, 2\})$ defined by $g(v) = g(v_1) = \{1\}$, $g(v_i) = f(v_i) \cup f(x_i)$ for $2 \leq i \leq r$, and $g(x) = f(x)$ otherwise, is a T2RDF of G of weight less than $\gamma_{tr2}(G')$ which leads to a contradiction.

Assume that $|f(v_1)|+|f(x_1)| = 2$ and so $|f(v_1)| = |f(x_1)| = 1$. Suppose without loss of generality that $f(v_1) = f(x_1) = \{1\}$. If $|f(x_i)| \geq 1$ for some $2 \leq i \leq r-1$, say $i = 2$, then the function $g : V(G) \to \mathcal{P}(\{1, 2\})$ defined by $g(v_1) = \{1\}$, $g(v) = f(x_2)$, $g(v_i) = f(v_i) \cup f(x_i)$ for $3 \leq i \leq r$, and $g(x) = f(x)$ otherwise, is a T2RDF of G of weight less than $\gamma_{tr2}(G')$ which leads to a contradiction. Suppose that $f(x_i) = \emptyset$ for each $i \in \{2, \ldots, r\}$. Then in order that x_i to be totally 2-rainbow dominated, we must have $f(v_i) = \{1, 2\}$ for each $i \in \{2, \ldots, r-1\}$. If there is some v_i $(2 \leq i \leq r-1)$, say $i = 2$, such that $f(w) \neq \emptyset$ or $\cup_{x\in N(w)\setminus\{v_i\}} f(x) = \{1, 2\}$ for each $w \in N_G(v_i) \setminus \{v\}$, then function $g : V(G) \to \mathcal{P}(\{1, 2\})$ defined by $g(v) = \{1\}$, $g(v_2) = \{1\}$, and $g(x) = f(x)$ otherwise, is a T2RDF of G of weight smaller than $\gamma_{tr2}(G')$, a contradiction. Thus for each $2 \leq i \leq r-1$, v_i has a private neighbor w_i with respect to $\{v_2, \ldots, v_{r-1}\}$. Clearly, the set $\{vv_1, v_2w_2, \ldots, v_{r-1}w_{r-1}\}$ is a matching of G and this implies that $\mathrm{sd}_{\gamma_{tr2}}(G) \leq r + 1 \leq \alpha'(G) + 1$ as desired. This completes the proof. □

3.2 A family of graphs with large total 2-rainbow domination subdivision number

In the section we will show that the total 2-rainbow domination subdivision number can be arbitrary large. Haynes et al. in [19] introduced the following graph to prove a similar result on $\mathrm{sd}_{\gamma_t}(G)$.

Let $X = \{1, 2, \ldots, 3(k-1)\}$, and let $\mathcal{Y}$ be the set that consists of all k-subsets of X. Clearly, $|\mathcal{Y}| = \binom{3(k-1)}{k}$. Let $\mathcal{G}$ be the graph with vertex set $X \cup \mathcal{Y}$ and with edge set constructed as follows: add an edge joining every two distinct vertices of X and for each $x \in X$ and $Y \in \mathcal{Y}$, add an edge joining x and Y if and only if $x \in Y$. Then, $\mathcal{G}$ is a connected graph of order $n = \binom{3(k-1)}{k} + 3(k-1)$. We observe that the set X induces a clique in $\mathcal{G}$, the set $\mathcal{Y}$ is an independent set and each vertex of $\mathcal{Y}$ has degree k in $\mathcal{G}$. It is proved in [19] that $\gamma_t(\mathcal{G}) = 2k-2$ and $\mathrm{sd}_{\gamma_t}(\mathcal{G}) = k$.

Lemma 2. *For any integer $k \geq 3$, $\gamma_{tr2}(\mathcal{G}) = 4(k-1)$.*

Proof. By Proposition A and the fact $\gamma_t(\mathcal{G}) = 2k-2$ we have $\gamma_{tr2}(\mathcal{G}) \leq 4(k-1)$. To prove the inverse inequality, let f be a $\gamma_{tr2}(\mathcal{G})$-function such that $|Z = \{v \in V : |f(v)| = 1\}|$ is as small as possible. We proceed with two claims.

Claim 1. For each $Y \in \mathcal{Y}$, $|f(Y)| \leq 1$.

Suppose, to the contrary, that $f(Y) = \{1, 2\}$ for some $Y \in \mathcal{Y}$. Since f is a T2RDF of $\mathcal{G}$, Y has a neighbor $x \in X$ with $|f(x)| \geq 1$. We assume without loss of generality that $1 \in f(x)$. Let $z \in Y - \{x\}$ and define the function $g : V(\mathcal{G}) \to \mathcal{P}(\{1, 2\})$ by $g(Y) = \emptyset$, $g(z) = \{2\} \cup f(z)$ and $g(x) = f(x)$ for all $x \in V(\mathcal{G}) - \{Y, z\}$. Since $\mathcal{Y}$ is independent and $\mathcal{G}[X]$ is a clique, g is a T2RDF of $\mathcal{G}$ with smaller weight than $\gamma_{tr2}(\mathcal{G})$ which is a contradiction.

Claim 2. $|Z \cap \mathcal{Y}| = 0$.

Suppose, to the contrary, that $|Z \cap \mathcal{Y}| \geq 1$. Let $Y_1 \in \mathcal{Y}$ such that $|f(Y_1)| = 1$. Since f is a T2RDF of $\mathcal{G}$, Y_1 must have a neighbor $x_1 \in X$, with $|f(x_1)| \geq 1$. Assume that Y_2 is a k-subset of X not containing x_1. In order that Y_2 to be totally rainbow dominated, it has a neighbor $x_2 \in X$ with $|f(x_2)| \geq 1$. Now the function g defined by $g(x_1) = \{1, 2\}$, $g(Y_1) = \emptyset$, and $g(x) = f(x)$ otherwise, is a $\gamma_{tr2}(\mathcal{G})$-function which contradicts the choice of f, and the claim follows.

Let X_i (i=1,2) be the set of vertices of X such that $f(x) = \{i\}$ and let X_3 be the set of vertices of X assigned $\emptyset$ by f. If $|X_1|+|X_3| \geq k$, then no k-subset of $X_1 \cup X_3$ is rainbow dominated under f, a contradiction. Hence, $|X_1|+|X_3| \leq k-1$. Likewise, we have $|X_2|+|X_3| \leq k-1$. Note that the other vertices of X are assigned $\{1, 2\}$ under f. Let $X_{1,2} = X - \{X_1, X_2, X_3\}$. Clearly, the following integer linear programming

$$\begin{array}{ll} Min & |X_1| + |X_2| + 2|X_{1,2}| \\ s.t. & |X_1| + |X_3| \leq k - 1 \\ & |X_2| + |X_3| \leq k - 1 \\ & |X_1| + |X_2| + |X_{1,2}| + |X_3| = 3k - 3 \\ & |X_i| \in \mathbb{Z} \geq 0 \end{array}$$

has the optimal value $4(k - 1)$, and this completes the proof. □

Theorem 3. *For any integer $k \geq 4$, $\mathrm{sd}_{\gamma_{tr2}}(\mathcal{G}) = k$.*

Proof. Let $F = \{e_1, e_2, \ldots, e_{k-1}\}$ be an arbitrary subset of $k-1$ edges of $\mathcal{G}$. Assume H is obtained from $\mathcal{G}$ by subdividing each edge in F.

We show that $\gamma_{tr2}(H) = \gamma_{tr2}(\mathcal{G})$. Since $\mathrm{sd}_{\gamma_t}(\mathcal{G}) = k$, we have $\gamma_t(\mathcal{G}) = \gamma_t(H) = 2(k-1)$, and we deduce from Proposition A and Lemma 2 that $\gamma_{tr2}(H) = \gamma_{tr2}(\mathcal{G}) = 4(k-1)$. Hence, $\mathrm{sd}_{\gamma_{tr2}}(\mathcal{G}) \geq k$. Now the result follows by Theorem 1. □

3.3 Complete bipartite graphs $K_{m,n}$

In this subsection we determine the total 2-rainbow domination subdivision number of complete bipartite graphs.

Proposition F. *If $G = K_{n,m}$ is the complete bipartite graph with $m \geq n \geq 1$, then*

$$\mathrm{sd}_{\gamma_{tr2}}(G) = \begin{cases} 1 & \text{if} \quad n = 1, 2 \\ 2 & \text{if} \quad n \geq 3. \end{cases}$$

Proof. Let $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_m\}$ be the bipartite sets of $K_{m,n}$. The result is trivial for $m = n = 1$. If $n = 1$ and $m \geq 2$, then, clearly, $\gamma_{tr2}(K_{1,m}) = 3$, and it follows from Proposition 2 that $\mathrm{sd}_{\gamma_{tr2}}(K_{1,m}) = 1$.

Assume next that $n = 2$. If $m = 2$, then the result follows from Corollary 1. Suppose $m \geq 3$. Then we have $\gamma_{tr2}(K_{2,m}) = 3$. Let G' be the graph obtained from $G = K_{2,m}$ by subdividing the edge x_1y_1 with vertex x and let f be a $\gamma_{tr2}(G')$-function. In order that x to be totally rainbow dominated, we must have $|f(x_1)| + |f(x)| + |f(y_1)| \geq 2$.
First, let $f(x) \neq \emptyset$. Then in order that x to be totally rainbow dominated, we may assume that $|f(x_1)| \geq 1$. If $|f(y_1)| \geq 1$, then the function f restricted to $G = K_{2,m}$ is a T2RDF of $K_{2,m}$ of weight less than $\omega(f)$, and so $\mathrm{sd}_{\gamma_{tr2}}(K_{2,m}) = 1$. Assume that $f(y_1) = \emptyset$. Then in order that y_1 to be rainbow dominated, we must have $|f(x)| + |f(x_2)| \geq 2$. If $f(x_2) = \emptyset$, then in order that x_2 to be rainbow dominated, we have $\sum_{i=2}^{m} |f(y_i)| \geq 2$, implying that $\gamma_{tr2}(G') \geq 4$, and if $|f(x_2)| \geq 1$, then in order that x_2 to be totally dominated, we have $\sum_{i=2}^{m} |f(y_i)| \geq 1$, yielding $\gamma_{tr2}(G') \geq 4$ again.
Now let $f(x) = \emptyset$. If $f(x_1) \neq \emptyset$ and $f(y_1) \neq \emptyset$, then in order that x_1, y_1 to be totally dominated, we must have $|f(x_2)| \geq 1$ and

$\sum_{i=2}^{m} |f(y_i)| \geq 1$, and so $\gamma_{tr2}(G') \geq 4$. Assume without loss of generality that $f(y_1) = \emptyset$. Now in order that x, y_1 to be rainbow dominated, we must have $f(x_1) = \{1, 2\}$ and $f(x_2) = \{1, 2\}$, yielding $\gamma_{tr2}(G') \geq 4$. This implies that $\mathrm{sd}_{\gamma_{tr2}}(K_{2,m}) = 1$.

Finally, let $n \geq 3$. Clearly, $\gamma_{tr2}(K_{n,m}) = 4$ in this case. First, we show that $\mathrm{sd}_{\gamma_{tr2}}(K_{n,m}) \geq 2$. Let $e = u_i v_j$ be an arbitrary edge of $K_{n,m}$. We may assume without loss of generality that $i = j = 1$. Assume G' is obtained from $K_{n,m}$ by subdividing the edge e. Then the function $g : V(G') \to \mathcal{P}(\{1, 2\})$ defined by $g(x_1) = g(y_2) = \{1\}$, $g(y_1) = g(x_2) = \{2\}$, and $g(x) = 0$ otherwise, is a total 2-rainbow dominating function of G' of weight 4, and so $\mathrm{sd}_{\gamma_{tr2}}(K_{n,m}) \geq 2$.

Next we show that $\mathrm{sd}_{\gamma_{tr2}}(K_{n,m}) \leq 2$. Assume that G' is the graph obtained from $K_{n,m}$ by subdividing the edges x_1y_1 and x_1y_2 with vertices z_1 and z_2, respectively, and let f be $\gamma_{tr2}(G')$-function. If $|f(x_1)| + \sum_{i=1}^{2} |f(z_i)| \geq 3$ and $|f(x_i)| \geq 1$ for each $2 \leq i \leq n$, then we have $\gamma_{tr2}(G') = \omega(f) \geq 5$, and if $|f(x_1)| + \sum_{i=1}^{2} |f(z_i)| \geq 3$ and $f(x_i) = \emptyset$ for some $2 \leq i \leq m$, then in order that x_i to be rainbow dominated, we must have $\sum_{j=1}^{n} |f(y_j)| \geq 2$, yielding $\gamma_{tr2}(G') = \omega(f) \geq 5$. Henceforth, we assume that $|f(x_1)| + \sum_{i=1}^{2} |f(z_i)| \leq 2$. We consider the following cases.

Case 1. $f(x_1) = \{1, 2\}$.
Then $f(z_1) = f(z_2) = \emptyset$. In order that x_1 to be totally dominated, we must have $|f(y_j)| \geq 1$ for some $j \geq 3$, say $j = 3$. If $f(y_j) = \emptyset$ for some $j \in \{1, 2\}$, then in order that y_j to be rainbow dominated, we must have $\sum_{i=2}^{m} |f(x_i)| \geq 2$, implying that $\gamma_{tr2}(G') = \omega(f) \geq 5$. Otherwise we have $|f(y_1)| \geq 1$ and $|f(y_2)| \geq 1$ and again $\gamma_{tr2}(G') = \omega(f) \geq 5$.

Case 2. $f(x_1) = \emptyset$.
Then in order that z_1, z_2 to be totally rainbow dominated, we must have $|f(z_1)| + |f(y_1)| \geq 2$ and $|f(z_2)| + |f(y_2)| \geq 2$. If $|f(y_3)| \geq 1$, then, clearly, $\gamma_{tr2}(G') = \omega(f) \geq 5$. Assume that $f(y_3) = \emptyset$. Then in order that y_3 to be rainbow dominated, we must have $\sum_{i=2}^{m} |f(x_i)| \geq 2$, and this implies that $\gamma_{tr2}(G') = \omega(f) \geq 6$.

Case 3. $|f(x_1)| = 1$.
Suppose without loss of generality that $f(x_1) = \{1\}$. Then in order

that z_i to be rainbow dominated, we must have $|f(z_i)| + |f(y_i)| \geq 1$ for each $i \in \{1, 2\}$. If $|f(z_1)| + |f(y_1)| + |f(z_2)| + |f(y_2)| \geq 3$ and $|f(y_i)| \geq 1$ for some $i \geq 3$, then we have $\gamma_{tr2}(G') = \omega(f) \geq 5$, and if $|f(z_1)|+|f(y_1)|+|f(z_2)|+|f(y_2)| \geq 3$ and $f(y_i) = \emptyset$ for some i, then in order that y_i to be rainbow dominated, we must have $\sum_{i=2}^{m} |f(x_i)| \geq 1$, implying that $\gamma_{tr2}(G') = \omega(f) \geq 5$. Hence, we assume that $|f(z_1)| + |f(y_1)| + |f(z_2)| + |f(y_2)| = 2$. We distinguish the following situations.

- $f(z_1) = f(z_2) = \emptyset$.
  Considering our assumption, in order that z_1, z_2 to be rainbow dominated, we have $f(y_1) = f(y_2) = \{2\}$. In order that y_1 to be totally dominated, we may assume without loss of generality that $|f(x_2)| \geq 1$. If $|f(x_i)| \geq 1$ for each $i \geq 3$, then, clearly, $\gamma_{tr2}(G') = \omega(f) \geq 5$. Assume that $f(x_i) = \emptyset$ for some $i \geq 3$, say $i = 3$. Then in order that x_3 to be rainbow dominated, we must have $1 \in f(y_j)$ for some $j \geq 3$, and so $\gamma_{tr2}(G') = \omega(f) \geq 5$.

- $f(z_1) = \emptyset$ and $|f(z_2)| = 1$.
  By assumption, we have $f(y_1) = \{2\}$ and $f(y_2) = \emptyset$. As above, we may assume that $|f(x_2)| \geq 1$. If $|f(x_i)| \geq 1$ for some $i \geq 3$, then, clearly, $\gamma_{tr2}(G') = \omega(f) \geq 5$. Otherwise, in order that x_3 to be rainbow dominated, we must have $\sum_{i=3}^{m} |f(y_i)| \geq 1$, yielding $\gamma_{tr2}(G') = \omega(f) \geq 5$ again.

- $|f(z_1)| = |f(z_2)| = 1$.
  Then $f(y_1) = f(y_2) = \emptyset$. In order that y_1, y_2 to be rainbow dominated, we may assume that $|f(x_2)| \geq 1$. Now in order that x_2 to be totally dominated, we must have $\sum_{i=3}^{m} |f(y_i)| \geq 1$, yielding $\gamma_{tr2}(G') = \omega(f) \geq 5$ again.

Thus $\mathrm{sd}_{\gamma_{tr2}}(K_{n,m}) = 2$ when $n \geq 3$. $\square$

4 Conclusion

In this paper, we initiated the study of the total k-rainbow domination subdivision number in graphs and presented some sharp bounds on the total k-rainbow domination subdivision number in terms of the order, maximum degree and total k-rainbow domination number. In the special case of $k = 2$, we proved that the total 2-rainbow domination subdivision number can be arbitrary large. For further study we pose the following open problems.

Problem 1. *Is it true that for any integer $k \geq 2$ and a connected graph G with $\delta(G) \geq 2$, $\mathrm{sd}_{\gamma_{tr2}}(G) \leq \alpha'(G) + 1$?*

Problem 2. *Is it true that for any integer $k \geq 2$ and a connected graph G with $\delta(G) \geq 2$, $\mathrm{sd}_{\gamma_{tr2}}(G) \leq \gamma_{tr2}(G) - 1$?*

By Theorem 1 and Proposition 2 we have that for any tree T of order $n \geq 3$, $\mathrm{sd}_{\gamma_{tr2}}(T) \leq 2$.

Problem 3. *Characterize all tree T with $\mathrm{sd}_{\gamma_{tr2}}(T) = 2$*

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