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Computation of general Randić polynomial and general Randić energy of some graphs

Harishchandra S. Ramane, Gouramma A. Gudodagi

Abstract

The general Randić matrix of a graph G, denoted by $GR(G)$ is an $n \times n$ matrix whose (i,j)-th entry is $(d_i d_j)^\alpha$, $\alpha \in \mathbb{R}$ if the vertices v_i and v_j are adjacent and 0 otherwise, where d_i is the degree of a vertex v_i and n is the order of G. The general Randić energy $E_{GR}(G)$ of G is the sum of the absolute values of the eigenvalues of $GR(G)$. In this paper, we compute the general Randić polynomial and the general Randić energy of path, cycle, complete graph, complete bipartite graph, friendship graph and Dutch windmill graph.

Keywords: General Randić eigenvalues, general Randić energy, Randić index, degree of a vertex.

MSC 2010: 05C50, 05C07.

1 Introduction

Topological indices are the numerical quantities of a graph which are invariant under graph isomorphism. The interest in topological indices is mainly related to their use in quantitative structure-property relationship (QSPR) and quantitative structure-activity relationship (QSAR) [16].

Throughout the paper we consider only simple finite graphs, without directed, multiple or weighted edges and without loops. Let G be a simple graph with n vertices and m edges. Let the vertex set of G be $V(G) = \{v_1, v_2, \ldots, v_n\}$. If two vertices v_i and v_j of G are adjacent, then we write $v_i \sim v_j$. For $v_i \in V(G)$, the degree of the vertex v_i, denoted by d_i, is the number of vertices adjacent to v_i.

The general Randić (GR) matrix of a graph G is a square matrix $GR(G) = (d_{ij})_{n\times n}$ in which

$$d_{ij} = \begin{cases} (d_i d_j)^\alpha & \text{if} \quad v_i \sim v_j \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha \in \mathbb{R}$.

If $\alpha = -1/2$, then the above definition reduces to the Randić matrix, which was invented by Milan Randić [23] in 1975 as a molecular structure descriptor. In 1998, Bollobás and Erdös [2] generalized this index as $R_\alpha = R_\alpha(G) = \sum_{v_i \sim v_j} (d_i d_j)^\alpha$, called general Randić index. The Randić index concept suggests that it is a purposeful to associate to the graph G a symmetric square matrix $R(G)$. The Randić matrix [3], [4], [9], [13] is denoted by $R(G) = (r_{ij})_{n\times n}$, where

$$r_{ij} = \begin{cases} \frac{1}{\sqrt{d_i d_j}} & \text{if} \quad v_i \sim v_j \\ 0 & \text{otherwise.} \end{cases}$$

Denote the eigenvalues of the GR matrix of G by $\lambda_1, \lambda_2, \ldots, \lambda_n$ and order them in nonincreasing order. Similar to the characteristic polynomial of a matrix, we consider the general Randić (GR) polynomial of G as $det(\lambda I - GR(G)) = \phi_{GR}(G, \lambda)$, where I is the identity matrix of order n. The general Randić energy is defined as $E_{GR}(G) = \sum_{i=1}^{n} |\lambda_i|$.

The $E_{GR}(G)$ is defined in analogous to the ordinary graph energy defined as the sum of the absolute values of the eigenvalues of the adjacency matrix [15]. The ordinary graph energy is closely related to the total π-electron energy of a non-saturated hydrocarbons as calculated with the Huckel molecular orbital (HMO) method in chemistry [11]. Detail information about the graph energy can be found in [12], [14], [20]. There are many other kinds of graph energies, such as incidence energy [5], [6], distance energy [18], Laplacian energy [17], matching energy [7], [19], [21], Randić energy [23] and skew energy [22].

In this paper we obtain the GR-polynomial and GR-energy of some specific graphs. These results generalise the results obtained in paper [1].

Remark 1. *Given graph G, its general Randić energy $E_{GR}(G)$, is directly obtained from its general Randić polynomial $\phi_{GR}(G)$ by :*

(a) finding the solutions, λ_i's, (which are eigenvalues) for the equation

$$\phi_{GR}(G) = 0,$$

(b) and computing $E_{GR}(G) = \sum_{i=1}^{n} |\lambda_i|$.

2 GR-polynomial and GR-energy:

Let P_n, C_n, K_n, $K_{p,q}$, and $S_n = K_{1,n-1}$ denote the path, the cycle, the complete graph, complete bipartite graph and star graph respectively on n vertices.

Theorem 2.1 For $n \geq 5$ and $\alpha \in \mathbb{R}$, the GR polynomial of the path P_n is

$\phi_{GR}(P_n, \lambda) = \lambda^2 \Lambda_{n-2} - 2(4)^\alpha \lambda \Lambda_{n-3} + (16)^\alpha \Lambda_{n-4}$, where for every $k \geq 3$, $\Lambda_k = \lambda \Lambda_{k-1} - (16)^\alpha \Lambda_{k-2}$ with $\Lambda_1 = \lambda$ and $\Lambda_2 = \lambda^2 - (16)^\alpha$.

Proof. For every $k \geq 3$, consider

$$B_k = \begin{bmatrix} \lambda & -4^\alpha & 0 & 0 & \dots & 0 & 0 & 0 \\ -4^\alpha & \lambda & -4^\alpha & 0 & \dots & 0 & 0 & 0 \\ 0 & -4^\alpha & \lambda & -4^\alpha & \dots & 0 & 0 & 0 \\ 0 & 0 & -4^\alpha & \lambda & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \lambda & -4^\alpha & 0 \\ 0 & 0 & 0 & 0 & \dots & -4^\alpha & \lambda & -4^\alpha \\ 0 & 0 & 0 & 0 & \dots & 0 & -4^\alpha & \lambda \end{bmatrix}_{k\times k},$$

and let $\Lambda_k = det(B_k)$. It is easy to see that $\Lambda_k = \lambda\Lambda_{k-1} - (16)^\alpha \Lambda_{k-2}$.

Therefore

$$\begin{aligned}\phi_{GR}(P_n,\lambda) &= det(\lambda I - GR(P_n))\\ &= \begin{vmatrix} \lambda & -2^\alpha & 0 & 0 & \dots & 0 & 0 & 0\\ -2^\alpha & \lambda & -4^\alpha & 0 & \dots & 0 & 0 & 0\\ 0 & -4^\alpha & \lambda & -4^\alpha & \dots & 0 & 0 & 0\\ 0 & 0 & -4^\alpha & \lambda & \dots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \dots & \lambda & -4^\alpha & 0\\ 0 & 0 & 0 & 0 & \dots & -4^\alpha & \lambda & -2^\alpha\\ 0 & 0 & 0 & 0 & \dots & 0 & -2^\alpha & \lambda \end{vmatrix}_{n\times n}.\end{aligned}$$

$$\begin{aligned}\phi_{GR}(P_n,\lambda) &= \lambda \begin{vmatrix} \lambda & -4^\alpha & 0 & \dots & 0 & 0 & 0\\ -4^\alpha & \lambda & -4^\alpha & \dots & 0 & 0 & 0\\ 0 & -4^\alpha & \lambda & \dots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & \lambda & -4^\alpha & 0\\ 0 & 0 & 0 & \dots & -4^\alpha & \lambda & -2^\alpha\\ 0 & 0 & 0 & \dots & 0 & -2^\alpha & \lambda \end{vmatrix}\\ &+ 2^\alpha \begin{vmatrix} -2^\alpha & 0 & 0 & \dots & 0 & 0 & 0\\ -4^\alpha & \lambda & -4^\alpha & \dots & 0 & 0 & 0\\ 0 & -4^\alpha & \lambda & \dots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & \lambda & -4^\alpha & 0\\ 0 & 0 & 0 & \dots & -4^\alpha & \lambda & -2^\alpha\\ 0 & 0 & 0 & \dots & 0 & -2^\alpha & \lambda \end{vmatrix}.\end{aligned}$$

Further,

$$\phi_{GR}(P_n,\lambda) = \lambda\left(\lambda\Lambda_{(n-2)} + 2^{\alpha}\begin{vmatrix} \lambda & -4^{\alpha} & 0 & \dots & 0 & 0 \\ -4^{\alpha} & \lambda & -4^{\alpha} & \dots & 0 & 0 \\ 0 & -4^{\alpha} & \lambda & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda & 0 \\ 0 & 0 & 0 & \dots & 0 & -2^{\alpha} \end{vmatrix}\right)$$

$$-(4)^{\alpha}\left(\lambda\begin{vmatrix} \lambda & -4^{\alpha} & \dots & 0 & 0 \\ (-4)^{\alpha} & \lambda & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \lambda & -4^{\alpha} \\ 0 & 0 & \dots & -4^{\alpha} & \lambda \end{vmatrix}\right.$$

$$\left. + \ 2^{\alpha}\begin{vmatrix} \lambda & -4^{\alpha} & \dots & 0 & 0 \\ -4^{\alpha} & \lambda & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \lambda & 0 \\ 0 & 0 & \dots & -4^{\alpha} & -2^{\alpha} \end{vmatrix}\right).$$

Hence,

$$\begin{aligned} \phi_{GR}(P_n,\lambda) &= \lambda^2\Lambda_{n-2} - 4^{\alpha}\lambda\Lambda_{n-3} - 4^{\alpha}\lambda\Lambda_{n-3} + 16^{\alpha}\Lambda_{n-4}. \\ &= \lambda^2\Lambda_{n-2} - 2(4)^{\alpha}\lambda\Lambda_{n-3} + 16^{\alpha}\Lambda_{n-4}. \end{aligned}$$

□

Theorem 2.2 For $n \geq 3$ and $\alpha \in \mathbb{R}$, the GR polynomial of the cycle C_n is

$$\phi_{GR}(C_n,\lambda) = \lambda\Lambda_{n-1} - 2(16)^{\alpha}\Lambda_{n-2} - (4)^{\alpha n}2,$$

where for every $k \geq 3$, $\Lambda_k = \lambda\Lambda_{k-1} - (16)^{\alpha}\Lambda_{k-2}$ with $\Lambda_1 = \lambda$ and $\Lambda_2 = \lambda^2 - (16)^{\alpha}$.

Proof. Similar to the proof of Theorem 2.1, for every $k \geq 3$, we consider

$$B_k = \begin{bmatrix} \lambda & -4^\alpha & 0 & 0 & \dots & 0 & 0 & 0 \\ -4^\alpha & \lambda & -4^\alpha & 0 & \dots & 0 & 0 & 0 \\ 0 & -4^\alpha & \lambda & -4^\alpha & \dots & 0 & 0 & 0 \\ 0 & 0 & -4^\alpha & \lambda & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \lambda & -4^\alpha & 0 \\ 0 & 0 & 0 & 0 & \dots & -4^\alpha & \lambda & -4^\alpha \\ 0 & 0 & 0 & 0 & \dots & 0 & -4^\alpha & \lambda \end{bmatrix}_{k\times k},$$

and let $\Lambda_k = det(B_k)$. It is easy to see that $\Lambda_k = \lambda\Lambda_{k-1} - (16)^\alpha\Lambda_{k-2}$.

Therefore

$$\begin{aligned} \phi_{GR}(C_n, \lambda) &= det(\lambda I - GR(C_n)) \\ &= \begin{vmatrix} \lambda & -4^\alpha & 0 & 0 & \dots & 0 & 0 & -4^\alpha \\ -4^\alpha & \lambda & -4^\alpha & 0 & \dots & 0 & 0 & 0 \\ 0 & -4^\alpha & \lambda & -4^\alpha & \dots & 0 & 0 & 0 \\ 0 & 0 & -4^\alpha & \lambda & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \lambda & -4^\alpha & 0 \\ 0 & 0 & 0 & 0 & \dots & -4^\alpha & \lambda & -4^\alpha \\ -4^\alpha & 0 & 0 & 0 & \dots & 0 & -4^\alpha & \lambda \end{vmatrix}_{n\times n}. \end{aligned}$$

$$\phi_{GR}(C_n,\lambda)=$$

$$=\lambda\Lambda_{(n-1)}+4^{\alpha}\begin{vmatrix} -4^{\alpha} & 0 & 0 & \dots & 0 & 0 & -4^{\alpha} \\ -4^{\alpha} & \lambda & -4^{\alpha} & \dots & 0 & 0 & 0 \\ 0 & -4^{\alpha} & \lambda & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda & -4^{\alpha} & 0 \\ 0 & 0 & 0 & \dots & -4^{\alpha} & \lambda & -4^{\alpha} \\ 0 & 0 & 0 & \dots & 0 & -4^{\alpha} & \lambda \end{vmatrix}_{(n-1)\times(n-1)}$$

$$+(-1)^{(n+1)}[-(4)^{\alpha}]\begin{vmatrix} -4^{\alpha} & 0 & 0 & \dots & 0 & 0 & -4^{\alpha} \\ \lambda & -4^{\alpha} & 0 & \dots & 0 & 0 & 0 \\ -4^{\alpha} & \lambda & -4^{\alpha} & \dots & 0 & 0 & 0 \\ 0 & -4^{\alpha} & \lambda & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda & -4^{\alpha} & 0 \\ 0 & 0 & 0 & \dots & -4^{\alpha} & \lambda & -4^{\alpha} \end{vmatrix}_{(n-1)\times(n-1).}$$

$$\phi_{GR}(C_n,\lambda)=\lambda\Lambda_{(n-1)}-16^{\alpha}\Lambda_{(n-2)}+$$

$$+(-1)^{n}(-16^{\alpha})\begin{vmatrix} -4^{\alpha} & \lambda & -4^{\alpha} & \dots & 0 & 0 \\ 0 & -4^{\alpha} & \lambda & \dots & 0 & 0 \\ 0 & 0 & -4^{\alpha} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & 0 & \dots & -4^{\alpha} & \lambda \\ 0 & 0 & 0 & \dots & 0 & -4^{\alpha} \end{vmatrix}_{(n-2)\times(n-2)}$$

$$+(-1)^{(n+1)}[-(4)^{\alpha}]\left((-4)^{\alpha}\begin{vmatrix} -4^{\alpha} & 0 & \dots & 0 \\ \lambda & -4^{\alpha} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -4^{\alpha} \end{vmatrix}_{(n-2)\times(n-2)} +(-1)^{n}[-4^{\alpha}]\Lambda_{n-2}\right).$$

Therefore,

$$\begin{aligned}\phi_{GR}(C_n,\lambda) &= \lambda\Lambda_{n-1} - 16^{\alpha}\Lambda_{n-2} + (-1)^n(-16^{\alpha})[-(4)^{\alpha}]^{n-2} \\ &+ (-1)^{n+1}[-(4)^{\alpha}]^n + (-1)^{2n+1}(16)^{\alpha}\Lambda_{n-2} \\ &= \lambda\Lambda_{n-1} - 2(16^{\alpha})\Lambda_{n-2} - (2)4^{(\alpha n)}.\end{aligned}$$

□

Lemma 2.3 [8] If M is a nonsingular square matrix, then

$$det\begin{pmatrix} M & N \\ P & Q \end{pmatrix} = det(M)\, det(Q - PM^{-1}N).$$

Theorem 2.4 For $n \geq 2$ and $\alpha \in \mathbb{R}$,

(i) the GR polynomial of the complete graph K_n is

$$\phi_{GR}(K_n,\lambda) = (\lambda - (n-1)^{2\alpha+1})(\lambda + (n-1)^{2\alpha})^{(n-1)},$$

(ii) the GRE of K_n is

$$E_{GR}(K_n) = 2(n-1)^{2\alpha+1}.$$

Proof. It is easy to see that the GR matrix of K_n is $(n-1)^{2\alpha}(J_n - I)$, where J_n is a matrix whose all entries are equal to one and I is an identity matrix. Therefore

$$\begin{aligned}\phi_{GR}(K_n,\lambda) &= \left|\lambda I - (n-1)^{2\alpha}J_n + (n-1)^{2\alpha}I\right| \\ &= \left|(\lambda + (n-1)^{2\alpha})I - (n-1)^{2\alpha}J_n\right|.\end{aligned}$$

Since the eigenvalues of J_n are n (once) and 0 ($n-1$ times), the eigenvalues of $(n-1)^{2\alpha}J_n$ are $n(n-1)^{2\alpha}$ (once) and 0 ($n-1$ times).
Hence

$$\phi_{GR}(K_n,\lambda) = (\lambda - (n-1)^{2\alpha+1})(\lambda + (n-1)^{2\alpha})^{(n-1)}.$$

(ii) It follows from Remark 1. □

Theorem 2.5 For any positive integers $p, q \geq 1$ and $\alpha \in \mathbb{R}$,

(i) The GR polynomial of complete bipartite graph $K_{p,q}$ is

$$\phi_{GR}(K_{p,q}, \lambda) = \lambda^{p+q-2}(\lambda^2 - (pq)^{2\alpha+1}),$$

(ii) $E_{GR}(K_{p,q}) = 2\sqrt{(pq)^{2\alpha+1}}$.

Proof. It is easy to see that the GR matrix of $K_{p,q}$ is

$$GR(K_{p,q}) = (pq)^{\alpha} \begin{pmatrix} O_{p\times p} & J_{p\times q} \\ J_{q\times p} & O_{q\times q} \end{pmatrix}.$$

Therefore,

$$\phi_{GR}(K_{p,q}, \lambda) = \begin{vmatrix} \lambda I_p & -(pq)^{\alpha} J_{p\times q} \\ -(pq)^{\alpha} J_{q\times p} & \lambda I_q \end{vmatrix}.$$

Using Lemma 2.3 we have

$$\begin{aligned} \phi_{GR}(K_{p,q}, \lambda) &= |\lambda I_p| \left| \lambda I_q - (-pq)^{\alpha} J_{q\times p} \frac{I_p}{\lambda} (-pq)^{\alpha} J_{p\times q} \right| \\ &= \lambda^{p-q} \left| \lambda^2 I_q - p\,(pq)^{2\alpha} J_q \right| \quad \text{since } J_{q\times p} J_{p\times q} = pJ_q. \end{aligned}$$

Since the eigenvalues of J_n are n (once) and 0 ($n-1$ times), the eigenvalues of $p(pq)^{2\alpha} J_q$ are $(pq)^{2\alpha+1}$ (once) and 0 ($q-1$ times). Therefore

$\phi_{GR}(K_{p,q}, \lambda) = \lambda^{p+q-2}(\lambda^2 - (pq)^{2\alpha+1})$.

(ii) It follows from Remark 1.

□

Corollary 2.6 For $n \geq 2$ and $\alpha \in \mathbb{R}$,

(i) the GR polynomial of the star $S_n = K_{1,n-1}$ is

$$\phi_{GR}(S_n, \lambda) = \lambda^{(n-2)}(\lambda^2 - (n-1)^{2\alpha+1}),$$

(ii) the GRE of S_n is

$$E_{GR}(S_n) = 2\sqrt{(n-1)^{2\alpha+1}}.$$

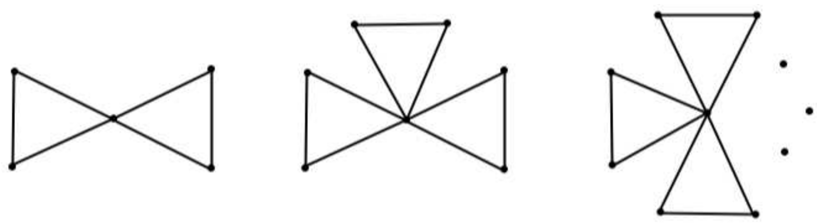

Figure 1. Friendship graphs F_2, F_3 and F_n respectively

Let n be any positive integer and F_n be a friendship graph with $2n+1$ vertices and $3n$ edges. In other words, the friendship graph F_n is a graph that can be constructed by coalescence n copies of the cycle C_3 of length 3 with common vertex. The Friendship theorem of Erdös et al. [10], states that graphs with the property that every two vertices have exactly one neighbour in common are exactly the friendship graphs. The Fig. 1 shows some examples of friendship graphs. Here we compute the GRE of friendship graphs.

Theorem 2.7 For $n \geq 2$ and $\alpha \in \mathbb{R}$,

(i) the GR polynomial of friendship graph F_n is

$$\begin{aligned}\phi_{GR}(F_n, \lambda) &= (\lambda^2 - 4^{2\alpha})^{n-1}(\lambda + 4^\alpha)\left(\lambda - \left[2^{2\alpha-1} + 2^{2\alpha-1}\sqrt{1+8n^{2\alpha+1}}\right]\right) \\ &\quad \left(\lambda - \left[2^{2\alpha-1} - 2^{2\alpha-1}\sqrt{1+8n^{2\alpha+1}}\right]\right),\end{aligned}$$

(ii) the GR energy of friendship graph F_n is

$$E_{GR}(F_n) = \begin{cases} 4^{\alpha}(2n-1)+2^{2\alpha} & \text{if } n^{2\alpha+1} \leq 0 \\ 4^{\alpha}(2n-1)+2^{2\alpha}\sqrt{1+8n^{2\alpha+1}} & \text{if } n^{2\alpha+1} > 0. \end{cases}$$

Proof. The GR matrix of F_n is

$$GR(F_n) = \begin{bmatrix} 0 & (4n)^{\alpha} & (4n)^{\alpha} & \dots & (4n)^{\alpha} & (4n)^{\alpha} \\ (4n)^{\alpha} & 0 & 4^{\alpha} & \dots & 0 & 0 \\ (4n)^{\alpha} & 4^{\alpha} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (4n)^{\alpha} & 0 & 0 & \dots & 0 & 4^{\alpha} \\ (4n)^{\alpha} & 0 & 0 & \dots & 4^{\alpha} & 0 \end{bmatrix}_{(2n+1)\times(2n+1)}.$$

Now, for computing $|\lambda I - GR(F_n)|$, we consider its first row. The cofactor of the first array in this row is

$$\begin{vmatrix} \lambda & -4^{\alpha} & \dots & 0 & 0 \\ -4^{\alpha} & \lambda & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \lambda & -4^{\alpha} \\ 0 & 0 & \dots & -4^{\alpha} & \lambda \end{vmatrix}_{(2n)\times(2n)}$$

and the cofactor of another arrays in the first row are similar to

$$\begin{vmatrix} -(4n)^{\alpha} & -4^{\alpha} & \dots & 0 & 0 \\ -(4n)^{\alpha} & \lambda & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -(4n)^{\alpha} & 0 & \dots & \lambda & -4^{\alpha} \\ -(4n)^{\alpha} & 0 & \dots & -4^{\alpha} & \lambda \end{vmatrix}_{(2n)\times(2n)}.$$

Now solving the above two determinants, we get

$$\begin{aligned}\phi_{GR}(F_n,\lambda) &= \lambda(\lambda^2-4^{2\alpha})^n+(4n)^\alpha 2n\left[(-(4n)^\alpha\lambda-(4n)^\alpha 4^\alpha)(\lambda^2-4^{2\alpha})^{(n-1)}\right]\\ &= (\lambda^2-4^{2\alpha})^{n-1}(\lambda+4^\alpha)\left(\lambda-\left[2^{2\alpha-1}+2^{2\alpha-1}\sqrt{1+8n^{2\alpha+1}}\right]\right)\\ &\quad\left(\lambda-\left[2^{2\alpha-1}-2^{2\alpha-1}\sqrt{1+8n^{2\alpha+1}}\right]\right).\end{aligned}$$

(ii) It follows from Remark 1.

□

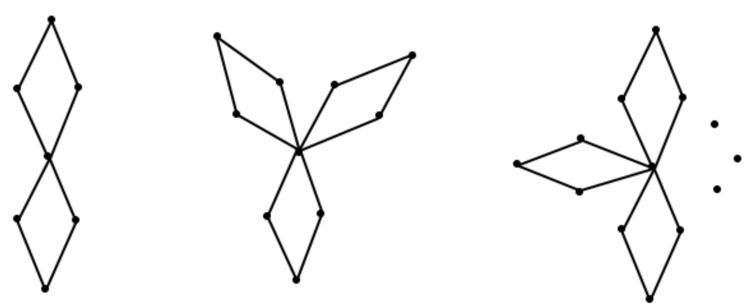

Figure 2. Dutch Windmill graph D_4^2 , D_4^3 and D_4^n respectively

Let n be any positive integer and D_4^n be Dutch Windmill graph with $3n+1$ vertices and $4n$ edges. In other words, the graph D_4^n is a graph that can be constructed by coalescing n copies of the cycle C_4 of length 4 with a common vertex. Figure 2 shows some examples of Dutch Windmill graphs. Here we compute the GRE of Dutch Windmill graphs.

Theorem 2.8 For $n \geq 2$ and $\alpha \in \mathbb{R}$,

(i) the GR polynomial of Dutch Windmill graph D_4^n is

$$\phi_{GR}(D_4^n,\lambda) = \lambda^{n+1}(\lambda^2-(2)4^{2\alpha})^{n-1}[\lambda^2-(2)4^{2\alpha}-2n(4n)^{(2\alpha)}],$$

(ii) $E_{GR}(D_4^n) = 2\sqrt{2}\ 4^\alpha(n-1) + 2^{2\alpha+1}\sqrt{2(1+n^{2\alpha+1})}$.

Proof. The GR matrix of D_4^n is

$$\begin{bmatrix} 0 & (4n)^\alpha & (4n)^\alpha & 0 & \dots & (4n)^\alpha & (4n)^\alpha & 0 \\ (4n)^\alpha & 0 & 0 & 4^\alpha & \dots & 0 & 0 & 0 \\ (4n)^\alpha & 0 & 0 & 4^\alpha & \dots & 0 & 0 & 0 \\ 0 & 4^\alpha & 4^\alpha & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ (4n)^\alpha & 0 & 0 & 0 & \dots & 0 & 0 & 4^\alpha \\ (4n)^\alpha & 0 & 0 & 0 & \dots & 0 & 0 & 4^\alpha \\ 0 & 0 & 0 & 0 & \dots & 4^\alpha & 4^\alpha & 0 \end{bmatrix}_{(3n+1)\times(3n+1)}.$$

Let $A = \begin{pmatrix} \lambda & 0 & -4^\alpha \\ 0 & \lambda & -4^\alpha \\ -4^\alpha & -4^\alpha & \lambda \end{pmatrix}$, $B = \begin{pmatrix} -(4n)^\alpha & 0 & 0 \\ -(4n)^\alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $C = \begin{pmatrix} -(4n)^\alpha & 0 & -4^\alpha \\ -(4n)^\alpha & \lambda & -4^\alpha \\ 0 & -4^\alpha & \lambda \end{pmatrix}$.

Then

$$\begin{aligned} \phi_{GR}(D_4^n, \lambda) &= det(\lambda I - GR(D_4^n)) \\ &= \lambda(det(A))^n + 2n(4n)^\alpha det \begin{pmatrix} C & O & O & \dots & O \\ B & A & O & \dots & O \\ B & O & A & \dots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B & O & O & \dots & A \end{pmatrix}_{(3n)\times(3n)}. \end{aligned}$$

Now, by the straightforward computation we have the result.

(ii) It follows from Remark 1.

□

Let n be any positive integer and D_5^n be Dutch Windmill graph with $4n+1$ vertices and $5n$ edges. In other words, the graph D_5^n is a graph that can be constructed by coalescing n copies of the cycle C_5

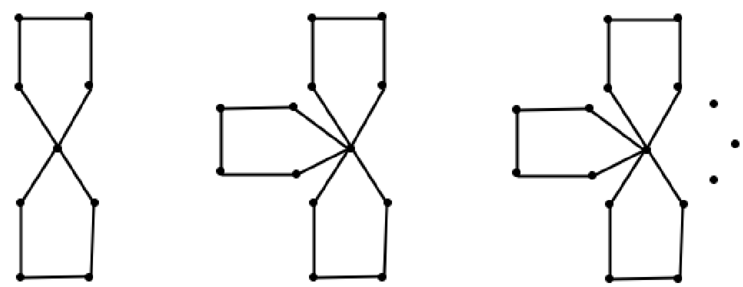

Figure 3. Dutch Windmill graph D_5^2, D_5^3 and D_5^n respectively

of length 5 with a common vertex. Figure 3 shows some examples of Dutch Windmill graphs.

Theorem 2.9 For $n \geq 2$ and $\alpha \in \mathbb{R}$, the GR polynomial of Dutch Windmill graph D_5^n is

$$\begin{aligned}\phi_{GR}(D_5^n,\lambda) &= (\lambda^4-3\lambda^2 4^{2\alpha}+4^{4\alpha})^{(n-1)}(\lambda^5-3\lambda^3 4^{2\alpha}+\lambda 4^{4\alpha}- \\ &\quad -2n\lambda^3(4n)^{(2\alpha)}+4n\lambda(4n)^{(2\alpha)}4^{2\alpha}-2n(4n)^{(2\alpha)}4^{3\alpha}).\end{aligned}$$

Proof. The GR matrix of D_5^n is

$$\begin{bmatrix} 0 & (4n)^\alpha & (4n)^\alpha & 0 & 0 & \ldots & (4n)^\alpha & (4n)^\alpha & 0 & 0 \\ (4n)^\alpha & 0 & 0 & 0 & 4^\alpha & \ldots & 0 & 0 & 0 & 0 \\ (4n)^\alpha & 0 & 0 & 4^\alpha & 0 & \ldots & 0 & 0 & 0 & 0 \\ 0 & 0 & 4^\alpha & 0 & 4^\alpha & \ldots & 0 & 0 & 0 & 0 \\ 0 & 4^\alpha & 0 & 4^\alpha & 0 & \ldots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ (4n)^\alpha & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 4^\alpha \\ (4n)^\alpha & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 4^\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 4^\alpha & 0 & 4^\alpha \\ 0 & 0 & 0 & 0 & 0 & \ldots & 4^\alpha & 0 & 4^\alpha & 0 \end{bmatrix}_{(3n+1)\times(3n+1)}.$$

Let

$$A = \begin{pmatrix} \lambda & 0 & 0 & -4^{\alpha} \\ 0 & \lambda & -4^{\alpha} & 0 \\ 0 & -4^{\alpha} & \lambda & -4^{\alpha} \\ -4^{\alpha} & 0 & -4^{\alpha} & \lambda \end{pmatrix}, \ B = \begin{pmatrix} -(4n)^{\alpha} & 0 & 0 & 0 \\ -(4n)^{\alpha} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{and } \ C = \begin{pmatrix} -(4n)^{\alpha} & 0 & 0 & -4^{\alpha} \\ -(4n)^{\alpha} & \lambda & -4^{\alpha} & 0 \\ 0 & -4^{\alpha} & \lambda & -4^{\alpha} \\ 0 & 0 & -4^{\alpha} & \lambda \end{pmatrix}.$$

Then

$$\begin{aligned} \phi_{GR}(D_5^n, \lambda) &= det(\lambda I - GR(D_5^n)) \\ &= \lambda(det(A))^n + 2n(4n)^{\alpha} det \begin{pmatrix} C & O & O & O & \dots & O \\ B & A & O & O & \dots & O \\ B & O & A & O & \dots & O \\ B & O & O & A & \dots & O \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ B & O & O & O & \dots & A \end{pmatrix}_{(4n)\times(4n)}. \end{aligned}$$

Now, by the straightforward computation we have the result. □

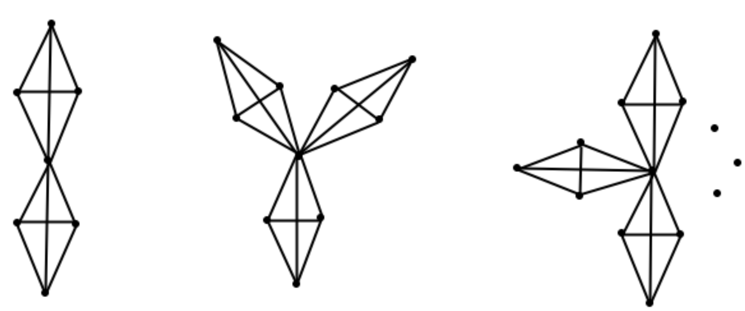

Figure 4. K_4-Windmill graph K_4^2, K_4^3 and K_4^n respectively

Let n be any positive integer and K_4^n be K_4-Windmill graph with $4n+1$ vertices and $6n$ edges. In other words, the graph K_4^n is a graph that can be constructed by coalescing n copies of the complete graph K_4 with a common vertex. Figure 4 shows some examples of K_4-Windmill graphs.

Theorem 2.10 For $n \geq 2$ and $\alpha \in \mathbb{R}$,

(i) the GR polynomial of K_4^n-Windmill graph is

$$\begin{aligned} \phi_{GR}(K_4^n, \lambda) &= [(\lambda + 9^\alpha)^2(\lambda - 2(9)^\alpha)]^{(n-1)}(\lambda + 9^\alpha) \\ &\quad \left(\lambda - \left[9^\alpha + 9^\alpha\sqrt{1+3n^{2\alpha+1}}\right]\right) \\ &\quad \left(\lambda - \left[9^\alpha - 9^\alpha\sqrt{1+3n^{2\alpha+1}}\right]\right). \end{aligned}$$

(ii) the GR energy of K_4-Windmill graph is

$$E_{GR}(K_4^n) = \begin{cases} 4n(9)^\alpha & \text{if } n^{2\alpha+1} \leq 0 \\ 2(9)^\alpha[(2n-1) + \sqrt{1+3n^{2\alpha+1}} & \text{if } n^{2\alpha+1} > 0. \end{cases}$$

Proof. The GR matrix of K_4^n is

$$\begin{bmatrix} 0 & (9n)^\alpha & (9n)^\alpha & (9n)^\alpha & \dots & (9n)^\alpha & (9n)^\alpha & (9n)^\alpha \\ (9n)^\alpha & 0 & 9^\alpha & 9^\alpha & \dots & 0 & 0 & 0 \\ (9n)^\alpha & 9^\alpha & 0 & 9^\alpha & \dots & 0 & 0 & 0 \\ (9n)^\alpha & 9^\alpha & 9^\alpha & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ (9n)^\alpha & 0 & 0 & 0 & \dots & 0 & 9^\alpha & 9^\alpha \\ (9n)^\alpha & 0 & 0 & 0 & \dots & 9^\alpha & 0 & 9^\alpha \\ (9n)^\alpha & 0 & 0 & 0 & \dots & 9^\alpha & 9^\alpha & 0 \end{bmatrix}_{(3n+1)\times(3n+1)}.$$

Let

$$A = \begin{pmatrix} \lambda & -9^{\alpha} & -9^{\alpha} \\ -9^{\alpha} & \lambda & -9^{\alpha} \\ -9^{\alpha} & -9^{\alpha} & \lambda \end{pmatrix}, \quad B = \begin{pmatrix} -(9n)^{\alpha} & 0 & 0 \\ -(9n)^{\alpha} & 0 & 0 \\ -(9n)^{\alpha} & 0 & 0 \end{pmatrix}$$

and $C = \begin{pmatrix} -(9n)^{\alpha} & -9^{\alpha} & -9^{\alpha} \\ -(9n)^{\alpha} & \lambda & -9^{\alpha} \\ -(9n)^{\alpha} & -9^{\alpha} & \lambda \end{pmatrix}$.

Then

$$\begin{aligned} \phi_{GR}(K_4^n, \lambda) &= det(\lambda I - GR(K_4^n)) \\ &= \lambda(det(A))^n + 3n(9n)^{\alpha} det \begin{pmatrix} C & O & O & \dots & O \\ B & A & O & \dots & O \\ B & O & A & \dots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B & O & O & \dots & A \end{pmatrix}_{(3n)\times(3n)}. \end{aligned}$$

Now, by the straightforward computation we have the result.

(ii) It follows from Remark 1.

□

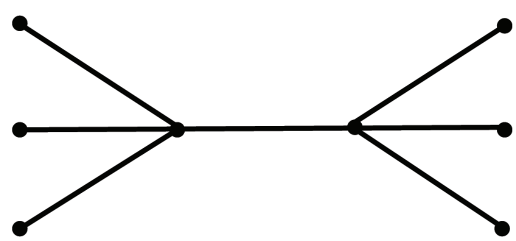

Figure 5. Double star $S(3, 3)$

For $p, q \geq 1$ the double star $S(p, q)$ is the graph on the points $\{v_0, v_1, \dots, v_p, w_0, w_1, \dots, w_q\}$ with lines $\{(v_0, w_0), (v_0, v_i), (w_0, w_j) : 1 \leq i \leq p, 1 \leq j \leq q\}$ (see Fig. 5).

Theorem 2.11 For $p, q \geq 1$ and $\alpha \in \mathbb{R}$,

(i) the GR polynomial of double star graph $S(p,q)$ is

$$\begin{aligned}\phi_{GR}(S(p,q),\lambda) &= \lambda^{p+q-4}\left[\lambda^4-\lambda^2\left((p-1)p^{2\alpha}+(q-1)q^{2\alpha}\right.\right.\\ &\left.\left.+(pq)^{2\alpha}\right)+(p-1)(q-1)(pq)^{2\alpha}\right].\end{aligned}$$

(ii)

$$\begin{aligned}E_{GR}(S(p,q)) &= \sqrt{2}\sqrt{X+\sqrt{X^2-4(p-1)(q-1)(pq)^{2\alpha}}}\\ &+\sqrt{2}\sqrt{X-\sqrt{X^2-4(p-1)(q-1)(pq)^{2\alpha}}},\end{aligned}$$

where $X=(p-1)\,p^{2\alpha}+(q-1)\,q^{2\alpha}+(pq)^{2\alpha}$.

Proof. The GR polynomial of $S(p,q)$ is

$$\phi_{GR}(S(p,q),\lambda)=det(\lambda I-GR(S(p,q)))=$$
$$\begin{vmatrix}\lambda & -(pq)^\alpha & -q^\alpha & \dots & -q^\alpha & 0 & \dots & 0\\ -(pq)^\alpha & \lambda & 0 & \dots & 0 & -p^\alpha & \dots & -p^\alpha\\ -q^\alpha & 0 & \lambda & \dots & 0 & 0 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots\\ -q^\alpha & 0 & 0 & \dots & \lambda & 0 & \dots & 0\\ 0 & -p^\alpha & 0 & \dots & 0 & \lambda & \dots & 0\\ 0 & -p^\alpha & 0 & \dots & 0 & 0 & \dots & 0\\ 0 & -p^\alpha & 0 & \dots & 0 & 0 & \dots & \lambda\end{vmatrix}_{(p+q)\times(p+q)}.$$

Using Lemma 2.3,

$$\phi_{GR}(S(p,q),\lambda) = \lambda^{p+q-4}\begin{vmatrix}\lambda^2-(q-1)q^{2\alpha} & -\lambda(pq)^\alpha\\ -\lambda(pq)^\alpha & \lambda^2-(p-1)p^{2\alpha}\end{vmatrix}.$$

Now, by the straightforward computation we have the result.

(ii) It follows from Remark 1. ☐

3 Conclusion

In this paper we obtained the expression for the GR polynomial and GR energy of some specific graphs. These results generalise the results obtained in paper [1]. The results in [1] follow from our work by letting $\alpha = -(1/2)$.

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Harishchandra S. Ramane
Department of Mathematics,
Karnatak University,
Dharwad - 580003, India
Phone:+919945031752
E–mail: hsramane@yahoo.com

Gouramma A. Gudodagi
Department of Mathematics,
KLE's G. I. Bagewadi Arts, Science and Commerce College,
Nipani – 591237, India
Phone:+918095319099
E–mail: gouri.gudodagi@gmail.com