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On the signed Italian domination of graphs

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Abstract

A signed Italian dominating function on a graph $G = (V, E)$ is a function $f : V \to \{-1, 1, 2\}$ satisfying the condition that for every vertex u, $f[u] \geq 1$. The weight of signed Italian dominating function is the value $f(V) = \sum_{u \in V} f(u)$. The signed Italian domination number of a graph G, denoted by $\gamma_{sI}(G)$, is the minimum weight of a signed Italian dominating function on a graph G. In this paper, we determine the signed Italian domination number of some classes of graphs. We also present several lower bounds on the signed Italian domination number of a graph. In particular, for a graph G without isolated vertex we show that $\gamma_{sI}(G) \geq \frac{3n-4m}{2}$ and characterize all graphs attaining equality in this bound. We show that if G is a graph of order $n \geq 2$, then $\gamma_{sI}(G) \geq 3\sqrt{\frac{n}{2}} - n$ and this bound is sharp.

Keywords: Domination, Signed Italian Dominating Function, Signed Italian Domination Number.

MSC 2010: 05C69.

1 Introduction

Throughout this paper we consider (non trivial) simple graphs, that are finite and undirected graphs without loops or multiple edges.

Let $G = (V, E)$ be a graph of order n and size m. For every vertex $v \in V$, the *open neighborhood* of v is defined by $N_G(v) = \{u \in V \mid uv \in E(G)\}$. Also the *closed neighborhood* of v is defined by $N_G[v] = N_G(v) \cup \{v\}$. For a subset $S \subset V$ we denoted the number of neighbors of a vertex $v \in S$ by $d_S(v)$. In particular, $d(v) = deg_G(v) = |N(v)|$. The *minimum* and *maximum* degree among the vertices of G are denoted by δ and Δ, respectively.

A graph G is *k-colorable* if there exists the function $f : V(G) \to \{1, 2, \ldots, k\}$ such that $f(u) \neq f(v)$ for any edge $uv \in E(G)$. The minimum positive integer k for which G is k-colorable is the *Chromatic number* of G and is denoted by $\chi(G)$.

A set $S \subset V$ in a graph G is called a *dominating set* if every vertex of G is either in S or adjacent to a vertex of S. The *domination number* $\gamma(G)$ equals the minimum cardinality of a dominating set on G.

For a subset $T \subset \mathbb{Z}$, the *weight* of function $f : V \to T$ is denoted by $w(f)$ and defined by $w(f) = \sum_{v \in V} f(v)$. For $S \subset V$, we set $f(S) = \sum_{v \in S} f(v)$.

A *signed dominating function* (SDF) on a graph $G = (V, E)$ is a function $f : V \to \{-1, 1\}$ such that $f(N[v]) \geq 1$ for every vertex $v \in V$.

The *signed domination number*, denoted by $\gamma_s(G)$, is the minimum weight of a SDF on G; that is, $\gamma_s(G) = min\{w(f) \mid f \textit{ is a SDF on } G\}$.

Recently, Ahangar et al. [1] defined a *signed Roman dominating function* (SRDF) on a graph $G = (V, E)$ as a function $f : V \to \{-1, 1, 2\}$ such that $f(N[v]) \geq 1$ for every vertex $v \in V(G)$ and every vertex u with $f(u) = -1$ is adjacent to a vertex v with $f(v) = 2$. The signed domination number, denoted by $\gamma_{sR}(G)$, is the minimum weight of a SRDF on G; that is, $\gamma_s(G) = min\{w(f) \mid f \textit{ is a SRDF on } G\}$.

Mustapha Chellali et al. (2016) [4] defined an *Italian dominating function* (IDF) on a graph $G = (V, E)$ to be a function $f : V \to \{0, 1, 2\}$ with the property that for every vertex $v \in V(G)$ with $f(v) = 0$, $f(N[v]) \geq 2$. The *Italian domination number* denoted by $\gamma_I(G)$, is the minimum weight of a IDF on graph G; that is, $\gamma_I(G) = min\{w(f) \mid f \textit{ is a IDF on } G\}$. For further results on Italian domination see [7] and [6].

A *signed Italian dominating function* (SIDF) on a graph $G = (V, E)$ is a function $f : V \to \{-1, 1, 2\}$ with the property that for every vertex $v \in V$, $f(N[v]) \geq 1$. Thus a signed Italian dominating function combines the properties of both an Italian dominating function and a signed dominating function. The *signed Italian domination number*, denoted by $\gamma_{sI}(G)$, is the minimum weight of a SIDF on G; that is,

$\gamma_{sI}(G) = min\{w(f) \mid f\ is\ a\ SIDF\ on\ G\}$. A SIDF of weight $\gamma_{sI}(G)$ is called a $\gamma_{sI}(G)$-function. For a vertex $v \in V$, we denote $f(N[v])$ by $f[v]$ for notational convenience. For a SIDF f on G, let $V_i = \{v \in V(G) \mid f(v) = i\}$ for $i = -1, 1, 2$. Since this partition determines f, we can equivalently write $f = (V_{-1}, V_1, V_2)$.

Firstly note that if $f = (V_{-1}, V_1, V_2)$ is a SIDF on a graph G of order n, then

(i) $|V_{-1}| + |V_1| + |V_2| = n$,
(ii) $w(f) = |V_1| + 2|V_2| - |V_{-1}|$,
(iii) $V_1 \cup V_2$ is a dominating set of G.

A function $f : V(G) \to \{-1, 1, 2, 3\}$ is a *signed double Roman dominating function* (SDRDF) on graph G if (i) every vertex v with $f(v) = -1$ is adjacent to at least two vertices assigned a 2 or to at least one vertex w with $f(w) = 3$, (ii) every vertex v with $f(v) = 1$ is adjacent to at least one vertex w with $f(w) \geq 2$ and (iii) $f[v] = \sum_{u \in N[v]} f(u) \geq 1$ holds for any vertex v. The *signed double Roman domination number* $\gamma_{sdR}(G)$ is the minimum weight of a SIDF on G. The signed double Roman domination was introduced by Ahangar et al. [2]

A *cycle* on n vertices is denoted by C_n, while a *path* on n vertices is denoted by P_n. We denoted by K_n the *complete graph* on n vertices and $K_{n,m}$ the *complete bipartite graph* with one partite set of cardinality n and the other of cardinality m. A *star* is a complete bipartite graph of the form $S_n = K_{1,n-1}$. A *double star* with respectively p and q leaves attached at each support vertex is denoted by $DS_{p,q}$. The *distance* $d_G(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u-v$ path in G. The *diameter* of a graph G, denoted by $diam(G)$, is the greatest distance between two vertices of G. The *corona product* of two graphs G_1 and G_2, denoted by $G = G_1 \odot G_2$, is a graph obtained by taking one copy of G_1 and $|V(G_1)|$ copies of G_2 and joining the i^{th}-vertex of G_1 with all the vertices of the i^{th}-copy of G_2.

In this paper, we present various bounds on the signed Italian domination number of graph. In addition, we determine the signed Italian domination number for special classes of graphs including complete graphs, cycles, paths and complete bipartite graphs.

2 Special Classes Of Graphs

In this section we determine the signed Italian domination number for complete graphs, cycles, paths and complete bipartite graphs.

Lemma 1. *Let G be a graph of order n such that $\gamma(G) = 1$. Then $\gamma_{sI}(G) \geq 1$.*

Proof. Let v be a vertex of G with $deg(v) = n - 1$ and f be a γ_{sI}-function of G. Hence $\gamma_{sI}(G) = w(f) = f[v] \geq 1$. □

Proposition 1. *For $n \geq 1$, $\gamma_{sI}(K_n) = 1$.*

Proof. Let f be a γ_{sI}-function on K_n. Since $\gamma(K_n) = 1$, hence $\gamma_{sI}(K_n) \geq 1$ by Lemma 1.

Now define the functions $f : V(K_n) \to \{-1, 1, 2\}$ as following:
If n is even, then let $f(v_1) = 2$, $f(v_i) = -1$ for $2 \leq i \leq \frac{n+2}{2}$ and $f(v_i) = 1$ for $\frac{n+4}{2} \leq i \leq n$. If n is odd, then let $f(v_i) = 1$ for $1 \leq i \leq \frac{n+1}{2}$ and $f(v_i) = -1$ for $\frac{n+3}{2} \leq i \leq n$. It is clear that in any case we have defined a SIDF on K_n of weight 1. Hence $\gamma_{sI}(K_n) \leq 1$ and consequently, $\gamma_{sI}(K_n) = 1$. □

Proposition 2. *For $n \geq 3$,*

$$\gamma_{sI}(P_n) = \begin{cases} \lceil \frac{n}{3} \rceil + 1 & \textit{if } n \equiv 0 \textit{ or } 2 \ (\textit{mod } 3), \\ \lceil \frac{n}{3} \rceil & \textit{if } n \equiv 1 \ (\textit{mod } 3). \end{cases}$$

Proof. Let $P_n = v_1 v_2 \ldots v_n$. Define the function $f : V(P_n) \to \{-1, 1, 2\}$ as follows:
If $n \equiv 0(mod\ 3)$, then let $f(v_{3i+1}) = -1$ for $0 \leq i \leq \frac{(n-3)}{3}$, $f(v_2) = 2$, $f(v_{3i}) = 1$ for $1 \leq i \leq \frac{n}{3}$ and $f(v_{3i+2}) = 1$ for $1 \leq i \leq \frac{(n-3)}{3}$. If $n \equiv 1(mod\ 3)$, then let $f(v_{3i}) = -1$ for $0 \leq i \leq \frac{(n-1)}{3}$, $f(v_1) = f(v_{n-2}) = 2$, $f(v_{3i+2}) = 1$ for $0 \leq i \leq \frac{(n-7)}{3}$ and $f(v_{3i+1}) = 1$ for $1 \leq i \leq \frac{(n-4)}{3}$. If $n \equiv 2(mod\ 3)$, then let $f(v_{3i+1}) = -1$ for $0 \leq i \leq \frac{(n-5)}{3}$, $f(v_n) = -1$,

$f(v_2) = f(v_{n-1}) = 2$, $f(v_{3i}) = 1$ for $1 \leq i\frac{(n-2)}{3}$ and $f(v_{3i+2}) = 1$ for $1 \leq i \leq \frac{(n-5)}{3}$. Clearly, f is a SIDF of P_n and thus $\gamma_{sI}(P_n) \leq \lceil \frac{n}{3} \rceil$ if $n \equiv 1\ (mod\ 3)$ and $\gamma_{sI}(P_n) \leq \lceil \frac{n}{3} \rceil + 1$ when $n \not\equiv 1\ (mod\ 3)$.

To prove the inverse inequality, let f be a γ_{sI}-function on P_n. First, assume that $n \equiv 0 (mod\ 3)$. If $f(v_1) + f(v_2) \geq 2$, then $f(v_1) \geq 1$ and we have

$$\gamma_{sI}(P_n) = f(v_1) + \sum_{i=0}^{\frac{(n-3)}{3}} f[v_{3i+3}] \geq 1 + \frac{n}{3}.$$

Hence we assume that $f(v_1) + f(v_2) = 1$. Then we must have $f(v_1) = -1$, $f(v_2) = 2$ and to Italian dominate v_2, we must have $f(v_3) \geq 1$, and so $f[v_2] \geq 2$. Therefore

$$\gamma_{sI}(P_n) = f[v_2] + \sum_{i=1}^{\frac{(n-3)}{3}} f[v_{3i+2}] \geq 2 + \frac{(n-3)}{3} = 1 + \frac{n}{3}.$$

By the same argument, the result is obtained in cases $n \equiv 1, 2\ (mod\ 3)$. $\square$

Proposition 3. *For $n \geq 3$,*

$$\gamma_{sI}(C_n) = \begin{cases} \lceil \frac{n}{3} \rceil = t & \text{if } n = 3t, \\ \lceil \frac{n}{3} \rceil = t+1 & \text{if } n = 3t+1, \\ \lceil \frac{(n+4)}{3} \rceil = t+2 & \text{if } n = 3t+2. \end{cases}$$

Proof. Let $C_n = v_1 v_2 ... v_n v_1$ and let f be a γ_{sI}-function on C_n.

Assume first that $n = 3t$ with an integer $t \geq 1$. We deduce from the fact $f(v_{3i-2}) + f(v_{3i-1}) + f(v_{3i}) \geq 1$ for $1 \leq i \leq t$ that

$$\gamma_{sI}(C_n) = \gamma_{sI}(C_{3t}) = \sum_{i=1}^{t} f(v_{3i-2}) + f(v_{3i-1}) + f(v_{3i}) \geq t.$$

Now define the function $f : V(C_{3t}) \to \{-1, 1, 2\}$ by $f(v_{3i-1}) = -1$ and $f(v_{3i-2}) = f(v_{3i}) = 1$ for $1 \leq i \leq t$. Then $f[v_j] \geq 1$ for each $0 \leq j \leq 3t-1$ and therefore f is a SIDF on C_{3t} of weight t. Thus $\gamma_{sI}(C_{3t}) \leq w(f) = t$. Consequently, $\gamma_{sI}(C_{3t}) = t = \lceil \frac{n}{3} \rceil$.

Assume next that $n = 3t + 1$ with an integer $t \geq 1$. If $f(v_i) \geq 1$ for all $1 \leq i \leq n$, then $\gamma_{sI}(C_n) \geq n > \lceil \frac{n}{3} \rceil$. Hence assume now that $f(v_1) = -1$. By definition v_1 must have a neighbor with label 2 or two neighbors with label 1. Let v_1 have a neighbor with label 2, say v_{3t+1}. Since $f[v_1] \geq 1$, then we must have $f(v_2) \geq 1$. It follows that

$$\gamma_{sI}(C_n) = \gamma_{sI}(C_{3t+1}) = \sum_{i=1}^{t} f[v_{3i-1}] + f(v_{3t+1}) \geq t + 2 > t + 1.$$

Let v_1 have two neighbors with label 1, i.e $f(v_{3t+1}) = f(v_2) = 1$. Since $f[v_{3t+1}] \geq 1$ and $f[v_2] \geq 1$, we must have $f(v_{3t}) \geq 1$ and $f(v_3) \geq 1$. It follows that

$$\gamma_{sI}(C_n) = \gamma_{sI}(C_{3t+1}) = f[v_2] + \sum_{i=1}^{t-1} f[v_{3i+2}] + f(v_{3t+1}) \geq t + 1.$$

On the other hand define $f : V(C_{3t+1}) \to \{-1, 1, 2\}$ by $f(v_{3i-1}) = -1$, $f(v_{3i}) = f(v_{3i-2}) = 1$ for $1 \leq i \leq t$ and $f(v_{3t+1}) = 1$. Then $f[v_j] \geq 1$ for each $1 \leq j \leq 3t + 1$ and therefore f is a SIDF on C_{3t+1} of weight $t + 1$. Thus $\gamma_{sI}(C_{3t+1}) \leq w(f) = t + 1$. Consequently, $\gamma_{sI}(C_{3t+1}) = \lceil \frac{n}{3} \rceil = t + 1$.

Finally, assume that $n = 3t + 2$ with an integer $t \geq 1$. The result holds if $f(v) \geq 1$ for all $v \in V(C_{3t+2})$. Thus without loss of generality, assume that $f(v_1) = -1$. By definition v_1 must have a neighbor with label 2 or two neighbors with label 1. Let v_1 have a neighbor with label 2, say v_2. Since $f[v_2] \geq 1$ and $f[v_{3t+2}] \geq 1$, we must have $f(v_3) \geq 1$. It follows that

$$\gamma_{sI}(C_{3t+2}) = \sum_{i=1}^{3} f(v_i) + \sum_{i=1}^{t-1} f[v_{3i+2}] + f[v_{3t+2}] \geq 2 + (t-1) + 1 = t + 2.$$

Let v_1 have two neighbors with label 1, i.e $f(v_{3t+2}) = f(v_2) = 1$. Since $f[v_{3t+2}] \geq 1$ and $f[v_2] \geq 1$, we must have $f(v_{3t+1}) \geq 1$ and $f(v_3) \geq 1$. It follows that

$$\begin{aligned} \gamma_{sI}(C_{3t+2}) &= f[v_2] + \sum_{i=1}^{t-1} f[v_{3i+2}] + f(v_{3t+1}) + f(v_{3t+2}) \geq \\ &\geq 1 + (t-1) + 2 = t - 2. \end{aligned} \tag{1}$$

Now define the function $f : V(C_{3t+2}) \to \{-1, 1, 2\}$ by $f(v_{3i}) = -1$, $f(v_{3i-1}) = f(v_{3i-2}) = 1$ for each $1 \le i \le t$ and $f(v_{3t+1}) = f(v_{3t+2}) = 1$. Then $f[v_j] \ge 1$ for each $1 \le j \le 3t + 2$ and therefore f is a SIDF on C_{3t+2} of weight $t + 2$. Thus $\gamma_{sI}(C_{3t+2}) \le w(f) = t + 2$.

Thus the proof is complete. $\square$

Proposition 4. *For $n \ge 2$,*

$$\gamma_{sI}(K_{1,n-1}) = \begin{cases} 1 & \textit{if } n \textit{ is even,} \\ 2 & \textit{if } n \textit{ is odd.} \end{cases}$$

Proposition 5. *For $2 \le m \le n$,*

$$\gamma_{sI}(K_{m,n}) = \begin{cases} 2 & \textit{if } m = 2 \textit{ and } n \ge 2, \\ 3 & \textit{if } m = 3 \textit{ and } n \ge 3, \\ 4 & \textit{if } n, m \ge 4. \end{cases}$$

Proof. Let $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ be the bipartite sets of $K_{m,n}$.

First assume that $m = 2$. For $n = 2, 3$ the result is obvious. Assume that $n \ge 4$. Let f be a γ_{sI}-function on $K_{2,n}$. If $f(x_1) = -1$, then $f(y_i) > 0$ for any $y_i \in Y$. In addition $f(x_2) > 0$. Hence $w(f) = f[x_1] + f(x_2) \ge 2$. Now assume that $f(x_1), f(x_2) > 0$. If $f(x_1) = 2$, then $\sum_{i=1}^{n} f(y_i) \ge -1$ and hence $w(f) = f[x_1] + f(x_2) \ge 2$. If $f(x_1) = f(x_2) = 1$, then $\sum_{i=1}^{n} f(y_i) \ge 0$, and so $w(f) = f[x_1] + f(x_2) \ge 2$. Since we have discussed all possible cases, we obtain $\gamma_{sI}(K_{2,n}) \ge 2$.

To prove $\gamma_{sI}(K_{2,n}) \le 2$, define the function $f : V(K_{2,n}) \to \{-1, 1, 2\}$ by $f(x_1) = f(x_2) = 1$ and $f(y_i) = (-1)^i$ for $1 \le i \le n$ when n is even, and by $f(x_1) = f(x_2) = 1$, $f(y_1) = 2$, $f(y_2) = f(y_3) = -1$ and $f(y_i) = (-1)^{i+1}$ for $4 \le i \le n$ when n is odd. It is clear that f is a SIDF on $K_{2,n}$ of weight 2, and so $\gamma_{sI}(K_{2,n}) \le 2$. Therefore $\gamma_{sI}(K_{2,n}) = 2$.

Now assume that $m = 3$. Let f be a γ_{sI}-function on $K_{3,n}$. If for any $x_i \in X$, $f(x_i) > 0$, then we have $w(f) = \sum_{i=1}^{m-1} f(x_i) + f[x_m] \ge (m-1)+1 \ge 3$. Hence we assume that there are $x_i \in X$ and $y_i \in Y$ such that $f(x_i) = f(y_i) = -1$. It follows from $f[x_i] \ge 1$ and $f[y_i] \ge 1$ that

$\sum_{i=1}^{m} f(x_i) \geq 2$ and $\sum_{i=1}^{n} f(y_i) \geq 2$. Therefore $w(f) = \sum_{i=1}^{m} f(x_i) + \sum_{i=1}^{n} f(y_i) \geq 4$. In any case, $\gamma_{sI}(K_{3,n}) \geq 3$.

To prove $\gamma_{sI}(K_{3,n}) \leq 3$, define the function $f : V(K_{3,n}) \rightarrow \{-1, 1, 2\}$ as follows:
If $n = 3$, then let $f(x_1) = f(x_2) = f(x_3) = 1$, $f(y_1) = 2$ and $f(y_2) = f(y_3) = -1$. If $n > 3$ is odd, then $f(x_1) = f(x_2) = f(x_3) = 1$, $f(y_1) = 2$, $f(y_2) = f(y_3) = -1$ and $f(y_i) = (-1)^{i+1}$ for $4 \leq i \leq n$. If $n > 3$ is even, then $f(x_1) = f(x_2) = f(x_3) = 1$ and $f(y_i) = (-1)^i$ for $1 \leq i \leq n$. It is clear that f is a SIDF on $K_{3,n}$ of weight 3, and so $\gamma_{sI}(K_{3,n}) \leq 3$. Therefore $\gamma_{sI}(K_{3,n}) = 3$.

Finally, assume that $m \geq 4$. Let f be a γ_{sI}-function on $K_{m,n}$. If for any $x_i \in X$, $f(x_i) > 0$ (the case $f(y_i) > 0$, for any $y_i \in Y$ is similar), then we have $w(f) = \sum_{i=1}^{m-1} f(x_i) + f[x_m] \geq (m-1)+1 \geq 4$. Hence we assume that there are $x_i \in X$ and $y_i \in Y$ such that $f(x_i) = f(y_i) = -1$. It follows from $f[x_i] \geq 1$ and $f[y_i] \geq 1$ that $\sum_{i=1}^{m} f(x_i) \geq 2$ and $\sum_{i=1}^{n} f(y_i) \geq 2$. Therefore $w(f) = \sum_{i=1}^{m} f(x_i) + \sum_{i=1}^{n} f(y_i) \geq 4$. In any case, $\gamma_{sI}(K_{m,n}) \geq 4$ when $m \geq 4$.

To prove the inverse inequality, define the function $f : V(K_{m,n}) \rightarrow \{-1, 1, 2\}$ as follows:
If $n = m$ and n is even, then let $f(x_i) = f(y_i) = 1$ for $1 \leq i \leq \frac{n}{2} + 1$ and $f(x_i) = f(y_i) = -1$ for $\frac{n}{2} + 2 \leq i \leq n$. If $n = m$ and n is odd, then let $f(x_1) = f(y_1) = 2$, $f(x_i) = f(y_i) = 1$ for $2 \leq i \leq \frac{n-1}{2} + 1$ and $f(x_i) = f(y_i) = -1$ for $\frac{n-1}{2} + 2 \leq i \leq n$. If $m \neq n$ and m, n are odd, then let $f(x_1) = f(y_1) = 2$, $f(x_i) = (-1)^{i+1}$ for $2 \leq i \leq m$ and $f(y_i) = (-1)^{i+1}$ for $2 \leq i \leq n$. If $m \neq n$ and m, n are even, then let $f(x_1) = f(x_2) = 2$, $f(y_1) = f(y_2) = 2$, $f(x_3) = f(x_4) = -1$, $f(y_3) = f(y_4) = -1$, $f(x_i) = (-1)^{i+1}$ for $5 \leq i \leq m$ and $f(y_i) = (-1)^{i+1}$ for $5 \leq i \leq n$. If $m \neq n$, m is even and n is odd (the case when m is odd and n is even is similar), then let $f(x_1) = f(x_2) = f(y_1) = 2$, $f(y_i) = (-1)^{i+1}$, $f(x_3) = f(x_4) = -1$ and $f(x_i) = (-1)^{i+1}$ for $5 \leq i \leq n$. It is clear that in any case we have defined a SIDF of weight 4, and thus $\gamma_{sI}(K_{m,n}) \leq 4$. Therefore $\gamma_{sI}(K_{m,n}) = 4$ and the proof is complete. $\square$

3 Preliminary Results And Some Bounds

In this section we present basic properties of the signed Italian dominating functions, the signed Italian domination numbers and bounds on the signed Italian domination number.

In Proposition 6 we show that some bounds for SIDF are based on Δ and δ.

Proposition 6. *Let $f = (V_{-1}, V_1, V_2)$ be a SIDF on a graph G of order n, then*

(i) $(2\Delta + 1)|V_2| + \Delta|V_1| \geq (\delta + 2)|V_{-1}|$,
(ii) $(2\Delta + \delta + 3)|V_2| + (\Delta + \delta + 2)|V_1| \geq (\delta + 2)n$,
(iii) $(\Delta + \delta + 2)w(f) \geq (\delta - \Delta + 2)n + (\delta - \Delta)|V_2|$,
(iv) $w(f) \geq \frac{(\delta-2\Delta+1)n}{(2\Delta+\delta+3)} + |V_2|$.

In addition all inequalities are sharp.

Proof. *(i)* We have that

$$\begin{aligned} n =&|V_{-1}| + |V_1| + |V_2| \leq \sum_{v\in V} f[v] = \sum_{v\in V}(d(v) + 1)f(v) \\ =& \sum_{v\in V_2} 2(d(v) + 1) + \sum_{v\in V_1}(d(v) + 1) - \sum_{v\in V_{-1}}(d(v) + 1) \\ \leq& 2(\Delta + 1)|V_2| + (\Delta + 1)|V_1| - (\delta + 1)|V_{-1}|. \end{aligned}$$

and the desired result follows.
(ii) This follows immediately from Part *(i)* by substituting $|V_{-1}| = n - |V_1| - |V_2|$.
(iii) Since $w(f) = |V_1| + 2|V_2| - |V_{-1}|$ and $|V_{-1}| + |V_1| + |V_2| = n$, we have that

$$\begin{aligned} (\Delta + \delta + 2)w(f) &= (\Delta + \delta + 2)(2(|V_1| + |V_2|) - n + |V_2|) \\ &\geq 2(\delta + 2)n - 2(\Delta + 1)|V_2| + (\Delta + \delta + 2)(|V_2| - n) \\ &= (\delta - \Delta + 2)n + (\delta - \Delta)|V_2|. \end{aligned}$$

(iv) From the proof of Part *(i)* we have

$$\begin{aligned} n &\leq 2(\Delta + 1)|V_1 \cup V_2| - (\delta + 1)|V_{-1}| \\ &= (2\Delta + \delta + 3)|V_1 \cup V_2| - (\delta + 1)n, \end{aligned}$$

and so

$$|V_1 \cup V_2| \geq \frac{n(\delta+2)}{(2\Delta+\delta+3)}.$$

Therefore

$$w(f) = 2|V_1 \cup V_2| - n + |V_2| \geq \frac{(\delta - 2\Delta + 1)n}{(2\Delta+\delta+3)} + |V_2|.$$

If $G = K_n$ or $G = C_{3t}$, where $t \geq 1$, then Parts (i), (ii) and (iii) are sharp and also if $G = nK_2$, where $n \geq 1$, then Part (iv) is sharp. $\square$

As an immediate consequence of Proposition 6, we obtain a lower bound on the signed Italian domination number of graphs.

Corollary 1. *If G is a graph of order n such that $\delta < \Delta$, then*

$$\gamma_{sI}(G) \geq \left(\frac{-2\Delta^2 + 2\Delta\delta + \Delta + 2\delta + 3}{(\Delta+1)(2\Delta+\delta+3)} \right) n.$$

Proof. Multiplying both sides of the inequality in Proposition 6 (iv), by $\Delta - \delta$ and adding the resulting inequality to the inequality in Proposition 6 (iii), we yield the desired result. $\square$

Proposition 7. *For $r \geq 1$, if G is an r-regular graph of order n, then*

$$\gamma_{sI}(G) \geq \frac{n}{(r+1)}.$$

Proof. Let $f = (V_{-1}, V_1, V_2)$ be a SIDF on G. We have that

$$n \leq \sum_{v \in V} f[v] = (r+1) \sum_{v \in V} f(v) = (r+1)w(f).$$

Hence $\gamma_{sI}(G) \geq \frac{n}{(r+1)}$.
If $G = (K_n)$, then $\gamma_{sI}(K_n) = 1$ and the equality holds. $\square$

Theorem 1. *If G is a graph of order n such that $\delta \geq 1$, then*

$$\gamma_{sI}(G) \geq 2 + \Delta - n.$$

Proof. Let $u \in V(G)$ be a vertex of degree Δ and let f be a $\gamma_{sI}(G)$-function. Then the definitions imply that

$$\begin{aligned}\gamma_{sI}(G) &= \sum_{x\in V(G)} f(x) = \sum_{x\in N[u]} f(x) + \sum_{x\in V(G)-N[u]} f(x)\\ &\geq 1 + \sum_{x\in V(G)-N[u]} f(x) \geq 1-(n-(\Delta+1))\\ &= 2+\Delta-n.\end{aligned}$$

□

Note that the inequality of Theorem 1, is sharp for $G = K_n$.

In the following result we present a relation between Chromatic number and signed Italian domination.

Corollary 2. *Let G be a connected graph. If G is not an odd cycle or a complete graph, then $\gamma_{sI}(G) \geq 2+\chi(G)-n$, otherwise $\gamma_{sI}(G) \geq 1+\chi(G)-n$.*

Proof. Since G is a connected graph, then by Brooks' Theorem [3] $\chi(G) \leq \Delta(G)$ if G is not an odd cycle or a complete graph. Now by applying Theorem 3.4, we conclude that $\gamma_{sI}(G) \geq 2+\chi(G)-n$, otherwise $\gamma_{sI}(G) \geq 1+\chi(G)-n$. □

A set $S \subset V(G)$ is a 2-*packing* of the graph G if $N[u] \cap N[v] = \emptyset$ for any two distinct vertices $u, v \in S$. The 2-*packing number* $\rho(G)$ of G is defined by

$$\rho(G) = max\{|S| \ : \ S \ is \ a \ 2-packing \ of \ G\}.$$

Clearly, for all graphs G, $\rho(G) \leq \gamma(G)$.

Theorem 2. *Let G be a graph of order n such that $\delta \geq 1$. Then*

$$\gamma_{sI}(G) \geq \rho(G)(2+\delta)-n.$$

Proof. Let $\{v_1, v_2, \ldots, v_{\rho(G)}\}$ be a 2-packing of G and let f be a γ_{sI}-function on G. If we define the set $A = \cup_{i=1}^{\rho(G)} N[v_i]$, then since $\{v_1, v_2, \ldots, v_{\rho(G)}\}$ is a 2-packing, we have that

$$|A| = \sum_{i=1}^{\rho(G)} (d(v_i) + 1) \geq \rho(G)(\delta(G) + 1).$$

Now we have

$$\begin{aligned}\gamma_{sI}(G) &= \sum_{x \in V(G)} f(x) = \sum_{i=1}^{\rho(G)} f[v_i] + \sum_{x \in V(G)-A} f(x) \\ &\geq \rho(G) - (n - |A|) \geq \rho(G) - n + \rho(G)(\delta(G) + 1) \\ &= \rho(G)(2 + \delta) - n.\end{aligned}$$

□

Example 1. *Now we show that the bound in Theorem 2, is sharp. Let F be an arbitrary graph of order $t \geq 1$. Let G be a graph of order st, where $s \geq 2$ is obtained as follows:*
For every vertex $v \in V(F)$ add a vertex-disjoint copy of a complete graph K_s and identify the vertex v with one vertex of added complete graph. Let $G_1, G_2, \ldots, G_t$ be the added copies of K_s and let v_i be the vertex of G_i for $1 \leq i \leq t$ that is identified with a vertex of F. Let $f_i : V(G_i) \to \{-1, 1, 2\}$ be the SIDF on the complete graph $G_i \cong K_s$ defined as in Proposition 1. We note that the function f_i assigns to at least one vertex of G_i the value 2 or 1 respectively when $|G_i|$ is even or $|G_i|$ is odd. We choose v_i be one such vertex of G_i, and so if $|G_i|$ is even, then $f(v_i) = 2$, otherwise $f(v_i) = 1$. As shown in Proposition 1, we have $w(f_i) = 1$. Now we define the function $f : V(G) \to \{-1, 1, 2\}$ by $f(v) = f_i(v)$ for each vertex $v \in V(G_i)$. If $v = v_i$ for $1 \leq i \leq t$, then $f[v] \geq f_i[v]$ with strict inequality if the vertex corresponding to v_i is not isolated in F. If $v \neq v_i$ for $1 \leq i \leq t$, then $f[v] = f_i[v]$. Therefore the function $f = \cup_{i=1}^{t} f_i$ is a SIDF on G, and so $\gamma_{sI}(G) \leq w(f) = \sum_{i=1}^{t} f_i = t$. On the other hand, by Theorem 2, and noting that here $\delta(G) = s - 1$, $\rho(G) = t$ and $n(G) = st$ we

have $\gamma_{sI}(G) \geq \rho(G)(2+\delta(G)) - n(G) = t$. *Consequently,* $\gamma_{sI}(G) = \rho(G)(2+\delta(G)) - n(G) = t$.

Corollary 3. *Let G be a graph of order n such that $\delta \geq 1$. Then*

$$\gamma_{sI}(G) \geq \left(1 + \lfloor \frac{diam(G)}{3} \rfloor\right)(2+\delta) - n.$$

Proof. We assume that $diam(G) = 3t + r$ is such that $t \geq 0$ and $0 \leq r \leq 2$. Let $x_0x_1 \dots x_n$ be a diametral path and define the set $A = \{x_0, x_3, \dots, x_{3t}\}$. Then A is a 2-packing set of G such that $|A| = 1 + \lfloor \frac{diam(G)}{3} \rfloor$. Since $\rho(G) \geq |A|$, by Theorem 2, we have

$$\gamma_{sI}(G) \geq \rho(G)(2+\delta) - n \geq \left(1 + \lfloor \frac{diam(G)}{3} \rfloor\right)(2+\delta) - n.$$

□

Now in the following we find bounds for signed Italian domination for cubic graph.

Theorem 3. *Let G be a connected cubic graph of order n. Then*

$$\frac{n}{4} \leq \gamma_{sI}(G) \leq \frac{3n}{4}.$$

Proof. The lower bound follows from Proposition 7. Now we prove the theorem for the upper bound. Let G be the Petersen graph. Consider the labeling of the Petersen graph in Figure 1. Then f is a SIDF on G of weight $w(f) = 5 = \frac{4n}{8}$ which implies that $\gamma_{sI}(G) \leq \frac{n}{2} < \frac{3n}{4}$. Now assume that G is not a Petersen graph. Since every signed dominating function is a signed Italian dominating function, then by Theorem 2 [5] the proof is complete. □

Example 2. *To see that the lower bound presented in Theorem 3 is sharp, consider a cycle $C_{3t} : v_1v_2 \dots v_{3t}v_1$, where $t \geq 1$, add t new vertices $x_1, x_2, \dots, x_t$ and join x_i to the $v_{3i-2}, v_{3i-1}, v_{3i}$ for $1 \leq i \leq t$. Let G denote the resulting cubic graph of order $n = 4t$. Define the*

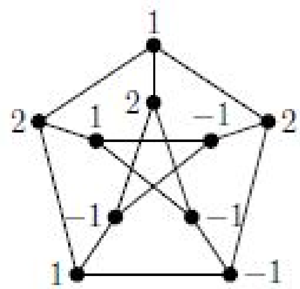

Figure 1. A labeling of the Petersen graph

function $f : V(G) \to \{-1, 1, 2\}$ *by* $f(x_i) = 1$ *for* $1 \le i \le t$, $f(v_{3i-2}) = f(v_{3i}) = -1$ *for* $1 \le i \le t$ *and* $f(v_{3i-1}) = 2$ *for* $1 \le i \le t$. *Then* f *is a SIDF on* G *of weight* t, *and so* $\gamma_{sI}(G) \le t$. *By Proposition 7, we have that* $\gamma_{sI}(G) \ge t$. *Consequently,* $\gamma_{sI}(G) = t = \frac{n}{4}$.

Remark 1. *If* f *is a signed Italian dominating function on* G *and* $u \in V(G)$, *then there exists a signed Italian dominating function* g *on* G, *with* $g(u) > 0$ *and* $w(g) - w(f) \le 2$.

Lemma 2. *Let* G *be a graph of order* n. *If* $uv \in E(G)$, *then*

$$\gamma_{sI}(G \setminus uv) - 4 \le \gamma_{sI}(G) \le \gamma_{sI}(G \setminus uv) + 2.$$

Proof. For the upper bound we assume that f is a γ_{sI}-function on $(G \setminus uv)$. It follows from Remark 1, that there exists a signed Italian dominating function g on $(G \setminus uv)$, with $g(u) > 0$ and $w(g) - w(f) \le 2$. Now we define the function $h : V(G) \to \{-1, 1, 2\}$ such that $h(x) = g(x)$ for each $x \in V(G)$.

Now assume that f is a γ_{sI}-function on G. For the lower bound we define the function $g : V(G \setminus uv) \to \{-1, 1, 2\}$ and consider the following cases:

Case 1. Assume that $f(u) = -1$ and $f(v) = 2$ such that $f[u] = 1$ and $f[v] \ge 1$. If there is no vertex with value -1 under f such $x \in N[v]$ where $x \ne u$ and $f[x] = 1$, then $g(u) = 1$, $g(v) = 1$ and $g(y) = f(y)$ for any $y \in V(G \setminus uv)$. Thus g is a SIDF on $(G \setminus uv)$, and so $\gamma_{sI}(G \setminus uv) \leq w(g) \leq \gamma_{sI}(G) + 1$, which implies that $\gamma_{sI}(G \setminus uv) - 1 \leq \gamma_{sI}(G)$.

Case 2. Assume that $f(u) = f(v) = 1$. If $f[u], f[v] \geq 2$, then $g = f$. If $f[u] \geq 1$ and $f[v] \geq 2$, then $g(u) = 2$ and $g(x) = f(x)$ for any $x \in V(G \setminus uv)$, where $x \neq u$. If $f[u], f[v] \geq 1$, then $g(u) = g(v) = 2$ and $g(x) = f(x)$ for any $x \in V(G \setminus uv)$, where $x \neq u, v$. Thus g is a SIDF on $(G \setminus uv)$, and so $\gamma_{sI}(G \setminus uv) \leq w(g) \leq \gamma_{sI}(G) + 2$, which implies that $\gamma_{sI}(G \setminus uv) - 2 \leq \gamma_{sI}(G)$.

Case 3. Assume that $f(u) = 1$ and $f(v) = 2$. If $f[u] = 2$ and $f[v] = 1$, then $g(u) = 2$, $g(x) = 1$, where $x \in N[v]$ assigned a -1 under f and $g(y) = f(y)$ for any $y \in V(G \setminus uv)$, where $y \neq u, x$. Thus g is a SIDF on $(G \setminus uv)$, and so $\gamma_{sI}(G \setminus uv) \leq w(g) \leq \gamma_{sI}(G) + 3$, which implies that $\gamma_{sI}(G \setminus uv) - 3 \leq \gamma_{sI}(G)$.

Case 4. Assume that $f(u) = f(v) = 2$. If $f[u] = f[v] = 1$ or $f[u] = f[v] = 2$, then $g(x) = g(y) = 1$, where $x \in N[u]$, $y \in N[v]$ assigned a -1 under f respectively and $g(z) = f(z)$ for any $z \in V(G \setminus uv)$, where $z \neq x, y$. Thus g is a SIDF on $(G \setminus uv)$, and so $\gamma_{sI}(G \setminus uv) \leq w(g) \leq \gamma_{sI}(G) + 4$, which implies that $\gamma_{sI}(G \setminus uv) - 4 \leq \gamma_{sI}(G)$. $\square$

Remark 2. *We present several examples of graphs that satisfy the bounds in Lemma 2. Notice that the edge uv is denoted by $----$.*

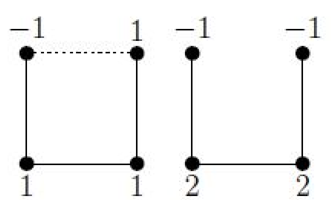

Figure 2. $\gamma_{sI}(G) = \gamma_{sI}(G \setminus uv)$

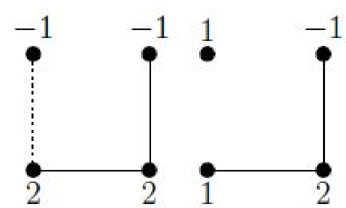

Figure 3. $\gamma_{sI}(G) = \gamma_{sI}(G \setminus uv) - 1$

Proposition 8. *For every graph G of order n, $2\gamma(G) - n \leq \gamma_{sI}(G) \leq \gamma_{sR}(G)$.*

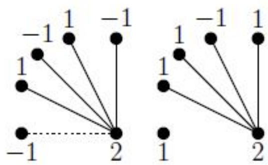

Figure 4. $\gamma_{sI}(G) = \gamma_{sI}(G \setminus uv) - 2$

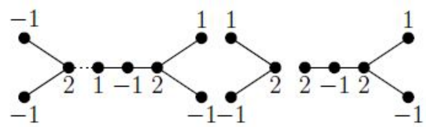

Figure 5. $\gamma_{sI}(G) = \gamma_{sI}(G \setminus uv) - 3$

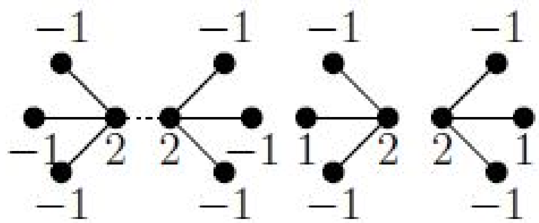

Figure 6. $\gamma_{sI}(G) = \gamma_{sI}(G \setminus uv) - 4$

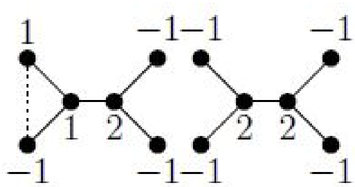

Figure 7. $\gamma_{sI}(G) = \gamma_{sI}(G \setminus uv) + 1$

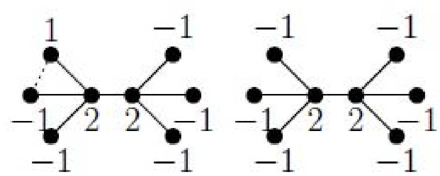

Figure 8. $\gamma_{sI}(G) = \gamma_{sI}(G \setminus uv) + 2$

Proof. Every signed Roman dominating function is a signed Italian dominating function, so the upper bound holds. For the lower bound, assume that f is a γ_{sI}-function of G. Since $V_1 \cup V_2$ is a dominating set for G, then

$$\begin{aligned}\gamma_{sI}(G) = w(f) = |V_1| + 2|V_2| - |V_{-1}| &= 2|V_1| + 3|V_2| - n \\ &\geq 2|V_1 \cup V_2| - n \geq 2\gamma(G) - n.\end{aligned}$$

□

Proposition 9. *For each graph G of order n, $\gamma_I(G) - \gamma_{sI}(G) + \gamma(G) \leq$*

n.

Proof. Let $f = (V_{-1}, V_1, V_2)$ be a γ_{sI}-function on G. We have $\gamma_{sI}(G) = w(f) = |V_1| + 2|V_2| - |V_{-1}|$ and $\gamma(G) \leq |V_{12}|$ since V_{12} dominates G. Define the function $g : V(G) \to \{0, 1, 2\}$ as $g(v) = 0$ for any $v \in V_{-1}$ and $g(v) = f(v)$ for any $v \in V \setminus V_{-1}$. It is straightforward to check that g is a IDF on G and hence

$$\gamma_I(G) \leq |V_1| + 2|V_2| = \gamma_{sI}(G) + |V_{-1}|.$$

This implies that

$$\gamma_I(G) \leq \gamma_{sI}(G) + (n - |V_{12}|) \leq \gamma_{sI}(G) + n - \gamma(G).$$

□

Proposition 10. *For any graph G, $\gamma_{sdR}(G) \leq 2\gamma_{sI}(G) + n - \gamma(G)$.*

Proof. Let $f = (V_{-1}^f, V_1^f, V_2^f)$ be an arbitrary γ_{sI}-function on G. Then the function $g = (V_{-1}^f, \emptyset, V_1^f, V_2^f)$ is a SDRDF for G. Hence

$$\begin{aligned}\gamma_{sdR}(G) &\leq 3|V_2^f| + 2|V_1^f| - |V_{-1}^f| \leq 4|V_2^f| + 2|V_1^f| - 2|V_{-1}^f| + V_{-1}^f \\ &= 2\gamma_{sI}(G) + |V_{-1}^f| \leq 2\gamma_{sI}(G) + n - \gamma(G).\end{aligned}$$

□

Now we present two sharp bounds on the signed Italian domination number graphs. We introduce some notation for convenience. Let $V_{-1}^{'} = \{v \in V_{-1} \mid N(v) \cap V_2 \neq \emptyset\}$ and $V_{-1}^{''} = V_{-1} - V_{-1}^{'}$. For disjoint subsets U and W of vertices, let $[U, W]$ denote the set of edges between U and W. Also let $V_{12} = V_1 \cup V_2$, $|V_{12}| = n_{12}$, $|V_1| = n_1$ and $|V_2| = n_2$. Then $n_{12} = n_1 + n_2$. In addition set $n_{-1} = |V_{-1}|$, and so $n_{-1} = n - n_{12}$. Let $G_{12} = G[V_{12}]$ be the subgraph induced by the set V_{12}, and G_{12} have size m_{12}. For $i = 1, 2$, if $V_i \neq \emptyset$, then $G_i = G[V_i]$ be the subgraph induced by the set V_i, and G_i have size m_i. Hence $m_{12} = m_1 + m_2 + |[V_1, V_2]|$.

For $k \geq 1$, let L_k be a graph obtained from a graph H of order k by adding $2d_H(v) + 1$ pendant edges to each vertex v of H. Note that $L_1 = K_2$. Let $\mathcal{H} = \{L_k \mid k \geq 1\}$.

Theorem 4. *Let G be a graph of order n and size m without isolated vertex. Then $\gamma_{sI}(G) \geq \frac{3n-4m}{2}$, with equality holds if and only if $G \in \mathcal{H}$.*

Proof. The proof is by induction on n. The result is obvious for $n = 2, 3$. Suppose that $n \geq 4$ and assume that the statement is true for all graphs of order less than n having no isolated vertices. Let G be a graph of order n with no isolated vertex and let $f = (V_{-1}, V_1, V_2)$ be a γ_{sI}-function. If $V_{-1} = \emptyset$, then $\gamma_{sI}(G) \geq n > \frac{3n-4m}{2}$, since G has no isolated vertex. Suppose that $V_{-1} \neq \emptyset$. We consider the following cases:

Case 1. $V_2 \neq \emptyset$.
Now, we consider the following subcases:

Subcase 1. $V_1 \neq \emptyset$.
By the definition of a SIDF, each vertex in V_{-1} is adjacent to at least one vertex in V_2 or at least two vertices in V_1, and so

$$|[V_{-1}, V_{12}]| = |[V_{-1}, V_2]| + |[V_{-1}, V_1]| \geq |V_{-1}^{'}| + 2|V_{-1}^{''}| \geq |V_{-1}| = n_{-1}.$$

Furthermore we have

$$2n_{-1} \leq 2|[V_{-1}, V_2]| + |[V_{-1}, V_1]| = 2\sum_{v \in V_2} d_{V_{-1}}(v) + \sum_{v \in V_1} d_{V_{-1}}(v).$$

For each vertex $v \in V_2$, we have

$$1 \leq f[v] = f(v) + 2d_{V_2}(v) + d_{V_1}(v) - d_{V_{-1}}(v), \tag{*}$$

and so $d_{V_{-1}}(v) \leq 2d_{V_2}(v) + d_{V_1}(v) + f(v) - 1 = 2d_{V_2}(v) + d_{V_1}(v) + 1$. Similarly for each vertex $v \in V_1$, we have that $d_{V_{-1}}(v) \leq 2d_{V_2}(v) + d_{V_1}(v)$. Hence

$$\begin{aligned} 2n_{-1} &\leq 2\sum_{v \in V_2}(2d_{V_2}(v) + d_{V_1}(v) + 1) + \sum_{v \in V_1}(2d_{V_2}(v) + d_{V_1}(v)) \\ &= (8m_2 + 2|[V_1, V_2]| + 2n_2) + (2|[V_1, V_2]| + 2m_1) \\ &= 8m_2 + 2m_1 + 4|[V_1, V_2]| + 2n_2. \end{aligned}$$

Since $m_{12} = m_1 + m_2 + |[V_1, V_2]|$, we have

$$\begin{aligned}2n_{-1} &\leq 8m_{12} - 8m_1 - 8|[V_1, V_2]| + 2m_1 + 4|[V_1, V_2]| + 2n_2 \\ &= 8m_{12} - 6m_1 - 4|[V_1, V_2]| + 2n_2.\end{aligned}$$

Therefore

$$m_{12} \geq \frac{1}{8}(2n_{-1} + 6m_1 + 4|[V_1, V_2]| - 2n_2).$$

We have $m \geq m_{12} + |[V_{-1}, V_{12}]|$. Then

$$\begin{aligned}m &\geq \frac{1}{8}(2n_{-1} + 6m_1 + 4|[V_1, V_2]| - 2n_2) + n_{-1} \\ &= \frac{1}{8}(10n_{-1} + 6m_1 + 4|[V_1, V_2]| + 2(n_1 - n_{12})) \\ &= \frac{1}{8}(10n - 10n_{12} - 2n_{12} + 2n_1 + 6m_1 + 4|[V_1, V_2]|) \\ &= \frac{1}{8}(10n - 12n_{12} + 2n_1 + 6m_1 + 4|[V_1, V_2]|)\end{aligned}$$

or equivalently,

$$n_{12} \geq \frac{1}{12}(10n - 8m + 2n_1 + 6m_1 + 4|[V_1, V_2]|).$$

In addition

$$\begin{aligned}\gamma_{sI}(G) &= 2n_2 + n_1 - n_{-1} = 3n_2 + 2n_1 - n = 3n_{12} - n - n_1 \\ &\geq \frac{1}{4}(10n - 8m + 2n_1 + 6m_1 + 4|[V_1, V_2]|) - n - n_1 \qquad (1) \\ &= \frac{1}{4}(6n - 8m) + \frac{1}{4}(6m_1 + 4|[V_1, V_2]| - 2n_1).\end{aligned}$$

If $(3m_1 + 2|[V_1, V_2]| - n_1) \geq 0$, then the result is obtained. Suppose that $\phi(n_1) = \frac{1}{2}(3m_1 + 2|[V_1, V_2]| - n_1)$. If $n_1 = 0$, then $\phi(n_1) = 0$, and we are done. Hence we may suppose that $n_1 \geq 1$. Let $v \in V_1$. If $d_{V_{12}}(v) = 0$, then $f[v] \leq 0$, since by assumption the graph G has not isolated vertex. But this is a contradiction, and we conclude that

$d_{V_{12}}(v) > 0$. Therefore

$$\begin{aligned}\phi(n_1) &= \frac{1}{2}(3m_1 + 2|[V_1, V_2]| - n_1)\\ &= \frac{3}{4}\sum_{v\in V_1} d_{V_1}(v) + \sum_{v\in V_1} d_{V_2}(v) - \frac{n_1}{2}\\ &\geq \frac{3}{4}\left(\sum_{v\in V_1} d_{V_{12}}(v)\right) - \frac{n_1}{2} \geq \frac{3n_1}{4} - \frac{n_1}{2} = \frac{n_1}{4} > 0. \qquad (2)\end{aligned}$$

And so $\gamma_{sI}(G) > \frac{3n-4m}{2}$.

Subcase 2. $V_1 = \emptyset$.
Since $V_{-1} \neq \emptyset$, we conclude that $V_2 \neq \emptyset$. By definition of a SIDF, each vertex in V_{-1} is adjacent to at least one vertex in V_2, and so

$$|[V_{-1}, V_2]| \geq |V_{-1}| = n_{-1}.$$

Therefore we have

$$n_{-1} \leq |[V_{-1}, V_2]| = \sum_{v\in V_2} d_{V_{-1}}(v).$$

For each vertex $v \in V_2$, we have $f(v) + 2d_{V_2}(v) - d_{V_{-1}}(v) = f[v] \geq 1$, and so $d_{V_{-1}}(v) \leq 2d_{V_2}(v) + 1$. It follows that

$$n_{-1} \leq \sum_{v\in V_2} d_{V_{-1}}(v) \leq \sum_{v\in V_2}(2d_{V_2}(v) + 1) = 4m_2 + n_2$$

which implies that

$$m_2 \geq \frac{1}{4}(n_{-1} - n_2).$$

Hence

$$\begin{aligned}m &\geq m_2 + |[V_{-1}, V_2]| \geq \frac{1}{4}(n_{-1} - n_2) + n_{-1}\\ &= \frac{1}{4}(5n_{-1} - n_2) = \frac{1}{4}(5n - 5n_2 - n_2) = \frac{1}{4}(5n - 6n_2),\end{aligned}$$

and so
$$n_2 \geq \frac{1}{6}(5n - 4m).$$
Now we have
$$\begin{aligned}\gamma_{sI}(G) &= 2n_2 - n_{-1} = 3n_2 - n \\ &\geq \frac{1}{2}(5n - 4m) - n = \frac{1}{2}(3n - 4m). \end{aligned} \tag{3}$$
Therefore $\gamma_{sI}(G) \geq \frac{3n-4m}{2}$.

Case 2. $V_2 = \emptyset$.
Since $V_{-1} \neq \emptyset$, we conclude that $V_1 \neq \emptyset$. By definition of a SIDF, each vertex in V_{-1} is adjacent to at least two vertices in V_1, and so
$$|[V_{-1}, V_1]| \geq 2|V_{-1}| = 2n_{-1}.$$
Therefore we have
$$2n_{-1} \leq |[V_{-1}, V_1]| = \sum_{v \in V_1} d_{V_{-1}}(v).$$
For each vertex $v \in V_1$, we have $f(v) + d_{V_1}(v) - d_{V_{-1}}(v) = f[v] \geq 1$, and so $d_{V_{-1}}(v) \leq d_{V_1}(v)$. It follows that
$$2n_{-1} \leq \sum_{v \in V_1} d_{V_{-1}}(v) \leq \sum_{v \in V_1} d_{V_1}(v) = 2m_1.$$
We have $m \geq m_1 + |[V_{-1}, V_1]| + m_{-1}$, then
$$\begin{aligned} m &\geq m_1 + |[V_{-1}, V_1]| \geq n_{-1} + 2n_{-1} \\ &= 3n_{-1} = 3n - 3n_1, \end{aligned}$$
and so
$$n_1 \geq \frac{1}{3}(3n - m).$$
Now we have
$$\begin{aligned}\gamma_{sI}(G) &= n_1 - n_{-1} = 2n_1 - n \\ &\geq \frac{1}{3}(6n - 2m) - n = \frac{1}{3}(3n - 2m). \end{aligned} \tag{4}$$

Therefore $\gamma_{sI}(G) \geq \frac{1}{3}(3n-2m)$ implies that $\gamma_{sI}(G) \geq \frac{1}{3}(3n-2m) > \frac{1}{2}(3n-4m)$, which completes the proof of the lower bound.

Now if $\gamma_{sI}(G) = \frac{3n-4m}{2}$, then all the inequalities (1), (2), (3) and (4) must be equalities. Hence $n_1 = 0$ and $n_2 = n_{12}$, and so $V_{12} = V_2$ and $V = V_{-1} \cup V_2$. Furthermore $m = m_2 + |[V_{-1}, V_2]| + m_{-1}$, $m_2 = \frac{(n_{-1}-n_2)}{4}$ and $n_{-1} = |[V_{-1}, V_2]|$. This implies that for each vertex $v \in V_{-1}$ we have $d_{V_{-1}}(v) = 0$ and $d_{V_2}(v) = 1$, and hence every vertex of V_{-1} is a leaf in G. Also for every vertex $v \in V_2$ we have $d_{V_{-1}}(v) = 2d_{V_2}(v) + 1$. Therefore $G \in \mathcal{H}$.

On the other hand, suppose that $G \in \mathcal{H}$. Then $G = L_k$ for some $k \geq 1$. Thus G is obtained from a graph H of order k by adding $2d_H(v) + 1$ pendant edges to each vertex v of H. Let G have order n and size m. Then

$$n = \sum_{v \in V(H)} (2d_H(v) + 2) = 4m(H) + 2n(H)$$

and

$$m = m(H) + \sum_{v \in V(H)} (2d_H(v) + 1) = 5m(H) + n(H).$$

Assigning to every vertex of H the weight 2 and to each vertex in $V(G) \setminus V(H)$ the weight -1 produces a SIDF f of weight $w(f) = 2n(H) - (4m(H) + n(H)) = n(H) - 4m(H) = \frac{3n-4m}{2}$. Hence $\gamma_{sI}(G) \leq w(f) = \frac{3n-4m}{2}$. Consequently, $\gamma_{sI}(G) = \frac{3n-4m}{2}$. $\square$

Theorem 5. *Let G be a graph of order $n \geq 2$. Then*

$$\gamma_{sI}(G) \geq 3\sqrt{\frac{n}{2}} - n.$$

Proof. Let $f = (V_{-1}, V_1, V_2)$ be a γ_{sI}-function on G. If $V_{-1} = \emptyset$, then $\gamma_{sI}(G) \geq n \geq 3\sqrt{\frac{n}{2}} - n$ for $n \geq 2$. Hence suppose that $|V_{-1}| \geq 1$. We consider the following cases:

Case 1. $V_2 \neq \emptyset$.
Since each vertex of $V_{-1}^{'}$ is adjacent to at least one vertex in V_2. Hence by the Pigeonhole Principle, we conclude that at least one vertex v of

V_2 is adjacent to at least $\frac{n'_{-1}}{n_2}$ vertices of V'_{-1}. It follows that $1 \leq f[v] \leq 2n_2 + n_1 - \frac{n'_{-1}}{n_2}$ and thus

$$0 \leq 2n_2^2 + n_1 n_2 - n'_{-1} - n_2. \tag{1}$$

Likewise, since each vertex in V''_{-1} is adjacent to at least two vertices in V_1, we deduce that at least one vertex u of V_1 is adjacent to at least $\frac{2n''_{-1}}{n_1}$ vertices of V''_{-1}. As above we have $1 \leq f[u] \leq 2n_2 + n_1 - \frac{2n''_{-1}}{n_1}$, and thus

$$0 \leq 2n_2 n_1 + n_1^2 - 2n''_{-1} - n_1. \tag{2}$$

Now by multiplying the inequality (1) by 2 and summing it with the inequality (2) we obtain

$$0 \leq 4n_2^2 + 2n_1 n_2 - 2n'_{-1} - 2n_2 + 2n_1 n_2 + n_1^2 - 2n''_{-1} - n_1.$$

Since $n = n_2 + n_1 + n_{-1}$, we have

$$0 \leq 4n_2^2 + 4n_1 n_2 - 2n + n_1^2 + n_1.$$

Equivalently

$$\begin{aligned} 0 &\leq 9n_2^2 + 9n_1 n_2 - \frac{9}{2}n + \frac{9}{4}n_1^2 + \frac{9}{4}n_1 \\ &\leq 9n_2^2 + 12n_1 n_2 + 4n_1^2 - \frac{9}{2}n \\ &= (3n_2 + 2n_1)^2 - \frac{9}{2}n \end{aligned}$$

which implies that $3\sqrt{\frac{n}{2}} \leq (3n_2 + 2n_1)$. Therefore

$$\begin{aligned} \gamma_{sI}(G) &= 2n_2 + n_1 - n_{-1} \\ &= 3n_2 + 2n_1 - n \\ &\geq 3\sqrt{\frac{n}{2}} - n. \end{aligned}$$

Case 2. $V_2 = \emptyset$.
Since $V_{-1} \neq \emptyset$, we conclude that $V_1 \neq \emptyset$. As in Case 1, at least one vertex u of V_1 is adjacent to at least $\frac{2n_{-1}}{n_1}$ vertices of V_{-1}. Then $1 \leq f[u] \leq n_1 - \frac{2n_{-1}}{n_1}$ which implies that

$$0 \leq n_1^2 - n_1 - 2n_{-1}.$$

Since $n = n_1 + n_{-1}$, we have

$$0 \leq n_1^2 + n_1 - 2n.$$

We have $\frac{7}{9}n_1^2 - n_1 \geq 0$, since $n_1 \geq 2$. Therefore

$$\frac{16}{9}n_1^2 - 2n = (n_1^2 + n_1 - 2n) + (\frac{7}{9}n_1^2 - n_1) \geq 0,$$

Equivalently

$$0 \leq \frac{16}{9}n_1^2 - 2n,$$

which implies that $3\sqrt{\frac{n}{8}} \leq n_1$. Therefore

$$\begin{aligned}\gamma_{sI}(G) &= n_1 - n_{-1} \\ &= 2n_1 - n \\ &\geq 3\sqrt{\frac{n}{2}} - n.\end{aligned}$$

□

The following example demonstrates that the lower bound in Theorem 5, is sharp.

Example 3. *Let $k \geq 1$ be an integer and F_k be the graph obtained from the corona product of two graphs K_{k+1} and $\overline{K_{2k+1}}$. Assigning to the all vertices of K_{k+1} the weight 2 and to the remaining vertices the weight -1, produces a SIDF of weight* $2(k+1)-(k+1)(2k+1) = (k+1)(1-2k)$

on F_k. Since $n(F_k) = (k+1)(2k+2)$, Theorem 5 implies that

$$\begin{aligned}\gamma_{sI}(F_k) &\geq 3\sqrt{\frac{n(F_k)}{2}} - n(F_k)\\ &= 3\sqrt{\frac{(k+1)(2k+2)}{2}} - (k+1)(2k+2)\\ &= 3(k+1) - (k+1)(2k+2) = (k+1)(1-2k).\end{aligned}$$

Therefore $\gamma_{sI}(F_k) = 3\sqrt{\frac{n(F_k)}{2}} - n(F_k) = (k+1)(1-2k)$.

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