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New Bounds For Degree Sequence Of Graphs

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Abstract

Let $G = (V, E)$ be a simple graph with n vertices, m edges, and vertex degrees $d_1, d_2, ..., d_n$. Let d_1, d_n be the maximum and and minimum degree of vertices. In this paper, we present lower and upper bounds for $\sum_{i=1}^{n} d_i^2$ and $\sum_{i=1}^{n} d_i^3$ and relations between them. Also, we improve the bounds given in (2) and (3).

Keywords: Degree sequence of graph, Maximum degree, Minimum degree.

MSC 2010: 68R10, 68Q25, 05C35, 05C05.

1 Introduction

Let $G = (V, E)$ be a simple undirected graph with vertex set $V = V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$, $\mid E(G) \mid= m$. The *order* and *size* of G are $n = |V|$ and $m = |E|$, respectively. For a vertex $v_i \in V$, the *degree* of v_i, denoted by $\deg(v_i)$ (or just d_i), is the number of edges incident to v. We denote by $\Delta(G) = d_1$, the *maximum degree* among the vertices of G, and by $\delta(G) = d_n$, the *minimum degree* among the vertices of G. Let m_i be the average degree of the vertices adjacent to vertex v_i in G. A graph G is *regular* of degree r if all the vertices of G have the same degree r. A *complete* graph is a graph in which every two distinct vertices are joined by exactly one edge. A *walk* from a vertex u to a vertex v is a finite alternating sequence $v_0(= u)e_1v_1e_2 \ldots v_{k-1}e_kv_k(= v)$ of vertices and edges such that $e_i = v_{i-1}v_i$ for $i = 1, 2, \ldots, k$. The number k is the *length* of the walk. In particular, if the vertexes $v_i, i = 0, 1, ..., k$ in the walk are all distinct, then the walk is called a *path*. A path of order n is denoted by P_n. A *closed path* or *cycle*, is obtained from a path $v_1, \ldots, v_k$ (where $k \geqslant 3$) by adding the edge v_1v_k. A cycle of order n is denoted by C_n. A graph is *unicyclic* if it contains precisely one cycle. A graph is *connected* if each pair of vertices in a graph is joined by a walk. For other graph theory notation and terminology we refer to [21].

The *adjacency matrix* $A(G)$ of a graph G is defined by its entries as $a_{ij} = 1$ if $v_iv_j \in E(G)$ and 0 otherwise. Let $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_{n-1} \geqslant \lambda_n$ denote the *eigenvalues* of $A(G)$. Then λ_1 is called the *spectral radius* of G.

The degree sequence of a graph G is denoted by $d_1, d_2, ..., d_n$ and assumed to be labelled in a non-increasing manner:

$$d_1 \geqslant d_2 \geqslant ... \geqslant d_n.$$

We denote by d_1 the *highest* and d_n the *lowest* degree of vertices of G.

The inverse degree first attracted attention through conjectures of the computer program Graffti [13]. This vertex-degree-based graph invariant is defined as follows:

$$ID = ID(G) = \sum_{v_i \in V}^{n} \frac{1}{d_i}. \tag{1}$$

The *first* Zagreb indices of a graph G are defined as $M_1(G) = \sum_{u \in V} d_u^2$. For further study on the Zagreb indices and their properties, we refer to [23]. We recall two upper bounds for $\sum_{i=1}^{n} d_i^2$:

In [8], [17], [18]: Let G be a connected graph with n vertices, m edges. Then

$$M_1(G) \leqslant \frac{2m^2}{n} + \left(\frac{d_1}{d_n} + \frac{d_n}{d_1}\right)\frac{2m^2}{n}. \tag{2}$$

In [17], [18]: Let G be a connected graph with n vertices and m edges. If $\delta = 1$, then

$$M_1(G) \leqslant \frac{nm^2}{n-1}. \tag{3}$$

Since then, numerous other bounds for degree sequence of a graph were found (see, [2], [7], [9], [10], [20]).

The organization of the paper is as follows. In Section 2, we give a list of some previously known results. In Section 3, we present our upper and lower bounds for the $\sum_{i=1}^{n} d_i^2$ of a graph G and improve the bounds given in (2) and (3). In Section 4, we present our bounds for the $\sum_{i=1}^{n} d_i^3$ of a graph G. In Section 5, we investigate relations between $\sum_{i=1}^{n} d_i^2$ and $\sum_{i=1}^{n} d_i^3$.

2 Preliminaries and known results

In this section, we list some previously known results that will be needed in the next sections. We recall some known bounds for $d_1^2 + ... + d_n^2$. The Cauchy-Schwarz inequality yields a lower bound

$$\frac{4m^2}{n} = \frac{1}{n}\left(d_1 + ... + d_n\right)^2 \leqslant d_1^2 + ... + d_n^2.$$

It is known by D. de Caen [10] that $d_1^2 + ... + d_n^2 \leqslant m\left(\frac{2m}{n-1} + n - 2\right)$.

In 2004, K. Das [7] obtained equivalent conditions to be

$$d_1^2 + ... + d_n^2 = m \max\{d_j + m_j | v_j \in V\}.$$

In 2004, K. C. Das [7] obtained an upper bound of the sum of squares of degrees of a graph, which is less then equal to Caen's upper bound as follows;

1. $d_1^2 + ... + d_n^2 \leqslant m \max\{d_j + m_j | v_j \in V\}$,
2. $m \max\{d_j + m_j | v_j \in V\} \leqslant \dfrac{2m}{n-1} + n - 2$.

We begin with the following lemma for the connected graphs.

Lemma 1. *(Collatz and Sinogowitz [5]). If G is a connected graph with n vertices, then*

$$\lambda_1(G) \leqslant \sqrt{n-1}.$$

Lemma 2. *[14] If G is a connected unicyclic graph, then*

$$\lambda_1(G) \leqslant \lambda_1(S_n^3),$$

where S_n^3 denotes the graph obtained by joining any two vertices of degree one of the star $K_{1,n-1}$ by an edge.

Lemma 3. *(Hong [15]). If G is a connected graph, then*

$$\lambda_1(G) \leqslant \sqrt{2m-n+1}.$$

Lemma 4. *(Berman and Zhang [3]). If G is a connected graph, then*

$$\lambda_1(G) \leqslant \max\{\sqrt{d_i d_j} : 1 \leqslant i, j \leqslant n, v_i v_j \in E\}.$$

Lemma 5. *(Favaron et al. [12]). For any graph without isolated vertices,*

$$\lambda_1(G) \leqslant \max\{m_i : v_i \in V\}.$$

Lemma 6. *(Favaron et al. [12]) For any simple graph*

$$\lambda_1(G) \geqslant \sqrt{d_1}.$$

Lemma 7. *[6] For a connected non-regular graph G with diameter D*

$$\lambda_1(G) < d_1 - \frac{1}{nD}.$$

Lemma 8. *[22] For a connected non-regular graph G with diameter D*

$$\lambda_1(G) < d_1 - \frac{\left(\sqrt{d_1}-\sqrt{d_n}\right)^2}{nDd_1}.$$

3 Lower and upper bounds for $\sum_{i=1}^n d_i^2$

In this section, we obtain some new upper and lower bounds for $\sum_{i=1}^n d_i^2$ in terms of graph invariants such as the number of vertices, the number of edges, the highest and the lowest degree of vertices.

The following lemma is a well-known result called the handshaking lemma.

Lemma 9. *(The Handshaking Lemma) Let G be a graph with n vertices, m edges and the degree sequence of $d_1, d_2, ..., d_n$. Then*

$$d_1 + ... + d_n = 2m.$$

We begin with the following upper bound in terms of vertices, edges, the highest and the lowest degree of vertices.

Theorem 1. *Let G be a connected graph with n vertices and m edges, then*

$$\sum_{i=1}^{n} d_i^2 \leqslant \frac{m^2}{n} \left(\sqrt{\frac{d_1}{d_n}} + \sqrt{\frac{d_n}{d_1}} \right)^2, \tag{4}$$

equality holds if and only if G is a regular graph.

Proof. Let a_i be positive numbers for $i = 1, 2, ..., n$, such that there exist positive numbers A, a satisfying:

$$0 < a \leq a_i \leq A. \tag{5}$$

Then the following inequality is valid (see [19] p. 71–72):

$$\frac{n \sum_{i=1}^{n} a_i^2}{\left(\sum_{i=1}^{n} a_i\right)^2} \leq \frac{1}{4} \left(\sqrt{\frac{A}{a}} + \sqrt{\frac{a}{A}} \right)^2. \tag{6}$$

The inequality becomes an equality if and only if $a = A$.

For $a_i := d_i$, $i = 1, 2, ..., n$, inequality (6) becomes

$$\frac{n \sum_{i=1}^{n} d_i^2}{\left(\sum_{i=1}^{n} d_i\right)^2} \leq \frac{1}{4} \left(\sqrt{\frac{d_1}{d_n}} + \sqrt{\frac{d_n}{d_1}} \right)^2.$$

By Lemma 9, we have, $\sum_{i=1}^{n} d_i = 2m$, from the above inequality it follows directly the assertion of Theorem 1, i.e. inequality (4).

Let us consider the case when G is a regular graph of degree r. Since $m = \frac{1}{2}nr$ and $d_i = r$, we know that $d_1^2 + ... + d_n^2 = nr^2$. Therefore, for a regular graph of degree r we have,

$$\frac{m^2}{n} \left(\sqrt{\frac{d_1}{d_n}} + \sqrt{\frac{d_n}{d_1}} \right)^2 = \frac{4m^2}{n}.$$

Thus we have $d_1^2 + ... + d_n^2 = \frac{4m^2}{n}$.

Now, suppose that the equality holds in (4). Then the equality holds in (6). From the equality in (6), we get $d_1 = d_n$ and by (5), we have $d_1 = d_2 = ... = d_n$, then G is a regular graph. □

Note that since

$$\frac{m^2}{n}\left(\sqrt{\frac{d_1}{d_n}} + \sqrt{\frac{d_n}{d_1}}\right)^2 \leqslant \frac{2m^2}{n} + \left(\frac{d_1}{d_n} + \frac{d_n}{d_1}\right)\frac{2m^2}{n}.$$

The bound of Theorem 1 is another improvement of the bound given in (2) for connected graphs.

Theorem 2. *Let G be a connected graph with $n > 4$ vertices, then*

$$\sum_{i=1}^{n} d_i^2 \leqslant \frac{4m^2}{n-1} + d_n^2. \tag{7}$$

Proof. Let a_i be real numbers, for $i = 1, 2, ..., n$. Then the following inequality is valid (see [4]):

$$\sqrt{(n-1)(a_1^2 + a_2^2 + ... + a_{n-1}^2)} \leqslant a_1 + a_2 + ... + a_n. \tag{8}$$

For $a_i := d_i$, inequality (8) becomes

$$\sqrt{(n-1)(d_1^2 + d_2^2 + ... + d_{n-1}^2)} \leqslant d_1 + d_2 + ... + d_n$$
$$\sqrt{(n-1)(\sum_{i=1}^{n-1} d_i^2 - d_n)} \leqslant \sum_{i=1}^{n} d_i$$
$$\sum_{i=1}^{n} d_i^2 \leqslant \frac{4m^2}{n-1} + d_n^2.$$

□

Note that for $n > 4$ we have:

$$\frac{4m^2}{n-1} + d_n^2 \leqslant \frac{nm^2}{n-1}.$$

The bound of Theorem 2 is another improvement of the bound given in (3) for connected graphs.

Remark 1. *Let G be a regular graph with n vertices and m edges, then*

$$\frac{m^2}{n}\left(\sqrt{\frac{d_1}{d_n}} + \sqrt{\frac{d_n}{d_1}}\right)^2 = \frac{4m^2}{n} \leqslant \frac{4m^2}{n-1} + d_n^2.$$

Therefore, the bound of Theorem 1 is another improvement of the bound in Theorem 2 for regular graphs.

Theorem 3. *Let G be a connected graph with n vertices and m edges, then*

$$\sum_{i=1}^{n} d_i^2 \leqslant 2md_1, \tag{9}$$

equality holds if and only if G is a regular graph.

Proof. Let a_i, b_i are decreasing non-negative sequences with $a_1, b_1 \neq 0$ for $i = 1, 2, ..., n$. Then the following inequality is valid (see [1]):

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \leqslant max\left\{b_1 \sum_{i=1}^{n} a_i, a_1 \sum_{i=1}^{n} b_i\right\} \sum_{i=1}^{n} a_i b_i. \tag{10}$$

Equality holds in (10) if and only if $a_1 = a_2 = ... = a_n$, and $b_1 = b_2 = ... = b_n$.

For $a_i, b_i := d_i$ and $w_i := 1$, $i = 1, 2, ..., n$, inequality (10) becomes

$$\sum_{i=1}^{n} d_i^2 \sum_{i=1}^{n} d_i^2 \leqslant max\left\{d_1 \sum_{i=1}^{n} d_i,\ d_1 \sum_{i=1}^{n} d_i\right\} \sum_{i=1}^{n} d_i^2. \tag{11}$$

By Lemma 9 and the equality $\sum_{i=1}^{n} 1 = n$, it follows directly the assertion of Theorem 3, i.e. inequality (9).

If G is a regular graph of degree r, then we know that $d_1^2+...+d_n^2 = nr^2$ and $d_1 = d_2 = ... = d_n = r$. Hence

$$2md_1 = 2(\frac{1}{2}nr)d_1 = nr^2.$$

Thus we have $d_1^2 + ... + d_n^2 = 2md_1$.

Now, suppose that the equality holds in (9). Then the equality holds in (10). From the equality in (10), we get $d_1 = d_2 = ... = d_n = d$, then G is a regular graph. □

The Theorem 4 is a Consequence of Theorem 3, by Lemmas 1 and 6.

Theorem 4. *Let G be a connected graph with n vertices and m edges, then*

$$\sum_{i=1}^{n} d_i^2 \leqslant 2m(n-1).$$

Similarly to the theorem above and by lemma 2, we can obtain upper bound for the unicyclic graphs as follows.

Lemma 10. *Let G be a connected unicyclic graph with n vertices, then*

$$\sum_{i=1}^{n} d_i^2 \leqslant 2m\left(\lambda_1(S_n^3)\right)^2.$$

Also, by lemmas 7 and 8, we can obtain two upper bounds for the non-regular graphs.

Lemma 11. *Let G be a connected non-regular graph with diameter D, then*

$$1) \sum_{i=1}^{n} d_i^2 < 2m\left(d_1 - \frac{1}{nD}\right)^2,$$

$$2) \sum_{i=1}^{n} d_i^2 < 2m\left(d_1 - \frac{\left(\sqrt{d_1}-\sqrt{d_n}\right)^2}{nDd_1}\right)^2.$$

Now, by lemma 5, we obtain upper bound for the graphs without isolated vertices.

Lemma 12. *Let G be a graph without isolated vertices, then*

$$\sum_{i=1}^{n} d_i^2 \leqslant 2m \left(\max\{m_i : v_i \in V\}\right)^2.$$

Again, by lemmas 3 and 4, we obtain two upper bounds for the connected graphs.

Lemma 13. *Let G be a connected graph with n vertices, then*

$$1) \sum_{i=1}^{n} d_i^2 \leqslant 2m(2m-n+1),$$

$$2) \sum_{i=1}^{n} d_i^2 \leqslant 2m \left(\max\{\sqrt{d_i d_j} : 1 \leqslant i,j \leqslant n, v_i v_j \in E\}\right)^2.$$

Theorem 5. *Let G be a connected graph with n vertices and m edges, then*

$$\sum_{i=1}^{n} d_i^2 \geqslant 4m - n. \tag{12}$$

Proof. Let a_i, b_i, c_i and e_i be real numbers and p_i, q_i be nonnegative numbers for $i = 1, 2, ..., n$. Then the following inequality is valid (see [11] p. 7)

$$\sum_{i=1}^{n} p_i a_i^2 \sum_{i=1}^{n} q_i b_i^2 + \sum_{i=1}^{n} p_i c_i^2 \sum_{i=1}^{n} q_i e_i^2 \geqslant 2 \sum_{i=1}^{n} p_i a_i c_i \sum_{i=1}^{n} q_i b_i e_i. \tag{13}$$

For $a_i := d_i$ and $b_i = c_i = e_i := 1$, $i = 1, 2, ..., n$, inequality (13) becomes

$$\sum_{i=1}^{n} d_i^2 \sum_{i=1}^{n} 1 + \sum_{i=1}^{n} 1 \sum_{i=1}^{n} 1 \geqslant 2 \sum_{i=1}^{n} d_i \sum_{i=1}^{n} 1.$$

By Lemma 9 and the equality $\sum_{i=1}^{n} 1 = n$, it follows directly the assertion of Theorem 5, i.e. inequality (12). ☐

4 Lower bounds for $\sum_{i=1}^n d_i^3$

In this section, we obtain some new lower bounds for $\sum_{i=1}^n d_i^3$ in terms of graph invariants such as the number of vertices, the number of edges.

Theorem 6. *Let G be a connected graph with n vertices and m edges, then*

$$\sum_{i=1}^{n} d_i^3 \geqslant 6m - 2n. \tag{14}$$

Proof. Let a_i, b_i, c_i and e_i be nonnegative numbers for $i = 1, 2, ..., n$. Then the following inequality is valid (see [11] p. 7):

$$\frac{1}{2}\left[\sum_{i=1}^{n} a_i^3 c_i \sum_{i=1}^{n} b_i^3 e_i + \sum_{i=1}^{n} a_i c_i^3 \sum_{i=1}^{n} b_i e_i^3\right] \geqslant \sum_{i=1}^{n} a_i^2 c_i^2 \sum_{i=1}^{n} b_i^2 e_i^2. \tag{15}$$

For $a_i := d_i$ and $b_i = c_i = e_i := 1$, $i = 1, 2, ..., n$, inequality (15) becomes

$$\frac{1}{2}\left[\sum_{i=1}^{n} d_i^3 \sum_{i=1}^{n} 1 + \sum_{i=1}^{n} d_i \sum_{i=1}^{n} 1\right] \geqslant \sum_{i=1}^{n} d_i^2 \sum_{i=1}^{n} 1. \tag{16}$$

By Lemma 9 and the equality $\sum_{i=1}^n 1 = n$, also from the Inequality (16), we have

$$n \sum_{i=1}^{n} d_i^3 + 2mn \geqslant \sum_{i=1}^{n} d_i^2 n.$$

Now by Theorem 5, the proof is completed. □

Theorem 7. *Let G be a connected graph with n vertices and m edges, then*

$$\sum_{i=1}^{n} d_i^3 \geqslant \frac{8m^2 - 2mn}{n}. \tag{17}$$

Proof. Let a_i and b_i be sequences of nonnegative real numbers and $\alpha, \beta > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. Then the following inequality is valid (see [11] p. 11):

$$\frac{1}{\alpha}\sum_{i=1}^{n} b_i \sum_{i=1}^{n} a_i^{\alpha+1} + \frac{1}{\beta}\sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i^{\beta+1} \geqslant \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2. \tag{18}$$

For $a_i := d_i$, $b_i := 1$ and $\alpha, \beta := 2$ for $i = 1, 2, ..., n$, inequality (18) becomes

$$\frac{1}{2}\sum_{i=1}^{n} 1 \sum_{i=1}^{n} d_i^3 + \frac{1}{2}\sum_{i=1}^{n} d_i \sum_{i=1}^{n} 1 \geqslant \sum_{i=1}^{n} d_i^2 \sum_{i=1}^{n} 1. \tag{19}$$

By Lemma 9 and the equality $\sum_{i=1}^{n} 1 = n$, it follows directly the assertion of Theorem 7, i.e. inequality (17). □

Theorem 8. *Let G be a connected graph with n vertices and m edges, then*

$$\sum_{i=1}^{n} d_i^3 \geqslant \frac{8m^3}{n^2}, \tag{20}$$

equality holds if and only if G is a regular graph.

Proof. Let a_i, b_i and c_i be positive real numbers, $i = 1, 2, ..., n$. Then the following inequality is valid (see [16] p.137):

$$\left(\sum_{i=1}^{n} a_i b_i c_i\right)^3 \leqslant \left[\sum_{i=1}^{n} a_i^3\right]\left[\sum_{i=1}^{n} b_i^3\right]\left[\sum_{i=1}^{n} c_i^3\right], \tag{21}$$

where equality holds if and only if $a_i = b_i = c_i, i = 1, 2, ..., n$.

For $a_i := d_i$ and $b_i = c_i := 1$, $i = 1, 2, ..., n$, inequality (20) becomes

$$\left(\sum_{i=1}^{n} d_i\right)^3 \leqslant \left[\sum_{i=1}^{n} d_i^3\right]\left[\sum_{i=1}^{n} 1\right]\left[\sum_{i=1}^{n} 1\right].$$

By Lemma 9 and the equality $\sum_{i=1}^{n} 1 = n$, it follows directly the assertion of Theorem 8, i.e. inequality (20).

If G is a regular graph of degree r, then we know that $d_1^3+...+d_n^3 = nr^3$. Moreover, for a regular graph of degree r we have , $d_1 = d_2 = ... = d_n = r$. Hence

$$\frac{8m^3}{n^2} = \frac{8(\frac{1}{2}nr)^3}{n^2} = nr^3.$$

Thus we have $d_1^3 + ... + d_n^3 = \frac{8m^3}{n^2}$.

Now, suppose that the equality holds in (20). Then the equality holds in (21). From the equality in (21), we get $d_1 = d_2 = ... = d_n = d$, then G is a regular graph. □

Theorem 9. *Let G be a connected graph with n vertices and m edges, then*

$$\sum_{i=1}^{n} d_i^3 \geqslant \frac{8mn - 2n^2 - 4m^2}{ID(G)}. \tag{22}$$

Proof. Let a_i and b_i be real numbers, $i = 1, 2, ..., n$. Then the following inequality is valid (see [11] p.8):

$$\sum_{i=1}^{n} \frac{a_i^3}{b_i} \sum_{i=1}^{n} \frac{b_i^3}{a_i} - \left(\sum_{i=1}^{n} a_i b_i\right)^2 \geqslant 2\left[\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2\right]. \tag{23}$$

For $a_i := d_i$ and $b_i := 1$, $i = 1, 2, ..., n$, inequality (23) becomes

$$\sum_{i=1}^{n} d_i^3 \sum_{i=1}^{n} \frac{1}{d_i} - \left(\sum_{i=1}^{n} a_i\right)^2 \geqslant 2\left[\sum_{i=1}^{n} d_i^2 \sum_{i=1}^{n} 1 - \left(\sum_{i=1}^{n} d_i\right)^2\right]. \tag{24}$$

By Lemma 9, the equality $\sum_{i=1}^{n} 1 = n$, from Equality (1) and Theorem 5 it follows directly the assertion of Theorem 9, i.e. inequality (22). □

5 Bounds for $\sum_{i=1}^{n} d_i^3$ involving $\sum_{i=1}^{n} d_i^2$

In this section, we investigate the relations between $\sum_{i=1}^{n} d_i^3$ and $\sum_{i=1}^{n} d_i^2$.

Theorem 10. *Let G be a connected graph with m edges, then*

$$\sum_{i=1}^{n} d_i^3 \geqslant 2\sum_{i=1}^{n} d_i^2 - 2m. \tag{25}$$

Proof. Let a_i, b_i, c_i and e_i be nonnegative numbers, $i = 1, 2, ..., n$. Then the following inequality is valid (see [11] p.7):

$$\sum_{i=1}^{n} a_i^3 c_i \sum_{i=1}^{n} b_i^3 e_i + \sum_{i=1}^{n} c_i^3 a_i \sum_{i=1}^{n} e_i^3 b_i \geqslant 2\sum_{i=1}^{n} a_i^2 c_i^2 \sum_{i=1}^{n} b_i^2 e_i^2. \tag{26}$$

For $a_i := d_i$ and $b_i = c_i = e_i := 1$, $i = 1, 2, ..., n$, inequality (26) becomes

$$\sum_{i=1}^{n} d_i^3 \sum_{i=1}^{n} 1 + \sum_{i=1}^{n} d_i \sum_{i=1}^{n} 1 \geqslant \sum_{i=1}^{n} d_i^2 \sum_{i=1}^{n} 1. \tag{27}$$

By Lemma 9 and the equality $\sum_{i=1}^{n} 1 = n$, it follows directly the assertion of Theorem 10, i.e. inequality (25).

□

Theorem 11. *Let G be a connected graph with m edges, then*

$$\sum_{i=1}^{n} d_i^3 \geqslant \frac{\left(\sum_{i=1}^{n} d_i^2\right)^2}{2m} - 2m. \tag{28}$$

Proof. Let a_i, b_i, c_i and e_i be real numbers, $i = 1, 2, ..., n$. Then the following inequality is valid (see [11] p.7):

$$\sum_{i=1}^{n} a_i^2 b_i e_i \sum_{i=1}^{n} b_i^2 a_i c_i + \sum_{i=1}^{n} c_i^2 b_i e_i \sum_{i=1}^{n} e_i^2 a_i c_i \geqslant \left(\sum_{i=1}^{n} a_i b_i c_i e_i\right)^2. \tag{29}$$

For $a_i = e_i := d_i$ and $b_i = c_i := 1$, $i = 1, 2, ..., n$, inequality (29) becomes

$$\sum_{i=1}^{n} d_i^3 \sum_{i=1}^{n} d_i + \sum_{i=1}^{n} d_i \sum_{i=1}^{n} d_i \geqslant \left(\sum_{i=1}^{n} d_i^2 \right)^2 . \tag{30}$$

By Lemma 9 and the equality $\sum_{i=1}^{n} 1 = n$, it follows directly the assertion of Theorem 11, i.e. inequality (28). □

Theorem 12. *Let G be a connected graph with m edges, then*

$$\sum_{i=1}^{n} d_i^3 \geqslant \frac{2n \sum_{i=1}^{n} d_i^2}{ID(G)} - \frac{4m^2}{ID(G)}. \tag{31}$$

Proof. Let a_i, b_i, c_i and e_i be real numbers, $i = 1, 2, ..., n$. Then the following inequality is valid (see [11] p.8):

$$\sum_{i=1}^{n} \frac{a_i^3}{c_i} \sum_{i=1}^{n} \frac{b_i^3}{e_i} + \sum_{i=1}^{n} a_i c_i \sum_{i=1}^{n} b_i e_i \geqslant 2 \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} a_i^2. \tag{32}$$

For $a_i = e_i := d_i$ and $b_i = c_i := 1$, $i = 1, 2, ..., n$, inequality (32) becomes

$$\sum_{i=1}^{n} d_i^3 \sum_{i=1}^{n} \frac{1}{d_i} + \sum_{i=1}^{n} d_i \sum_{i=1}^{n} d_i \geqslant 2 \sum_{i=1}^{n} d_i^2 \sum_{i=1}^{n} 1. \tag{33}$$

By Lemma 9, the equality $\sum_{i=1}^{n} 1 = n$ and from Equality (1) it follows directly the assertion of Theorem 12, i.e. inequality (31). □

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