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Completeness of the First-Order Logic of Partial Quasiary Predicates with the Complement Composition

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Abstract

Partial quasiary predicates are used in programming for representing program semantics and in logic for formalizing predicates over partial variable assignments. Such predicates do not have fixed arity therefore they may be treated as mappings over partial data. Obtained logics are not expressive enough to construct sound axiomatic systems of Floyd–Hoare type. To increase expressibility of such logics, oriented on quasiary predicates, we extend their language with the complement operation (composition). In the paper we define one of such logics called first-order logic of partial quasiary predicates with the complement composition. For this logic a special consequence relation called irrefutability consequence relation under undefinedness conditions is introduced. We study its properties, construct a sequent calculus for it and prove soundness and completeness of this calculus.

Keywords: partial predicate, quasiary predicate, program logic, predicate logic, soundness and completeness.

1 Introduction

Extensive usage of formal methods in Computer Science, Artificial Intelligence, and Software Engineering [1] leads to new logics that allow more adequate investigation of applied domains. Logic of partial quasiary predicates is one of such logics oriented on software verification. The class of partial quasiary predicates also appears in a natural way in other domains, in particular, in logic where it can be used for formalization of predicates defined over partial variable assignments. Algebras of such predicates serve as semantic base of logics of applied domains. An important question concerns expressibility of logic languages. It often happens that a chosen language is not expressive enough for effective usage. This question also concerns logics of partial quasiary predicates.

In our previous works [2]–[4] we studied logics with traditional compositions of disjunction, negation, renomination, and existential quantification. Application of such logics to software verification, in particular, to Floyd-Hoare program logic [5], [6], demonstrated that the logics are not expressive enough to construct a sound axiomatic system. This problem appeared due to necessity of introducing partial pre- and postconditions into Floyd-Hoare logic. Initially, this logic treats pre- and postconditions as total predicates, but being extended on class of partial predicates the logic becomes unsound [7]. There are different methods to solve this problem, in particular, a sound axiomatic system can be constructed for the logic language extended with the complement composition (discussion of the topic is presented in [8]–[10]). Introduction of this composition permits to modify rules of Floyd–Hoare logic in such a way that they become sound, but a negative side of this proposal is that the logic becomes more complicated. In this case, undefinedness conditions for predicates should be taken into account.

In [11]–[13] we constructed sound and complete sequent calculi for logics of propositional and renominative (quantifier-free) levels. Here we generalize the obtained results for the first-order logic of partial quasiary predicates extended with the complement composition. We additionally study semantic properties of quantifier elimination, of variable assignment composition (predicate), of a ternary consequence relation with undefinedness conditions. We define a sequent calculus for this logic and prove its soundness and completeness.

This paper is a refined and extended version of [14]. In particular, new simpler system of sequent forms and simpler sequent closeness conditions are defined and investigated.

Obtained results can be applied for software verification.

We use the following notations: $S \xrightarrow{p} S'$ ($S \xrightarrow{t} S'$) is the class of partial (total) mappings from S to S'; $p(d)\downarrow$ ($p(d)\uparrow$) means that p is defined (undefined) on d. The terms and notations, not defined here, are treated in the sense of [3], [4].

2 First-order Logic of Partial Quasiary Predicates with the Complement Composition

We treat a logic L as a tuple $(\mathcal{A}, Fr, \mathcal{I}, \models, \vdash)$ [2], where

– $\mathcal{A}$ is *a class of algebras of some signature* $\Sigma^{\mathcal{A}}$;

– Fr is *a language (based on the algebra signature* $\Sigma^{\mathcal{A}}$*)*;

– $\mathcal{I}$ is *a class of interpretations*;

– $\models$ is *a consequence relation*;

– $\vdash$ is *an inference relation based on some calculus*.

Here we define only pure (without functions) logic L^{QEC}. This logic is the next step of our construction of series of first-order logics of partial quasiary predicates. Earlier, we started with a basic logic L^{Q} with compositions of *disjunction* $\vee$, *negation* $\neg$, *renomination* $R_{\bar{x}}^{\bar{v}}$, and *existential quantification* $\exists x$ [2]–[4]. This logic was not expressive enough to prove its completeness, therefore a logic L^{QE} was constructed as an extension of L^{Q} with the *null-ary parametric composition (predicate) Ez of variable assignment* [3]. (Also, *variable unassignment predicate* εz can be used.) But again, logic L^{QE} was not expressive enough to construct sound program logics of Floyd–Hoare type, therefore L^{QE} is extended to a new logic L^{QEC} by adding *the composition of predicate complement* $\sim$ (discussion on the topic is presented in [9], [10]).

2.1 Predicate Algebras with the Complement Composition

Let V and A be sets of *variables* (*names*) and *values* respectively. The class of *nominative sets* (*partial assignments*, *partial data*) is defined as the class of all partial mappings from V to A, thus, ${}^{V}A = V \xrightarrow{p} A$.

The main operation for nominative sets is a total unary parametric *renomination* $r_{x_1,...,x_n}^{v_1,..,v_n} : {}^{V}A \xrightarrow{t} {}^{V}A$, where $v_1, ..., v_n, x_1, ..., x_n$ are variables, and $v_1, ..., v_n$ are distinct [2]–[4]. Intuitively, given a nominative set d this operation yields a new nominative set changing the values of $v_1, ..., v_n$ to the values of $x_1, ..., x_n$ respectively. For this operation we also use simpler notation $r_{\bar{x}}^{\bar{v}}$; $x \in \bar{v}$ means that x is a variable from $\bar{v}$; $\bar{v} \cup \bar{x}$ is the set of variables that occur in $\bar{v}$ and $\bar{x}$; $asn(d)$ is the set of *assigned variables (names)* in d.

Notation $d\nabla x \mapsto a$ defines a nominative set obtained from d by changing a value of x to a (or adding to d variable x with the value a).

The set $Pr_A^V = {}^{V}A \xrightarrow{p} Bool$ is called the set of *partial quasiary predicates*. For a partial quasiary predicate $p \in Pr_A^V$ its *truth*, *falsity*, and *undefinedness domains* are denoted $T(p)$, $F(p)$, and $\perp(p)$ respectively. These domains are formally defined by the following formulas:

$$T(p) = \{d \in {}^{V}A \mid p(d)\downarrow = T\},\ F(p) = \{d \in {}^{V}A \mid p(d)\downarrow = F\},$$
$$\perp(p) = \{d \in {}^{V}A \mid p(d)\uparrow\}.$$

For a partial quasiary predicate p we have that $T(p) \cap F(p) = \emptyset$ and $\perp(p) = {}^{V}A \setminus (T(p) \cup F(p))$. Thus, p is defined only by $T(p)$ and $F(p)$.

A predicate p is

– *irrefutable* (*partially valid*) if $F(p) = \emptyset$;

– *satisfiable* if $T(p) \neq \emptyset$.

A name (variable) z is *unessential* for $p \in Pr_A^V$, if for any $d \in {}^{V}A$ the value of p does not depend on the value of z in d [3], [4].

Operations over Pr_A^V are called *compositions*. Basic compositions of first-order level of partial quasiary predicates are *disjunction* $\vee$, *negation* $\neg$, *renomination* $R_{\bar{x}}^{\bar{v}}$, and *existential quantification* $\exists x$.

We define them via their definedness domains ($p, q \in Pr_A^V$):

– $T(p \vee q) = T(p) \cup T(q), F(p \vee q) = F(p) \cap F(q)$;

– $T(\neg p) = F(p), F(\neg p) = T(p)$;

– $T(R_{\bar{x}}^{\bar{v}}(p)) = \{d \in {}^{V}A \mid r_{\bar{x}}^{\bar{v}}(d) \in T(p)\}$,
$F(R_{\bar{x}}^{\bar{v}}(p)) = \{d \in {}^{V}A \mid r_{\bar{x}}^{\bar{v}}(d) \in F(p)\}$;

– $T(\exists x\ p) = \{d \in {}^{V}A \mid d\nabla x \mapsto a \in T(p) \text{ for some } a \in A\} =$
$= \bigcup_{a \in A}\{d \mid d\nabla x \mapsto a \in T(p)\} = \bigcup_{a \in A}\{d \mid p(d\nabla x \mapsto a)\downarrow = T\}$,
$F(\exists x\ p) = \{d \in {}^{V}A \mid d\nabla x \mapsto a \in F(p) \text{ for all } a \in A\} =$

$= \bigcap_{a\in A}\{d \mid d\nabla x{\mapsto}a{\in}F(p)\} = \bigcap_{a\in A}\{d \mid p(d\nabla x{\mapsto}a){\downarrow} = F\}$.

For underfinedness domains we have:

$\bot(p \vee q) = (\bot(p) \cap \bot(q)) \cup (\bot(p) \cap F(q)) \cup ((F(p) \cap \bot(q))$;

$\bot(\neg p) = \bot(p)$;

$\bot(R^{\bar{v}}_{\bar{x}}(p)) = \{d \mid r^{\bar{v}}_{\bar{x}}(d){\in}\bot(p)\}$;

$\bot(\exists x\ p) = \bigcap_{a\in A}\{d \mid p(d\nabla x{\mapsto}a) \neq T\} \cap \bigcup_{a\in A}\{d \mid p(d\nabla x{\mapsto}a){\uparrow}\}$.

Lemma 1. $(\exists x\ p)(d){\uparrow} \ \Leftrightarrow p(d\nabla x{\mapsto}b){\uparrow}$ for some $b{\in}A$ and it is not possible $(\exists x\ p)(d){\downarrow} = T$.

Proof follows directly from composition definitions. □

Please note that definitions of disjunction and negation are similar to *strong Kleene's connectives*; their properties are described in [15]. Also we use *variable assignment predicate Ez* defined as follows:

$T(Ez) = \{d \mid d(z){\downarrow}\} = \{d{\in}^{V}\!A \mid z{\in}asn(d)\}$,

$F(Ez) = \{d \mid d(z){\uparrow}\} = \{d{\in}^{V}\!A \mid z{\notin}asn(d)\}$.

Predicate Ez is total, thus $\bot(Ez) = \emptyset$. For any $x \neq z$ variable x is unessential for Ez.

At last, the *complement composition* is defined in the following way:

$$T(\sim p) = \bot(p),\ F(\sim p) = \emptyset.$$

From this follows that $\bot(\sim p) = T(p) \cup F(p)$. Therefore

$T(\neg \sim p) = F(\sim p)$; $F(\neg \sim p) = T(\sim p)$; $\bot(\neg \sim p) = \bot(\sim p)$.

We consider $\sim$ as a composition of propositional level. This composition differs from traditional compositions. The main difference lies in the fact that traditional compositions are applicative compositions [16]. Applicativity of composition C means that given predicates $p_1, ..., p_n$ the value of $C(p_1, ..., p_n)$ on some data is evaluated upon values of $p_1, ..., p_n$ on data from their definedness domains. The complement composition is not applicative because the value of $\sim p$ on some d may depend upon undefinedness domain of p. This fact complicates logics with such composition because the undefinedness domains should be explicitly involved in the definitions of consequence relations. Note, that applicative compositions are monotone with respect to predicate graph inclusion; but composition $\sim$ is not.

A tuple $\mathcal{A}^{QEC}(V, A) =< Pr^{V}_{A}; \vee, \neg, R^{\bar{v}}_{\bar{x}}, \exists x, Ez, \sim>$ is called *a first-order complemented algebra of partial quasiary predicates.*

A class of such algebras (with different A) forms a semantic base for logic L^{QEC}.

Now we describe the main properties of $\mathcal{A}^{QEC}(V, A)$.

We are interested in properties of the following types:

– equivalent transformation and simplification;

– properties based on truth tables of composition evaluations.

Equivalence properties induce three similar sequent rules, describing cases when a formula is true, false, or undefined.

Properties based on evaluation cases induce special rule for each case. For example, evaluation of $\sim \Phi$ leads to three cases:

– $\sim \Phi$ is true if Φ is undefined;

– $\sim \Phi$ is false. This case is not possible;

– $\sim \Phi$ is undefined if Φ is defined (Φ or $\neg\Phi$ is true).

The first and third cases lead to rules $\vdash\sim$ and $\perp \sim$, the second rule defines the closeness condition $CL_{\sim\dashv}$ (Section 3).

For the renomination compositions we identify the following properties [2]–[4].

Lemma 2. For any $p, q \in Pr_A^V$ we have:

$R\vee$) $R_{\bar{x}}^{\bar{v}}(p \vee q) = R_{\bar{x}}^{\bar{v}}(p) \vee R_{\bar{x}}^{\bar{v}}(q)$;

$R\neg$) $R_{\bar{x}}^{\bar{v}}(\neg p) = \neg R_{\bar{x}}^{\bar{v}}(p)$;

RR) $R_{\bar{x}}^{\bar{v}}(R_{\bar{y}}^{\bar{w}}(p)) = R_{\bar{x}}^{\bar{v}} \circ_{\bar{y}}^{\bar{w}}(p)$;

$R\exists$) $R_{\bar{x}}^{\bar{v}}(\exists y p) = \exists z R_{\bar{x}}^{\bar{v}}(R_z^y(p))$, $z \notin \bar{v} \cup \{y\}$, z is unessential for p;

RE) $R_{\bar{x}}^{\bar{v}}(Ez) = Ez$, $z \notin \bar{v}$;

RE_r) $R_{\bar{x},y}^{\bar{v},z}(Ez) = Ey$;

R) $R(p) = p$;

RI) $R_{z,\bar{x}}^{z,\bar{v}}(p) = R_{\bar{x}}^{\bar{v}}(p)$;

RU) $R_{y,\bar{x}}^{z,\bar{v}}(p) = R_{\bar{x}}^{\bar{v}}(p)$, z is unessential for p;

$R\sim$) $R_{\bar{x}}^{\bar{v}}(\sim p) = \sim R_{\bar{x}}^{\bar{v}}(p)$.

Proof. All properties are proved in the same manner, therefore we restrict ourselves by the property $R\sim$ only, which involves the complement composition.

To prove this property we should prove two equalities:

$$T(R_{\bar{x}}^{\bar{v}}(\sim p)) = T(\sim R_{\bar{x}}^{\bar{v}}(p)) \text{ and } F(R_{\bar{x}}^{\bar{v}}(\sim p)) = F(\sim R_{\bar{x}}^{\bar{v}}(p)).$$

By the definitions of renomination and complement compositions we get for the first equality that

$T(R_{\bar{x}}^{\bar{v}}(\sim p)) = \{d{\in}{}^{V}\!A \mid r_{\bar{x}}^{\bar{v}}(d){\in}T(\sim p)\} = \{d{\in}{}^{V}\!A \mid r_{\bar{x}}^{\bar{v}}(d){\in}\bot(p)\} =$
$= \{d{\in}{}^{V}\!A \mid d{\in}\bot(R_{\bar{x}}^{\bar{v}}(p)(d))\} = \{d{\in}{}^{V}\!A \mid d{\in}T(\sim R_{\bar{x}}^{\bar{v}}(p)(d))\} =$
$= T(\sim R_{\bar{x}}^{\bar{v}}(p))$.

In the similar way the second equality is proved. □

For the complement composition we identify the following properties.

Lemma 3. For any $p{\in}Pr_A^V$ we have:
$\sim \neg p = \sim p$; $\sim\sim p = p \vee \neg p$; $\sim\sim\sim p = \sim p$; $\sim\sim\sim\sim p = \sim\sim p$.

Proof follows directly from composition definitions.□

In L^{QEC} quantifier elimination is based on properties inherited from basic logic L^{Q} [3]:

$T\exists$) $T(R_y^x(P)) \cap T(Ey) \subseteq T(\exists x\ P)$;

$F\exists$) $F(\exists x\ P) \cap T(Ey) \subseteq F(R_y^x(P))$.

The properties presented in this subsection substantiate properties of the consequence relation and sequent rules for our logic.

2.2 Language (signature and formulas) of L^{QEC}

Let V be an infinite set of *variables* (*names*) and V_U be an infinite subset of V called *a set of unessential variables* [3], [4]. Let Ps be a set of *predicate symbols*. A tuple $\Sigma^{QEC} = (V, V_U; \vee, \neg, R_{\bar{x}}^{\bar{v}}, \exists x, Ez, \sim; Ps)$ is called the *language signature*.

For simplicity, we use the same notation for symbols of compositions and compositions themselves.

Given Σ^{QEC}, we define inductively the *language* of L^{QEC} – the *set of formulas* denoted $Fr(L^{QEC})$ or simply Fr:

– if $P{\in}Ps$, then $P{\in}Fr$;

– $Ez{\in}Fr$;

– if $\Phi, \Psi{\in}Fr$, then $\Phi \vee \Psi$, $\neg\Phi$, $R_{\bar{x}}^{\bar{v}}(\Phi)$, $\exists x\Phi$, $\sim \Phi{\in}Fr$.

Formulas of the forms P and Ez are called *atomic* ($P{\in}Ps$, $z{\in}V$); formulas of the form $R_{\bar{x}}^{\bar{v}}(P)$ are called *primitive*. Parentheses can be used to clarify formula structure.

Note that properties presented by Lemma 2 allow transforming any formula to special normal form in which renomination occurs only in primitive formulas.

2.3 L^{QEC}-interpretations

Let $\mathcal{A}^{QEC}(V,A) =< Pr^V_A; \vee, \neg, R^{\bar{v}}_{\bar{x}}, \exists x, Ez, \sim>$ be a first-order complemented algebra of partial quasiary predicates of a signature $\Sigma^{QEC} = (V, V_U; \vee, \neg, R^{\bar{v}}_{\bar{x}}, \exists x, Ez, \sim; Ps)$; $I^{Ps}_Q = Ps \xrightarrow{t} Pr^V_A$ be an *interpretation* mapping of predicate symbols that respects the set V_U of unessential variables. Then a pair $J(\Sigma^{QEC}) = (\mathcal{A}^{QEC}(V,A), I^{Ps}_Q)$ is called an L^{QEC}-*interpretation*. Note that this definition of interpretation is quite natural because the algebra $\mathcal{A}^{QEC}(V,A)$ defines interpretations of composition symbols (logical symbols) and I^{Ps}_Q defines interpretations of predicate symbols (descriptive symbols).

We simplify notation for L^{QEC}-interpretation $J(\Sigma^{QEC})$ omitting L^{QEC} and Σ^{QEC}.

For a given interpretation J and a formula Φ, we can define by induction on the structure of Φ its value in J. Obtained predicate is denoted Φ_J.

Formula Φ is *irrefutable in* J (denoted $J \models \Phi$), if predicate Φ_J is irrefutable. Formula Φ is *irrefutable* (denoted $\models \Phi$), if $J \models \Phi$ for any interpretation J. Irrefutability may be treated as *partial validity*.

Formula Φ is *satisfiable in* J (denoted $J \mid\!\sim \Phi$), if predicate Φ_J is satisfiable. Formula Φ is *satisfiable* (denoted $\mid\!\sim \Phi$), if $J \mid\!\sim \Phi$ for some interpretation J.

Variable x is *unessential for* Φ, if for any J variable x is unessential for Φ_J. Variable x is *unessential for* $\Gamma \subseteq Fr$, if for any J variable x is unessential for any formula $\Phi \in \Gamma$.

The set of all variables (names) that occur in Φ is denoted $nm(\Phi)$. The set $fu(\Phi) = V_U \setminus nm(\Phi)$ is called the set of fresh unessential variables for Φ.

For any $\Gamma \subseteq Fr$ we define

$$nm(\Gamma) = \bigcup_{\Phi \in \Gamma} nm(\Phi) \text{ and } fu(\Gamma) = \bigcap_{\Phi \in \Gamma} fu(\Phi).$$

We generalize notation $fu(\Gamma)$ on sequences of formulas and sets of formulas.

Lemma 4. Let $x \in V, \Phi \in Fr, \Gamma \subseteq Fr$. Then
1) x is unessential for Φ if $x \in fu(\Phi)$;
2) x is unessential for Γ if $x \in fu(\Gamma)$.

Proof. Induction on the structure of Φ.$\square$

2.4 Irrefutability Consequence Relation

Logic L^{QEC} is a logic of partial predicates, therefore the most natural consequence relation for this logic is the irrefutability relation, because it reflects partial validity.

Let $\Sigma \subseteq Fr$ and J be an interpretation. We denote:

$$\bigcap_{\Phi\in\Sigma} T(\Phi_J) \text{ as } T^{\cap}(\Sigma_J) \text{ and } \bigcap_{\Phi\in\Sigma} F(\Phi_J) \text{ as } F^{\cap}(\Sigma_J).$$

Let $\Gamma, U, \Delta \subseteq Fr$. Then Δ is called *an irrefutable consequence* of Γ *in interpretation* J (denoted $\Gamma_J \models_{IR} \Delta$) *if*

$$T^{\cap}(\Gamma_J) \cap F^{\cap}(\Delta_J) = \emptyset.$$

Δ is *logical irrefutable consequence* of Γ (denoted $\Gamma \models_{IR} \Delta$), if $\Gamma_J \models_{IR} \Delta$ for any interpretation J.

The basic properties of $\models_{IR}$ were presented in [3], [4]. They allow decomposition of complex formulas up to atomic or primitive formulas. In our case it is not always possible to make decomposition of formulas with the complement composition, therefore we need to define a new consequence relation $\models_{IR}^{\perp}$ which takes into consideration undefinedness domains.

In the sequel $\Phi, \Psi \in Fr$, $U, \Gamma, \Delta, \Sigma \subseteq Fr$, formulas may be signed or unsigned; variables (maybe with indexes) v, y, z, t belong to V; J is an interpretation.

2.5 Irrefutability Consequence Relation under Conditions of Undefinedness

Irrefutability consequence relation is a binary relation. In our case, introduction of composition $\sim$ requires more complicated ternary consequence relation, because formulas, treated as undefined, should be explicitly taken into consideration. Here we introduce such consequence relation denoted $\models_{IR}^{\perp}$ between three sets of formulas. The first set is called the set of *underfinedness conditions* ($_{\perp}$-conditions, $_{\perp}$-formulas);

the second set is called the set of *truth formulas* ($_{\vdash}$-formulas); and the third set is called the set of *falsity formulas* ($_{\dashv}$-formulas).

Relation $\models^{\perp}_{IR}$ will generalize the binary irrefutability relation. To define $\models^{\perp}_{IR}$ we additionally denote $\bigcap_{\Phi\in\Sigma}\perp(\Phi_J)$ as $\perp^{\cap}(\Sigma_J)$.

Let $U, \Gamma, \Delta \subseteq Fr$. Then Δ is called *an irrefutable consequence of* Γ *under undefinedness conditions* U *in interpretation* J (denoted $U/\Gamma_J \models^{\perp}_{IR} \Delta$) if

$$T^{\cap}(\Gamma_J)\cap\perp^{\cap}(U_J)\cap F^{\cap}(\Delta_J)=\emptyset.$$

Δ is *logical irrefutable consequence of* Γ *under undefinedness conditions* U (denoted $U/\Gamma \models^{\perp}_{IR} \Delta$), if $U/\Gamma_J \models^{\perp}_{IR} \Delta$ for any interpretation J.

We get traditional *logical irrefutability* $\Gamma \models_{IR} \Delta$ when $U = \emptyset$. Other consequence relations are studied in [4], [17], [18].

Relation $\models^{\perp}_{IR}$ is *monotone* in the following sense:

M) Let $\Gamma \subseteq \Lambda$, $U \subseteq W$, and $\Delta \subseteq \Sigma$; then

$$U/\Gamma \models^{\perp}_{IR} \Delta \Rightarrow W/\Lambda \models^{\perp}_{IR} \Sigma.$$

Let us introduce on Fr the binary relation $\simeq$ of logical strong equality. Namely, $\Phi \simeq \Psi$ if $\Phi_J = \Psi_J$ for any interpretation J.

Theorem 1. Let $\Phi \simeq \Psi$, then:

$U\ /\ \Phi, \Gamma \models^{\perp}_{IR} \Delta \Leftrightarrow U/\Psi, \Gamma \models^{\perp}_{IR} \Delta$;

$U\ /\ \Gamma \models^{\perp}_{IR} \Delta, \Phi \Leftrightarrow U/\Gamma \models^{\perp}_{IR} \Delta, \Psi$;

$U, \Phi/\Gamma \models^{\perp}_{IR} \Delta \Leftrightarrow U, \Psi/\Gamma \models^{\perp}_{IR} \Delta$.

Proof. Proof is based on the fact that $\Phi \simeq \Psi$ means $\Phi_J = \Psi_J$ for any J. $\square$

Let us formulate the properties that guarantee validity of the consequence relation $\models^{\perp}_{IR}$.

Theorem 2. For any $U, \Gamma, \Delta \subseteq Fr, \Phi\in Fr$ we have:

$C_{\vdash\dashv}$) $U/\Phi, \Gamma \models^{\perp}_{IR} \Delta, \Phi$;

$C_{\perp\vdash}$) $U, \Phi/\Phi, \Gamma \models^{\perp}_{IR} \Delta$;

$C_{\perp\dashv}$) $U, \Phi/\Gamma \models^{\perp}_{IR} \Delta, \Phi$;

$C_{\sim\dashv}$) $U/\Gamma \models^{\perp}_{IR} \Delta, \sim \Phi$;

$C_{E\perp}$) $U, Ey/\Gamma \models^{\perp}_{IR} \Delta$.

Proof. Property $C_{\vdash\dashv}$ follows from equality $T(\Phi_J) \cap F(\Phi_J) = \emptyset$.

Property $C_{\perp\vdash}$ follows from equality $\perp(\Phi_J) \cap T(\Phi_J) = \emptyset$.

Property $C_{\perp\dashv}$ follows from equality $\perp(\Phi_J) \cap F(\Phi_J) = \emptyset$.

Property $C_{\sim\dashv}$ holds because $F(\sim \Phi_J) = \emptyset$.

Property $C_{E\perp}$ holds because $\perp(Ey) = \emptyset$. □

Let us consider properties of $\models^{\perp}_{IR}$ induced by propositional compositions $\vee$, $\neg$, and $\sim$:

Theorem 3. For any U, $\Gamma, \Delta \subseteq Fr, \Phi, \Psi \in Fr$ the following properties of $\models^{\perp}_{IR}$ hold:

$\vee_{\vdash}$) $U/\Phi \vee \Psi, \Gamma \models^{\perp}_{IR} \Delta \Leftrightarrow U/\Phi, \Gamma \models^{\perp}_{IR} \Delta$ and $U\ /\ \Psi, \Gamma \models^{\perp}_{IR} \Delta$;

$\vee_{\dashv}$) $U/\Gamma \models^{\perp}_{IR} \Delta, \Phi \vee \Psi \Leftrightarrow U/\Gamma \models^{\perp}_{IR} \Delta, \Phi, \Psi$;

$\vee_{\perp}$) $U, \Phi \vee \Psi/\Gamma \models^{\perp}_{IR} \Delta \Leftrightarrow U, \Phi, \Psi/\Gamma \models^{\perp}_{IR} \Delta$ and
$U, \Phi/\Gamma \models^{\perp}_{IR} \Psi, \Delta$ and $U, \Psi/\Gamma \models^{\perp}_{IR} \Phi, \Delta$;

$\neg_{\vdash}$) $U/\neg\Phi, \Gamma \models^{\perp}_{IR} \Delta \Leftrightarrow U/\Gamma \models^{\perp}_{IR} \Delta, \Phi$;

$\neg_{\dashv}$) $U/\Gamma \models^{\perp}_{IR} \Delta, \neg\Phi \Leftrightarrow U/\Phi, \Gamma \models^{\perp}_{IR} \Delta$;

$\neg_{\perp}$) $U, \neg\Phi/\Gamma \models^{\perp}_{IR} \Delta \Leftrightarrow U, \Phi/\Gamma \models^{\perp}_{IR} \Delta$;

$\sim_{\vdash}$) $U/ \sim \Phi, \Gamma \models^{\perp}_{IR} \Delta \Leftrightarrow U, \Phi/\Gamma \models^{\perp}_{IR} \Delta$;

$\sim_{\perp}$) $U, \sim \Phi/\Gamma \models^{\perp}_{IR} \Delta \Leftrightarrow U/\Phi, \Gamma \models^{\perp}_{IR} \Delta$ and $U\ /\ \Gamma \models^{\perp}_{IR} \Delta, \Phi$.

Proof. The properties are proved in the same manner, therefore we demonstrate it proving $\vee_{\perp}$, $\neg_{\perp}$, $\sim_{\perp}$, and $\sim_{\vdash}$ only.

For property $\vee_{\perp}$ we have that $U, \Phi \vee \Psi/\Gamma_J \models^{\perp}_{IR} \Delta$ means that $T^{\cap}(\Gamma_J) \cap (\perp^{\cap}(U_J) \cap \perp(\Phi \vee \Psi)) \cap F^{\cap}(\Delta_J) = \emptyset$.

By definition of $\vee$ we get that
$\perp(\Phi_J \vee \Psi_J) = (\perp(\Phi_J) \cap \perp(\Psi_J)) \cup (\perp(\Phi_J) \cap F(\Psi_J) \cup (F(\Phi_J) \cap \perp(\Psi_J))$.

Substituting the right-hand side of this formula into the previous one we obtain that $T^{\cap}(\Gamma_J) \cap \perp^{\cap}(U_J) \cap ((\perp(\Phi_J) \cap \perp(\Psi_J)) \cup (\perp(\Phi_J) \cap F(\Psi_J)) \cup (F(\Phi_J) \cap \perp(\Psi_J))) \cap F^{\cap}(\Delta_J) = \emptyset$.

Transformation of this formula gives that
$T^{\cap}(\Gamma_J) \cap \perp^{\cap}(U_J) \cap (\perp(\Phi_J) \cap \perp(\Psi_J)) \cap F^{\cap}(\Delta_J) \cup$
$\cup T^{\cap}(\Gamma_J) \cap \perp^{\cap}(U_J) \cap (\perp(\Phi_J) \cap F(\Psi_J)) \cap F^{\cap}(\Delta_J) \cup$
$\cup T^{\cap}(\Gamma_J) \cap \perp^{\cap}(U_J) \cap (F(\Phi_J) \cap \perp(\Psi_J)) \cap F^{\cap}(\Delta_J) = \emptyset$.

Union of sets is empty, therefore each set is empty too. Thus,
$T^{\cap}(\Gamma_J) \cap \perp^{\cap}(U_J) \cap (\perp(\Phi_J) \cap \perp(\Psi_J)) \cap F^{\cap}(\Delta_J) = \emptyset$;
$\cup T^{\cap}(\Gamma_J) \cap \perp^{\cap}(U_J) \cap (\perp(\Phi_J) \cap F(\Psi_J)) \cap F^{\cap}(\Delta_J) = \emptyset$;
$\cup T^{\cap}(\Gamma_J) \cap \perp^{\cap}(U_J) \cap (F(\Phi_J) \cap \perp(\Psi_J)) \cap F^{\cap}(\Delta_J) = \emptyset$.

This means that
$U, \Phi, \Psi/\Gamma_J \models^{\perp}_{IR} \Delta$ and $U, \Phi/\Gamma_J \models^{\perp}_{IR} \Psi, \Delta$ and $U, \Psi/\Gamma_J \models^{\perp}_{IR} \Phi, \Delta$.

So, property $\vee_\perp$ holds because it holds for any interpretation J.
Property $\neg_\perp$ holds due to equality $\perp(\neg\Phi_J) = \perp(\Phi_J)$.
Property $\sim_\perp$ holds due to equality $\perp(\sim \Phi_J) = T(\Phi_J) \cup F(\Phi_J)$.
Property $\sim_\vdash$ holds due to equality $T(\sim \Phi_J) = \perp(\Phi_J)$. □

Let us consider properties of relation $\models^\perp_{IR}$ for renomination composition. Each of the properties $R\vee$, $R\neg$, RR, $R\exists$, R, RI, RU, $R\sim$ (Lemma 2) induces three corresponding properties for $\models^\perp_{IR}$, depending on the position of a formula (in the left side of $\models^\perp_{IR}$, in the right side of $\models^\perp_{IR}$, or in the undefinedness conditions of $\models^\perp_{IR}$). Such properties are formulated in a similar way. Properties RE and REr induce two cases because predicate Ex is a total predicate.

Theorem 4. For any U, $\Gamma, \Delta \subseteq Fr, \Phi, \Psi \in Fr$ renomination composition induces the following properties of $\models^\perp_{IR}$:
$R\vee_\vdash$) $U/R^{\bar v}_{\bar x}(\Phi \vee \Psi), \Gamma \models^\perp_{IR} \Delta \Leftrightarrow U/R^{\bar v}_{\bar x}(\Phi) \vee R^{\bar v}_{\bar x}(\Psi), \Gamma \models^\perp_{IR} \Delta$;
$R\vee_\dashv$) $U/\Gamma \models^\perp_{IR} R^{\bar v}_{\bar x}(\Phi \vee \Psi), \Delta \Leftrightarrow U/\Gamma \models^\perp_{IR} R^{\bar v}_{\bar x}(\Phi) \vee R^{\bar v}_{\bar x}(\Psi), \Delta$;
$R\vee_\perp$) $U, R^{\bar v}_{\bar x}(\Phi \vee \Psi)/\Gamma \models^\perp_{IR} \Delta \Leftrightarrow U, R^{\bar v}_{\bar x}(\Phi) \vee R^{\bar v}_{\bar x}(\Psi)/\Gamma \models^\perp_{IR} \Delta$;
$R\neg_\vdash$) $U/R^{\bar v}_{\bar x}(\neg\Phi), \Gamma \models^\perp_{IR} \Delta \Leftrightarrow U/\neg R^{\bar v}_{\bar x}(\Phi), \Gamma \models^\perp_{IR} \Delta$;
$R\neg_\dashv$) $U/\Gamma \models^\perp_{IR} R^{\bar v}_{\bar x}(\neg\Phi), \Delta \Leftrightarrow U/\Gamma \models^\perp_{IR} \neg R^{\bar v}_{\bar x}(\Phi), \Delta$;
$R\neg_\perp$) $U, R^{\bar v}_{\bar x}(\neg\Phi)/\Gamma \models^\perp_{IR} \Delta \Leftrightarrow U, \neg R^{\bar v}_{\bar x}(\Phi)/\Gamma \models^\perp_{IR} \Delta$;
$RR_\vdash$) $U \ / \ R^{\bar v}_{\bar x}(R^{\bar w}_{\bar y}(\Phi))\Gamma \models^\perp_{IR} \Delta \Leftrightarrow U/R^{\bar v}_{\bar x} \circ^{\bar w}_{\bar y} (\Phi), \Gamma \models^\perp_{IR} \Delta$;
$RR_\dashv$) $U \ / \ \Gamma \models^\perp_{IR} \Delta, R^{\bar v}_{\bar x}(R^{\bar w}_{\bar y}(\Phi)) \Leftrightarrow U/\Gamma \models^\perp_{IR} \Delta, R^{\bar v}_{\bar x} \circ^{\bar w}_{\bar y} (\Phi)$;
$RR_\perp$) $U, R^{\bar v}_{\bar x}(R^{\bar w}_{\bar y}(\Phi))/\Gamma \models^\perp_{IR} \Delta \Leftrightarrow U, R^{\bar v}_{\bar x} \circ^{\bar w}_{\bar y} (\Phi)/\Gamma \models^\perp_{IR} \Delta$;
$R\exists_\vdash$) if $z \in fu(R^{\bar v}_{\bar x}(\exists y\Phi))$ then
$U/R^{\bar v}_{\bar x}(\exists y\Phi), \Gamma \models^\perp_{IR} \Delta \Leftrightarrow U/\exists z R^{\bar v}_{\bar x}(R^y_z(\Phi)), \Gamma \models^\perp_{IR} \Delta$;
$R\exists_\dashv$) if $z \in fu(R^{\bar v}_{\bar x}(\exists y\Phi))$ then
$U/\Gamma \models^\perp_{IR} R^{\bar v}_{\bar x}(\exists y\Phi), \Delta \Leftrightarrow U/\Gamma \models^\perp_{IR} \exists z R^{\bar v}_{\bar x}(R^y_z(\Phi)), \Delta$;
$R\exists_\perp$) if $z \in fu(R^{\bar v}_{\bar x}(\exists y\Phi))$ then
$U, R^{\bar v}_{\bar x}(\exists y\Phi)/\Gamma \models^\perp_{IR} \Delta \Leftrightarrow U, \exists z R^{\bar v}_{\bar x}(R^y_z(\Phi))/\Gamma \models^\perp_{IR} \Delta$;
$R_\vdash$) $U/R(\Phi), \Gamma \models^\perp_{IR} \Delta \Leftrightarrow U/\Phi, \Gamma \models^\perp_{IR} \Delta$;
$R_\dashv$) $U/\Gamma \models^\perp_{IR} R(\Phi), \Delta \Leftrightarrow U/\Gamma \models^\perp_{IR} \Phi, \Delta$;
$R_\perp$) $U, R(\Phi)/\Gamma \models^\perp_{IR} \Delta \Leftrightarrow U, \Phi/\Gamma \models^\perp_{IR} \Delta$;
$RI_\vdash$) $U/R^{z,\bar v}_{z,\bar x}(\Phi), \Gamma \models^\perp_{IR} \Delta \Leftrightarrow U/R^{\bar v}_{\bar x}(\Phi), \Gamma \models^\perp_{IR} \Delta$;
$RI_\dashv$) $U/\Gamma \models^\perp_{IR} R^{z,\bar v}_{z,\bar x}(\Phi), \Delta \Leftrightarrow U/\Gamma \models^\perp_{IR} R^{\bar v}_{\bar x}(\Phi), \Delta$;
$RI_\perp$) $U, R^{z,\bar v}_{z,\bar x}(\Phi)/\Gamma \models^\perp_{IR} \Delta \Leftrightarrow U, R^{\bar v}_{\bar x}(\Phi)/\Gamma \models^\perp_{IR} \Delta$;

$RU_{\vdash}$) if $z \in fu(\Phi)$ then

$$U/R^{z,\bar{v}}_{y,\bar{x}}(\Phi), \Gamma \models^{\perp}_{IR} \Delta \Leftrightarrow U/R^{\bar{v}}_{\bar{x}}(\Phi), \Gamma \models^{\perp}_{IR} \Delta;$$

$RU_{\dashv}$) if $z \in fu(\Phi)$ then

$$U/\Gamma \models^{\perp}_{IR} R^{z,\bar{v}}_{y,\bar{x}}(\Phi), \Delta \Leftrightarrow U/\Gamma \models^{\perp}_{IR} R^{\bar{v}}_{\bar{x}}(\Phi), \Delta;$$

$RU_{\perp}$) if $z \in fu(\Phi)$ then

$$U, R^{z,\bar{v}}_{y,\bar{x}}(\Phi)/\Gamma \models^{\perp}_{IR} \Delta \Leftrightarrow U, R^{\bar{v}}_{\bar{x}}(\Phi)/\Gamma \models^{\perp}_{IR} \Delta;$$

$R^{\sim}_{\vdash}$) $U/R^{\bar{v}}_{\bar{x}}(\sim \Phi), \Gamma \models^{\perp}_{IR} \Delta \Leftrightarrow U/ \sim R^{\bar{v}}_{\bar{x}}(\Phi), \Gamma \models^{\perp}_{IR} \Delta;$

$R^{\sim}_{\dashv}$) $U/\Gamma \models^{\perp}_{IR} \Delta, R^{\bar{v}}_{\bar{x}}(\sim \Phi) \Leftrightarrow U/\Gamma \models^{\perp}_{IR} \Delta, \sim R^{\bar{v}}_{\bar{x}}(\Phi);$

$R^{\sim}_{\perp}$) $U, R^{\bar{v}}_{\bar{x}}(\sim \Phi)/\Gamma \models^{\perp}_{IR} \Delta \Leftrightarrow U, \sim R^{\bar{v}}_{\bar{x}}(\Phi)/\Gamma \models^{\perp}_{IR} \Delta;$

$RE_{\vdash}$) if $z \notin \bar{v}$ then $U/R^{\bar{v}}_{\bar{x}}(Ez), \Gamma \models^{\perp}_{IR} \Delta \Leftrightarrow U/Ez, \Gamma \models^{\perp}_{IR} \Delta;$

$RE_{\dashv}$) if $z \notin \bar{v}$ then $U/\Gamma \models^{\perp}_{IR} \Delta, R^{\bar{v}}_{\bar{x}}(Ez) \Leftrightarrow U/\Gamma \models^{\perp}_{IR} \Delta, Ez;$

$REr_{\vdash}$) $U/R^{\bar{v},z}_{\bar{x},y}(Ez), \Gamma \models^{\perp}_{IR} \Delta \Leftrightarrow U/Ey, \Gamma \models^{\perp}_{IR} \Delta;$

$REr_{\dashv}$) $U/\Gamma \models^{\perp}_{IR} \Delta, R^{\bar{v},z}_{\bar{x},y}(Ez) \Leftrightarrow U/\Gamma \models^{\perp}_{IR} \Delta, Ey.$

Proof. All properties hold due to Lemma 2. □

We add one more property RN that permits to substitute one unassigned variable by another unassigned variable. This allows to establish equivalence of formulas with unassigned variables.

Theorem 5. For any $U, \Gamma, \Delta \subseteq Fr, \Phi \in Fr, y, t \in V$ the renomination composition induces the following substitution properties of unassigned variables:

$RN_{\vdash}$) $U/R^{y,\bar{v}}_{z,\bar{x}}(\Phi), \Gamma \models^{\perp}_{IR} Ez, Et, \Delta \Leftrightarrow U/R^{y,\bar{v}}_{t,\bar{x}}(\Phi), \Gamma \models^{\perp}_{IR} Ez, Et, \Delta;$

$RN_{\dashv}$) $U/\Gamma \models^{\perp}_{IR} R^{y,\bar{v}}_{z,\bar{x}}(\Phi), Ez, Et, \Delta \Leftrightarrow U/\Gamma \models^{\perp}_{IR} R^{y,\bar{v}}_{t,\bar{x}}(\Phi), Ez, Et, \Delta;$

$RN_{\perp}$) $U, R^{y,\bar{v}}_{z,\bar{x}}(\Phi)/\Gamma \models^{\perp}_{IR} Ez, Et, \Delta \Leftrightarrow U, R^{y,\bar{v}}_{t,\bar{x}}(\Phi)/\Gamma \models^{\perp}_{IR} Ez, Et, \Delta.$

Proof is based on the fact that $F(Ey) = F(Et)$ in the case when both y and t are not assigned. □

The following theorem describes properties that will induce the quantifier elimination rules.

Theorem 6. For any $U, \Gamma, \Delta \subseteq Fr$ and $\Phi \in Fr$ we have:

$\exists_{\vdash}$) if $z \in fu(U, \Gamma, \Delta, \exists x\Phi)$, then

$$U/\exists x\Phi, \Gamma \models^{\perp}_{IR} \Delta \Leftrightarrow U/R^{x}_{z}(\Phi), Ez, \Gamma \models^{\perp}_{IR} \Delta;$$

$\exists_{\dashv}$) $U/\Gamma, Ey \models^{\perp}_{IR} \exists x\Phi, \Delta \Leftrightarrow U/\Gamma, Ey \models^{\perp}_{IR} \exists x\Phi, R^{x}_{y}(\Phi), \Delta;$

$\exists_{\perp}$) if $z \in fu(U, \Gamma, \Delta, \exists x\Phi)$, then

$$U, \exists x\Phi/\Gamma \models^{\perp}_{IR} \Delta \Leftrightarrow U, \exists x\Phi, R^{x}_{z}(\Phi)/Ez, \Gamma \models^{\perp}_{IR} \Delta.$$

Proof. We prove $\exists_\perp$ only, because other properties can be proved in the same manner.

By M we have $U, \exists x\Phi/\Gamma \models^{\perp}_{IR} \Delta \Rightarrow U, \exists x\Phi, R^x_z(\Phi)/Ez, \Gamma \models^{\perp}_{IR} \Delta$.

It is left to prove $(R) \Rightarrow (L)$, where
(L) is $\perp^\cap(U_J) \cap \perp(\exists x\Phi)_J \cap T^\cap(\Gamma_J) \cap F^\cap(\Delta_J) = \emptyset$ and (R) is
$\perp^\cap(U_J) \cap \perp(\exists x\Phi)_J \cap \perp(R^x_z(\Phi)_J) \cap\ T(Ez) \cap T^\cap(\Gamma_J) \cap F^\cap(\Delta_J) = \emptyset$.

Assume that (R) holds, but (L) does not hold. From this follows that there exists $d\in{}^VA$ such that $d\in\perp^\cap(U_J) \cap \perp((\exists x\Phi)_J) \cap T^\cap(\Gamma_J) \cap F^\cap(\Delta_J)$, therefore $d\in\perp((\exists x\Phi)_J)$ and $d \in\perp^\cap(U_J) \cap T^\cap(\Gamma_J) \cap F^\cap(\Delta_J)$. From $d\in\perp((\exists x\Phi)_J)$ we have that for some $a\in A$ there should be $d\nabla x{\mapsto}a\in\perp(\Phi_J)$. But $z\in fu(U, \Gamma, \Delta, \exists x\Phi)$, therefore $d\nabla z{\mapsto}a\in\perp^\cap(U_J)\cap \perp((\exists x\Phi)_J) \cap T^\cap(\Gamma_J) \cap F^\cap(\Delta_J)$ and $d\nabla x{\mapsto}a\nabla z{\mapsto}a\in\perp(\Phi_J)$. From this we obtain $d\nabla z{\mapsto}a\in\perp(R^x_z(\Phi)_J)$. By definition of Ez we have $d\nabla z{\mapsto}a\in T(Ez)$, therefore $d\nabla z{\mapsto}a\in\perp^\cap(U_J)\cap\perp((\exists x\Phi)_J)\cap\perp(R^x_z(\Phi)_J)\cap T(Ez) \cap T^\cap(\Gamma_J) \cap F^\cap(\Delta_J)$. But this contradicts (R).□

Having $\exists_\perp$ we obtain the special property that guarantees $\models^{\perp}_{IR}$: for any $U, \Gamma, \Delta \subseteq Fr$ and $\Phi\in Fr$

$C_{\exists\perp}$) $U, \exists x\Phi/R^x_y(\Phi), Ey, \Gamma \models^{\perp}_{IR} \Delta$.

So, we proved the following properties that guarantee $\models^{\perp}_{IR}$:

$C_{\vdash\dashv}$, $C_{\perp\vdash}$, $C_{\perp\dashv}$, $C_{\sim\dashv}$, $C_{E\perp}$, and $C_{\exists\perp}$.

Theorem 7. For any $U, \Gamma, \Delta \subseteq Fr, y\in V$ the variable assignment predicate induces the following insertion property :

$E^{\vdash\dashv}$) $U/\ \Gamma \models^{\perp}_{IR} \Delta \Leftrightarrow U/Ey, \Gamma \models^{\perp}_{IR} \Delta$ and $U/\Gamma \models^{\perp}_{IR} \Delta, Ey$.

Proof. Let J be an interpretation. Then $U/\Gamma_J \models^{\perp}_{IR} \Delta$ means that $T^\cap(\Gamma_J) \cap \perp^\cap(U_J) \cap F^\cap(\Delta_J) = \emptyset$. Further, $U/Ey, \Gamma \models^{\perp}_{IR} \Delta$ and $U/\Gamma \models^{\perp}_{IR} \Delta, Ey$ means that $(T^\cap(\Gamma_J)\cap T(Ey_J))\cap\perp^\cap(U_J)\cap F^\cap(\Delta_J) = \emptyset$ and $T^\cap(\Gamma_J) \cap \perp^\cap(U_J) \cap (F(Ey_J) \cap F^\cap(\Delta_J)) = \emptyset$ respectively. Since $T(Ey_J) \cup F(Ey_J) = {}^VA$, we obtain $T^\cap(\Gamma_J) \cap \perp^\cap(U_J) \cap F^\cap(\Delta_J) = \emptyset$. This proves the theorem because J was arbitrary interpretation.□

3 Sequent Calculus for L^{QEC}

Usually, an inference relation $\vdash$ is defined by some axiomatic system (calculus). We present here system C^{QEC} that adequately formalizes logical consequence relation $\models^{\perp}_{IR}$ between sets of formulas. Such systems are called *sequent calculi*.

The main objects of this calculus are *sequents*. Here we consider only the case with *finite* sequents. We treat them as sets of formulas signed by symbols ${}_\vdash$, ${}_\dashv$, and ${}_\perp$. Sequents are denoted ${}_\vdash\Gamma_\perp U_\dashv\Delta$, in abbreviated form Σ.

A sequent calculus is defined by *sequent forms* (*sequent rules*) and *closeness conditions* of sequents.

Sequent forms are syntactical analogs of the semantic properties of the logical consequence relation. Closed sequents are axioms of the sequent calculus.

A *closed sequent* is specified in such a way that the following condition should hold:

if sequent ${}_\vdash\Gamma_\perp U_\dashv\Delta$ is closed, then $U\ /\ \Gamma \models^{\perp}_{IR} \Delta$.

The following conditions are induced by the properties $C_{\vdash\dashv}$, $C_{\perp\vdash}$, $C_{\perp\dashv}$, $C_{\sim\dashv}$, $C_{E\perp}$, and $C_{\exists\perp}$ respectively:

$CL_{\vdash\dashv}$) there is Φ such that $\Phi\in\Gamma$ and $\Phi\in\Delta$;

$CL_{\perp\vdash}$) there is Φ such that $\Phi\in U$ and $\Phi\in\Gamma$;

$CL_{\perp\dashv}$) there is Φ such that $\Phi\in U$ and $\Phi\in\Delta$;

$CL_{\sim\dashv}$) there is Φ such that $\sim\Phi\in\Delta$;

$CL_{E\perp}$) there is Ey such that $Ey\in U$;

$CL_{\exists\perp}$) there are $\exists x\Phi$ and y such that $\exists x\Phi\in U, R^x_y(\Phi)\in\Gamma, Ey\in\Gamma$.

For C^{QEC} we take the following *closeness condition*: sequent ${}_\vdash\Gamma_\perp U_\dashv\Delta$ is closed if $CL_{\vdash\dashv}\vee CL_{\perp\vdash}\vee CL_{\perp\dashv}\vee CL_{\sim\dashv}\vee CL_{E\perp}\vee CL_{\exists\perp}$ holds.

Theorem 8. If sequent ${}_\vdash\Gamma_\perp U_\dashv\Delta$ is closed, then $U/\Gamma \models^{\perp}_{IR} \Delta$.

Proof follows directly from Theorem 2.□

Sequent forms are obtained directly from properties of $\models^{\perp}_{IR}$ presented by Theorems 3–7. The labels of the sequent forms are obtained from the labels of the corresponding properties by putting formula signs in front of the labels. Forms may have additional constraints.

Introduction of undefinedness conditions may lead to new sequent forms with three premises (rule ${}_\perp\vee$).

The sequent forms for *propositional compositions* $\vee,\neg,\sim$ are induced by the properties $\vee_\vdash,\vee_\dashv,\vee_\perp,\neg_\vdash,\neg_\dashv,\neg_\perp,\sim_\perp,\sim_\vdash$:

$$_{\vdash}\vee\ \frac{_{\vdash}\Phi,\Sigma \quad _{\vdash}\Psi,\Sigma}{_{\vdash}\Phi\vee\Psi,\Sigma}\ ; \qquad _{\dashv}\vee\ \frac{_{\dashv}\Phi,{_{\dashv}\Psi},\Sigma}{_{\dashv}\Phi\vee\Psi,\Sigma}\ ;$$

$$_{\perp}\vee\ \frac{_{\perp}\Phi,{_{\perp}\Theta},\Sigma \quad _{\perp}\Phi,{_{\dashv}\Theta},\Sigma \quad _{\dashv}\Phi,{_{\perp}\Theta},\Sigma}{_{\perp}\Phi\vee\Theta,\Sigma}\ ;$$

$$_{\vdash}\neg\ \frac{_{\dashv}\Phi,\Sigma}{_{\vdash}\neg\Phi,\Sigma}\ ; \qquad _{\dashv}\neg\ \frac{_{\vdash}\Phi,\Sigma}{_{\dashv}\neg\Phi,\Sigma}\ ; \qquad _{\perp}\neg\ \frac{_{\perp}\Phi,\Sigma}{_{\perp}\neg\Phi,\Sigma}\ ;$$

$$_{\vdash}\sim\ \frac{_{\perp}\Phi,\Sigma}{_{\vdash}\sim\Phi,\Sigma}\ ; \qquad _{\perp}\sim\ \frac{_{\vdash}\Phi,\Sigma \quad _{\dashv}\Phi,\Sigma}{_{\perp}\sim\Phi,\Sigma}\ .$$

The sequent forms for *renomination composition* are induced by the properties presented in Theorems 4 and 5:

$$_{\vdash}R_{\vee}\ \frac{_{\vdash}R^{\bar{v}}_{\bar{x}}(\Phi)\vee R^{\bar{v}}_{\bar{x}}(\Psi),\Sigma}{_{\vdash}R^{\bar{v}}_{\bar{x}}(\Phi\vee\Psi),\Sigma}\ ; \qquad _{\dashv}R_{\vee}\ \frac{_{\dashv}R^{\bar{v}}_{\bar{x}}(\Phi)\vee R^{\bar{v}}_{\bar{x}}(\Psi),\Sigma}{_{\dashv}R^{\bar{v}}_{\bar{x}}(\Phi\vee\Psi),\Sigma}\ ;$$

$$_{\perp}R_{\vee}\ \frac{_{\perp}R^{\bar{v}}_{\bar{x}}(\Phi)\vee R^{\bar{v}}_{\bar{x}}(\Psi),\Sigma}{_{\perp}R^{\bar{v}}_{\bar{x}}(\Phi\vee\Psi),\Sigma}\ ;$$

$$_{\vdash}R_{\neg}\ \frac{_{\vdash}\neg R^{\bar{v}}_{\bar{x}}(\Phi),\Sigma}{_{\vdash}R^{\bar{v}}_{\bar{x}}(\neg\Phi),\Sigma}\ ; \qquad _{\dashv}R_{\neg}\ \frac{_{\dashv}\neg R^{\bar{v}}_{\bar{x}}(\Phi),\Sigma}{_{\dashv}R^{\bar{v}}_{\bar{x}}(\neg\Phi),\Sigma}\ ; \qquad _{\perp}R_{\neg}\ \frac{_{\perp}\neg R^{\bar{v}}_{\bar{x}}(\Phi),\Sigma}{_{\perp}R^{\bar{v}}_{\bar{x}}(\neg\Phi),\Sigma}\ ;$$

$$_{\vdash}RR\ \frac{_{\vdash}R^{\bar{v}}_{\bar{x}}\circ^{\bar{w}}_{\bar{y}}(\Phi),\Sigma}{_{\vdash}R^{\bar{v}}_{\bar{x}}(R^{\bar{w}}_{\bar{y}}(\Phi)),\Sigma}\ ; \quad _{\dashv}RR\ \frac{_{\dashv}R^{\bar{v}}_{\bar{x}}\circ^{\bar{w}}_{\bar{y}}(\Phi),\Sigma}{_{\dashv}R^{\bar{v}}_{\bar{x}}(R^{\bar{w}}_{\bar{y}}(\Phi)),\Sigma}\ ; \quad _{\perp}RR\ \frac{_{\perp}R^{\bar{v}}_{\bar{x}}\circ^{\bar{w}}_{\bar{y}}(\Phi),\Sigma}{_{\perp}R^{\bar{v}}_{\bar{x}}(R^{\bar{w}}_{\bar{y}}(\Phi)),\Sigma}\ ;$$

$$_{\vdash}R\exists\ \frac{_{\vdash}\exists zR^{\bar{v}}_{\bar{x}}(R^{y}_{z}(\Phi)),\Sigma}{_{\vdash}R^{\bar{v}}_{\bar{x}}(\exists y\Phi),\Sigma}\ ,\ z\in fu(R^{\bar{v}}_{\bar{x}}(\exists y\Phi));$$

$$_{\dashv}R\exists\ \frac{_{\dashv}\exists zR^{\bar{v}}_{\bar{x}}(R^{y}_{z}(\Phi)),\Sigma}{_{\dashv}R^{\bar{v}}_{\bar{x}}(\exists y\Phi),\Sigma}\ ,\ z\in fu(R^{\bar{v}}_{\bar{x}}(\exists y\Phi));$$

$$_{\perp}R\exists\ \frac{_{\perp}\exists zR^{\bar{v}}_{\bar{x}}(R^{y}_{z}(\Phi)),\Sigma}{_{\perp}R^{\bar{v}}_{\bar{x}}(\exists y\Phi),\Sigma}\ ,\ z\in fu(R^{\bar{v}}_{\bar{x}}(\exists y\Phi));$$

$$_{\vdash}R\ \frac{_{\vdash}\Phi,\Sigma}{_{\vdash}R(\Phi),\Sigma}\ ; \qquad _{\dashv}R\ \frac{_{\dashv}\Phi,\Sigma}{_{\dashv}R(\Phi),\Sigma}\ ; \qquad _{\perp}R\ \frac{_{\perp}\Phi,\Sigma}{_{\perp}R(\Phi),\Sigma}\ ;$$

$$_{\vdash}RI\ \frac{_{\vdash}R^{\bar{v}}_{\bar{x}}(\Phi),\Sigma}{_{\vdash}R^{z,\bar{v}}_{z,\bar{x}}(\Phi),\Sigma}\ ; \qquad _{\dashv}RI\ \frac{_{\dashv}R^{\bar{v}}_{\bar{x}}(\Phi),\Sigma}{_{\dashv}R^{z,\bar{v}}_{z,\bar{x}}(\Phi),\Sigma}\ ; \qquad _{\perp}RI\ \frac{_{\perp}R^{\bar{v}}_{\bar{x}}(\Phi),\Sigma}{_{\perp}R^{z,\bar{v}}_{z,\bar{x}}(\Phi),\Sigma}\ ;$$

$$ {}_{\vdash}RU\ \frac{{}_{\vdash}R_{\bar{x}}^{\bar{v}}(\Phi),\Sigma}{{}_{\vdash}R_{z,\bar{x}}^{y,\bar{v}}(\Phi),\Sigma}\ ,y{\in}fu(\Phi); \qquad {}_{\dashv}RU\ \frac{{}_{\dashv}R_{\bar{x}}^{\bar{v}}(\Phi),\Sigma}{{}_{\dashv}R_{z,\bar{x}}^{y,\bar{v}}(\Phi),\Sigma}\ ,y{\in}fu(\Phi); $$

$$ {}_{\perp}RU\ \frac{{}_{\perp}R_{\bar{x}}^{\bar{v}}(\Phi),\Sigma}{{}_{\perp}R_{y,\bar{x}}^{z,\bar{v}}(\Phi),\Sigma}\ ,y{\in}fu(\Phi); $$

$$ {}_{\vdash}R^{\sim}\ \frac{{}_{\vdash}\sim R_{\bar{x}}^{\bar{v}}(\Phi),\Sigma}{{}_{\vdash}R_{\bar{x}}^{\bar{v}}(\sim\Phi),\Sigma}\ ; \qquad {}_{\dashv}R^{\sim}\ \frac{{}_{\dashv}\sim R_{\bar{x}}^{\bar{v}}(\Phi),\Sigma}{{}_{\dashv}R_{\bar{x}}^{\bar{v}}(\sim\Phi),\Sigma}\ ; \qquad {}_{\perp}R^{\sim}\ \frac{{}_{\perp}\sim R_{\bar{x}}^{\bar{v}}(\Phi),\Sigma}{{}_{\perp}R_{\bar{x}}^{\bar{v}}(\sim\Phi),\Sigma}\ ; $$

$$ {}_{\vdash}RE\ \frac{{}_{\vdash}Ez,\Sigma}{{}_{\vdash}R_{\bar{x}}^{\bar{v}}(Ez),\Sigma}\ ,z{\notin}\bar{v}; \qquad {}_{\dashv}RE\ \frac{{}_{\dashv}Ez,\Sigma}{{}_{\dashv}R_{\bar{x}}^{\bar{v}}(Ez),\Sigma}\ ,z{\notin}\bar{v}; $$

$$ {}_{\vdash}REr\ \frac{{}_{\vdash}Ey,\Sigma}{{}_{\vdash}R_{\bar{x},y}^{\bar{v},z}(Ez),\Sigma}\ ; \qquad {}_{\dashv}REr\ \frac{{}_{\dashv}Ey,\Sigma}{{}_{\dashv}R_{\bar{x},y}^{\bar{v},z}(Ez),\Sigma}\ ; $$

$$ {}_{\vdash}RN\ \frac{{}_{\vdash}R_{t,\bar{x}}^{y,\bar{v}}(\Phi),{}_{\dashv}E(z),{}_{\dashv}E(t),\Sigma}{{}_{\vdash}R_{z,\bar{x}}^{y,\bar{v}}(\Phi),{}_{\dashv}E(z),{}_{\dashv}E(t),\Sigma}\ ; \quad {}_{\dashv}RN\ \frac{{}_{\dashv}R_{t,\bar{x}}^{y,\bar{v}}(\Phi),{}_{\dashv}E(z),{}_{\dashv}E(t),\Sigma}{{}_{\dashv}R_{z,\bar{x}}^{y,\bar{v}}(\Phi),{}_{\dashv}E(z),{}_{\dashv}E(t),\Sigma}\ ; $$

$$ {}_{\perp}RN\ \frac{{}_{\perp}R_{t,\bar{x}}^{y,\bar{v}}(\Phi),{}_{\dashv}E(z),{}_{\dashv}E(t),\Sigma}{{}_{\perp}R_{z,\bar{x}}^{y,\bar{v}}(\Phi),{}_{\dashv}E(z),{}_{\dashv}E(t),\Sigma}\ . $$

Sequent forms of *quantifier elimination* are induced by truth cases for quantifiers:

$$ {}_{\vdash}\exists\ \frac{{}_{\vdash}R_{y}^{x}(\Phi),{}_{\vdash}Ey,\Sigma}{{}_{\vdash}\exists x\Phi,\Sigma}\ ,y{\in}fu(\Sigma,\exists x\Phi); $$

$$ {}_{\dashv}\exists\ \frac{{}_{\dashv}\exists x\Phi,{}_{\vdash}Ey,{}_{\dashv}R_{y}^{x}(\Phi),\Sigma}{{}_{\dashv}\exists x\Phi,{}_{\vdash}Ey,\Sigma}\ ; $$

$$ {}_{\perp}\exists\ \frac{{}_{\perp}\exists x\Phi,{}_{\perp}R_{z}^{x}(\Phi),{}_{\vdash}Ez,\Sigma}{{}_{\perp}\exists x\Phi,\Sigma}\ ,z{\in}fu(\Sigma,\exists x\Phi). $$

Sequent form for *insertion of variable assignment predicates* permits to specify a variable as assigned or unassigned:

$$ E^{\vdash\dashv}\ \frac{{}_{\vdash}Ex,\Sigma \quad {}_{\dashv}Ex,\Sigma}{\Sigma}\ . $$

The above-written sequent forms and closeness conditions define calculus C^{QEC}.

For sequent rules of C^{QEC} we have the following main properties.

Theorem 9. Let $k \in \{1,2,3\}$ and $\frac{\vdash \Gamma_1 {}_\perp U_1 {}_\dashv \Delta_1 \quad ... \quad \vdash \Gamma_k {}_\perp U_k {}_\dashv \Delta_k}{\vdash \Gamma {}_\perp U {}_\dashv \Delta}$ be basic sequent form. Then

$$U/\Gamma \models^{\perp}_{IR} \Delta \Leftrightarrow U_1/\Gamma_1 \models^{\perp}_{IR} \Delta_1 \text{ and ... } U_k/\Gamma_k \models^{\perp}_{IR} \Delta_k.$$

Proof. For each form the proof follows directly from its corresponding property formulated in Theorems 2–7. □

The *derivation* in C^{QEC} has the form of a tree, the vertices of which are sequents. Such trees are called *sequent trees*. A sequent tree is *closed*, if every its leaf is a closed sequent. A sequent Σ is *derivable*, if there is a closed sequent tree with the root Σ .

During construction of a sequent tree the following cases are possible:

– construction procedure is completed: all sequents on the leaves are closed; we have a finite closed tree;

– construction procedure is not completed; we have a finite or infinite unclosed tree. Such tree has at least one path called *unclosed*, all vertices of which are unclosed sequents.

We meet the first case while proving soundness and the second one while proving completeness of C^{QEC}.

Theorem 10 (soundness). Let sequent $\vdash \Gamma {}_\perp U {}_\dashv \Delta$ be derivable in C^{QEC}. Then $U/\Gamma \models^{\perp}_{IR} \Delta$.

Proof. Indeed, if $\vdash \Gamma {}_\perp U {}_\dashv \Delta$ is derivable, then a finite closed tree was constructed. Therefore, for any leaf of this tree its sequent $\vdash \Lambda {}_\perp \Omega {}_\dashv K$ is closed. Thus, by Theorem 8, $\Omega/\Lambda \models^{\perp}_{IR} K$ holds. Therefore, by Theorem 9 for the root of the tree (sequent $\vdash \Gamma {}_\perp U {}_\dashv \Delta$) we have that $U/\Gamma \models^{\perp}_{IR} \Delta$ holds. □

4 Completeness of C^{QEC}

Completeness is proved on the basis of theorems of the existence of a counter-model for the set of formulas of a non-closed path in the sequent tree. To do this we first define the notion of Hintikka set for L^{QEC}, then we prove that a Hintikka set is satisfiable, and at last, we prove that formulas from a non-closed path form a Hintikka set [19].

A *Hintikka set for* L^{QEC} is a set H of signed formulas satisfying two types of conditions:

1) uncloseness conditions derived from closeness conditions for sequents;

2) decomposition conditions derived from decomposition sequent forms.

Uncloseness conditions for H are the following conditions obtained by negation of closeness conditions of sequents:

$\overset{H}{_{\vdash\dashv}}CL$) there is no formula Φ such that ${}_{\vdash}\Phi\in H$ and ${}_{\dashv}\Phi\in H$;
$\overset{H}{_{\perp\vdash}}CL$) there is no formula Φ such that ${}_{\perp}\Phi\in H$ and ${}_{\vdash}\Phi\in H$;
$\overset{H}{_{\perp\dashv}}CL$) there is no formula Φ such that ${}_{\perp}\Phi\in H$ and ${}_{\dashv}\Phi\in H$;
$\overset{H}{_{\sim\dashv}}CL$) there is no formula $\Phi : {}_{\dashv} \sim \Phi\in H$;
$\overset{H}{_{E\perp}}CL$) there is no formula Ey such that ${}_{\perp}Ey\in H$;
$\overset{H}{_{\exists\perp}}CL$) it is not possible that ${}_{\perp}\exists x\Phi\in H$
and ${}_{\vdash}R_y^x(\Phi)\in H$ for some $y\in V$ such that ${}_{\vdash}Ey\in H$.

Decomposition conditions for H are the following conditions:

$\overset{H}{_{\vdash}}\vee$) ${}_{\vdash}\Phi\vee\Psi\in H \Rightarrow {}_{\vdash}\Phi\in H$ or ${}_{\vdash}\Psi\in H$;
$\overset{H}{_{\dashv}}\vee$) ${}_{\dashv}\Phi\vee\Psi\in H \Rightarrow {}_{\dashv}\Phi\in H$ and ${}_{\dashv}\Psi\in H$;
$\overset{H}{_{\perp}}\vee$) ${}_{\perp}\Phi\vee\Psi\in H \Rightarrow {}_{\perp}\Phi\in H$ and ${}_{\perp}\Psi\in H$ or ${}_{\perp}\Phi\in H$ and ${}_{\dashv}\Psi\in H$ or
${}_{\dashv}\Phi\in H$ and ${}_{\perp}\Psi\in H$;
$\overset{H}{_{\vdash}}\neg$) ${}_{\vdash}\neg\Phi\in H \Rightarrow {}_{\dashv}\Phi\in H$;
$\overset{H}{_{\dashv}}\neg$) ${}_{\dashv}\neg\Phi\in H \Rightarrow {}_{\vdash}\Phi\in H$;
$\overset{H}{_{\perp}}\neg$) ${}_{\perp}\neg\Phi\in H \Rightarrow {}_{\perp}\Phi\in H$;
$\overset{H}{_{\vdash}}\sim$) ${}_{\vdash} \sim \Phi\in H \Rightarrow {}_{\perp}\Phi\in H$;
$\overset{H}{_{\perp}}\sim$) ${}_{\perp} \sim \Phi\in H \Rightarrow {}_{\vdash}\Phi\in H$ or ${}_{\dashv}\Phi\in H$.
$\overset{H}{_{\vdash}}RR$) ${}_{\vdash}R_{\bar{x}}^{\bar{v}}(R_{\bar{y}}^{\bar{w}}(\Phi))\in H \Rightarrow {}_{\vdash}R_{\bar{x}}^{\bar{v}} \circ_{\bar{y}}^{\bar{w}} (\Phi)\in H$;
$\overset{H}{_{\dashv}}RR$) ${}_{\dashv}R_{\bar{x}}^{\bar{v}}(R_{\bar{y}}^{\bar{w}}(\Phi))\in H \Rightarrow {}_{\dashv}R_{\bar{x}}^{\bar{v}} \circ_{\bar{y}}^{\bar{w}} (\Phi)\in H$;
$\overset{H}{_{\perp}}RR$) ${}_{\perp}R_{\bar{x}}^{\bar{v}}(R_{\bar{y}}^{\bar{w}}(\Phi))\in H \Rightarrow {}_{\perp}R_{\bar{x}}^{\bar{v}} \circ_{\bar{y}}^{\bar{w}} (\Phi)\in H$;
$\overset{H}{_{\vdash}}R\exists$) ${}_{\vdash}R_{\bar{x}}^{\bar{v}}(\exists y\Phi)\in H \Rightarrow {}_{\vdash}\exists zR_{\bar{x}}^{\bar{v}}(R_z^y(\Phi))\in H$ for some $z\in fu(R_{\bar{x}}^{\bar{v}}(\exists y\Phi))$;
$\overset{H}{_{\dashv}}R\exists$) ${}_{\dashv}R_{\bar{x}}^{\bar{v}}(\exists y\Phi)\in H \Rightarrow {}_{\dashv}\exists zR_{\bar{x}}^{\bar{v}}(R_z^y(\Phi))\in H$ for some $z\in fu(R_{\bar{x}}^{\bar{v}}(\exists y\Phi))$;
$\overset{H}{_{\perp}}R\exists$) ${}_{\perp}R_{\bar{x}}^{\bar{v}}(\exists y\Phi)\in H \Rightarrow {}_{\perp}\exists zR_{\bar{x}}^{\bar{v}}(R_z^y(\Phi))\in H$ for some $z\in fu(R_{\bar{x}}^{\bar{v}}(\exists y\Phi))$;
$\overset{H}{_{\vdash}}R$) ${}_{\vdash}R(\Phi)\in H \Rightarrow {}_{\vdash}\Phi\in H$;
$\overset{H}{_{\dashv}}R$) ${}_{\dashv}R(\Phi)\in H \Rightarrow {}_{\dashv}\Phi\in H$;
$\overset{H}{_{\perp}}R$) ${}_{\perp}R(\Phi))\in H \Rightarrow {}_{\perp}\Phi\in H$;

${}^{H}_{\vdash}RI)$ ${}_{\vdash}R^{z,\bar{v}}_{z,\bar{x}}(\Phi)\in H \Rightarrow {}_{\vdash}R^{\bar{v}}_{\bar{x}}(\Phi)\in H$;
${}^{H}_{\dashv}RI)$ ${}_{\dashv}R^{z,\bar{v}}_{z,\bar{x}}(\Phi)\in H \Rightarrow {}_{\dashv}R^{\bar{v}}_{\bar{x}}(\Phi)\in H$;
${}^{H}_{\perp}RI)$ ${}_{\perp}R^{z,\bar{v}}_{z,\bar{x}}(\Phi))\in H \Rightarrow {}_{\perp}R^{\bar{v}}_{\bar{x}}(\Phi)\in H$;
${}^{H}_{\vdash}RU)$ if $z\in fu(\Phi)$ then ${}_{\vdash}R^{z,\bar{v}}_{y,\bar{x}}(\Phi)\in H \Rightarrow {}_{\vdash}R^{\bar{v}}_{\bar{x}}(\Phi)\in H$;
${}^{H}_{\dashv}RU)$ if $z\in fu(\Phi)$ then ${}_{\dashv}R^{z,\bar{v}}_{y,\bar{x}}(\Phi)\in H \Rightarrow {}_{\dashv}R^{\bar{v}}_{\bar{x}}(\Phi)\in H$;
${}^{H}_{\perp}RU)$ if $z\in fu(\Phi)$ then ${}_{\perp}R^{z,\bar{v}}_{y,\bar{x}}(\Phi)\in H \Rightarrow {}_{\perp}R^{\bar{v}}_{\bar{x}}(\Phi)\in H$;
${}^{H}_{\vdash}R\sim)$ ${}_{\vdash}R^{\bar{x}}_{\bar{x}}(\sim\Phi)\in H \Rightarrow {}_{\vdash}\sim R^{\bar{x}}_{\bar{x}}(\Phi)\in H$;
${}^{H}_{\dashv}R\sim)$ ${}_{\dashv}R^{\bar{x}}_{\bar{x}}(\sim\Phi)\in H \Rightarrow {}_{\dashv}\sim R^{\bar{x}}_{\bar{x}}(\Phi)\in H$;
${}^{H}_{\perp}R\sim)$ ${}_{\perp}R^{\bar{x}}_{\bar{x}}(\sim\Phi)\in H \Rightarrow {}_{\perp}\sim R^{\bar{x}}_{\bar{x}}(\Phi)\in H$;
${}^{H}_{\vdash}RE)$ if $z\notin\bar{v}$ then ${}_{\vdash}R^{\bar{v}}_{\bar{x}}(Ez) \Rightarrow {}_{\vdash}Ez$;
${}^{H}_{\dashv}RE)$ if $z\notin\bar{v}$ then ${}_{\dashv}R^{\bar{v}}_{\bar{x}}(Ez) \Rightarrow {}_{\dashv}Ez$;
${}^{H}_{\vdash}REr)$ ${}_{\vdash}R^{\bar{v},z}_{\bar{x},y}(Ez) \Rightarrow {}_{\vdash}Ey$;
${}^{H}_{\dashv}REr)$ ${}_{\dashv}R^{\bar{v},z}_{\bar{x},y}(Ez) \Rightarrow {}_{\dashv}Ey$;
${}^{H}_{\vdash}RN)$ ${}_{\vdash}R^{y,\bar{v}}_{z,\bar{x}}(\Phi), {}_{\dashv}Ez, {}_{\dashv}Et\in H \Rightarrow {}_{\vdash}R^{y,\bar{v}}_{t,\bar{x}}(\Phi)\in H$;
${}^{H}_{\dashv}RN)$ ${}_{\dashv}R^{y,\bar{v}}_{z,\bar{x}}(\Phi), {}_{\dashv}Ez, {}_{\dashv}Et\in H \Rightarrow {}_{\dashv}R^{y,\bar{v}}_{t,\bar{x}}(\Phi)\in H$;
${}^{H}_{\perp}RN)$ ${}_{\perp}R^{y,\bar{v}}_{z,\bar{x}}(\Phi), {}_{\dashv}Ez, {}_{\dashv}Et\in H \Rightarrow {}_{\perp}R^{y,\bar{v}}_{t,\bar{x}}(\Phi)\in H$.

Let W be the set of all assigned variables in H. Now we can define the following conditions of *quantifier elimination*:
${}^{H}_{\vdash}\exists)$ ${}_{\vdash}\exists x\Phi\in H \Rightarrow$ exists $y\in W$ such that ${}_{\vdash}Ey\in H$ and ${}_{\vdash}R^{x}_{y}(\Phi)\in H$;
${}^{H}_{\dashv}\exists)$ ${}_{\dashv}\exists x\Phi\in H \Rightarrow {}_{\vdash}Ey\in H$ and ${}_{\dashv}R^{x}_{y}(\Phi)\in H$ for all $y\in W$;
${}^{H}_{\perp}\exists)$ ${}_{\perp}\exists x\Phi\in H \Rightarrow$ exists $z\in W$ such that ${}_{\vdash}Ez\in H$ and ${}_{\perp}R^{x}_{z}(\Phi)\in H$ and there is no $y\in W$ such that ${}_{\vdash}Ey\in H$ and ${}_{\vdash}R^{x}_{y}(\Phi)\in H$.

Sequent form for insertion of variable assignment predicates induces the following condition:
${}^{H}E^{\vdash\dashv})$ if $x\in nm(H)$, then ${}_{\vdash}Ex\in H$ or ${}_{\dashv}Ex\in H$.

A set H of signed formulas is called *satisfiable*, if there exists an interpretation $J=(\mathcal{A}^{QEC}(V,A), I^{Ps}_{Q})$ and $\delta\in{}^{V}\!A$ such that for any formula Φ:

${}_{\vdash}\Phi\in H \Rightarrow \Phi_J(\delta)\downarrow = T$;
${}_{\dashv}\Phi\in H \Rightarrow \Phi_J(\delta)\downarrow = F$;
${}_{\perp}\Phi\in H \Rightarrow \Phi_J(\delta)\uparrow$.

Theorem 11. Let H be a Hintikka set for L^{QEC}. Then H is satisfiable.

Proof. First we define a set A that gives us an algebra $\mathcal{A}^{QEC}(V,A)$; then we construct $\delta\in{}^{V}\!A$. At last, we specify an interpretation of predicate symbols I_Q^{Ps} that gives us an interpretation J.

Let $W = \{\ x \mid {}_{\vdash}Ex\in H\}$ be the set of all assigned variables in H. Let a set A be such that $|A| = |W|$, i.e. A is a copy of W. Elements of A are denoted a_w, where $w\in W$.

Nominative set $\delta\in{}^{V}\!A$ is constructed in the following way:

– if $x\in W$, then a value of x in δ is defined and equal to a_x;

– if $x\notin W$, then a value of x is not defined.

Let us admit that any variable x from $nm(H)\setminus W$ is unassigned in H, i.e. ${}_{\dashv}Ex\in H$ by ${}^{H}E^{\vdash\dashv}$.

Let us specify values of basic predicate $P\in Ps$ on δ and on the nominative sets of the form $r_{\bar{x}}^{\bar{v}}(\delta)$:

– ${}_{\vdash}P\in H \Rightarrow P_J(\delta){\downarrow} = T$;

– ${}_{\dashv}P\in H \Rightarrow P_J(\delta){\downarrow} = F$;

– ${}_{\perp}P\in H \Rightarrow P_J(\delta){\uparrow}$;

– ${}_{\vdash}R_{\bar{x}}^{\bar{v}}(P)\in H \Rightarrow P_J(r_{\bar{x}}^{\bar{v}}(\delta)){\downarrow} = T$;

– ${}_{\dashv}R_{\bar{x}}^{\bar{v}}(P)\in H \Rightarrow P_J(r_{\bar{x}}^{\bar{v}}(\delta)){\downarrow} = F$;

– ${}_{\perp}R_{\bar{x}}^{\bar{v}}(P)\in H \Rightarrow P_J(r_{\bar{x}}^{\bar{v}}(\delta)){\uparrow}$.

Values of P on other data can be chosen in arbitrary way with respect to unessential variables from V_U.

No ambiguity arises in these definitions due to uncloseness conditions for H.

Let us note that formulas of the forms $R_{z,\bar{x}}^{y,\bar{v}}\Phi$ and $R_{t,\bar{x}}^{y,\bar{v}}\Phi$ with ${}_{\dashv}Ez, {}_{\dashv}Et\in H$ cannot lead to ambiguity either, because RN-rules specify for them equal values.

For atomic formulas and formulas of the form $R_{\bar{x}}^{\bar{v}}(p)$ the statement of the theorem follows from the definitions of the basic predicates and variable assignment predicates. The proof of the theorem is then done by induction on the formula structure.

Let us consider the cases with the complement and quantification compositions only. Other cases are proved in a similar manner.

Let ${}_{\vdash}\sim\Phi\in H$. From $\overset{H}{\vdash}\!\sim$ we have ${}_{\perp}\Phi\in H$. In accordance with the induction hypothesis $\Phi_J(\delta){\uparrow}$, so, $\sim\Phi_J(\delta){\downarrow} = T$.

Let $_{\perp} \sim \Phi{\in}H$. From $\overset{H}{\perp}\sim$ we have $_{\vdash}\Phi{\in}H$ or $_{\dashv}\Phi{\in}H$. In accordance with the induction hypothesis we have $\Phi_J(\delta){\downarrow} = T$ or $\Phi_J(\delta){\downarrow} = F$, therefore $\Phi_J(\delta){\downarrow}$, and $\sim \Phi_J(d){\uparrow}$.

Let $_{\vdash}\exists x\Phi{\in}H$. By $\overset{H}{\vdash}\exists$ there exists $y{\in}W$ such that $_{\vdash}R^x_y(\Phi){\in}H$. In accordance with the induction hypothesis we have $R^x_y(\Phi)_J(\delta){\downarrow} = T$, whence $\Phi_J(\delta\nabla x{\mapsto}\delta(y)){\downarrow} = T$. So, for $a_y = \delta(y)$ we have $\Phi_J(\delta\nabla x{\mapsto}a_y){\downarrow} = T$, whence $\exists x\Phi_J(\delta){\downarrow} = T$.

Let $_{\dashv}\exists x\Phi{\in}H$. By $\overset{H}{\dashv}\exists$ for all $y{\in}W$ $_{\dashv}R^x_y(\Phi){\in}H$. In accordance with the induction hypothesis we have $R^x_y(\Phi)_J(\delta){\downarrow} = F$ for all $y{\in}W$, whence $\Phi_J(\delta\nabla x{\mapsto}\delta(y)){\downarrow} = F$ for all $y{\in}W$. So, $\Phi_J(\delta\nabla x{\mapsto}a_y){\downarrow} = F$ for all $a_y{\in}A$, whence $\exists x\Phi_J(\delta){\downarrow} = F$.

Let $_{\perp}\exists x\Phi{\in}H$. By $\overset{H}{\perp}\exists$ there exists $z{\in}W$ such that $_{\perp}R^x_z(\Phi){\in}H$ and there is no $y{\in}W$ such that $_{\vdash}Ey{\in}H$ and $_{\vdash}R^x_y(\Phi){\in}H$. In accordance with the induction hypothesis for δ we have $R^x_z(\Phi)_J(\delta){\uparrow}$ for some $z{\in}W$ and $R^x_y(\Phi)_J(\delta) \neq T$ for all $y{\in}W$, whence $\Phi_J(\delta\nabla x{\mapsto}\delta(z)){\uparrow}$ for some $z{\in}W$ and $\Phi_J(\delta\nabla x{\mapsto}\delta(y)){\neq}T$ for all $y{\in}W$. Therefore $\Phi_J(\delta\nabla x{\mapsto}a_y){\neq}T$ for all $a_y{\in}A$. Thus, $\exists x\Phi_J(\delta){\uparrow}$.□

Theorem 12. For C^{QEC} there exists a sequent tree construction procedure such that unclosed paths form Hintikka sets.

Proof. Such procedure for constructing a sequent tree in C^{QEC} is defined in the same way as for other sequent calculi for finite sequents [20], therefore we will not go into details. In our case of logic of partial quasiary predicates with the complement composition, this procedure is more complicated. The reason is that we should take into account 1) the undefinedness conditions; 2) the assigned and unassigned variables. These features manifest themselves in various sequent forms, especially in quantifier elimination forms.□

Theorem 13 (completeness). Let $U/\Gamma \models^{\perp}_{IR} \Delta$ holds. Then sequent $_{\vdash}\Gamma_{\perp}U_{\dashv}\Delta$ is derivable in C^{QEC}.

Proof. Let $U/\Gamma \models^{\perp}_{IR} \Delta$ and $_{\vdash}\Gamma_{\perp}U_{\dashv}\Delta$ be not derivable. Then a sequent tree for $_{\vdash}\Gamma_{\perp}U_{\dashv}\Delta$ is not closed. Thus, an unclosed path exists in this tree. Let H be the set of all formulas of this path. By Theorem 12, H is a Hintikka set. By Theorem 11, H is satisfiable. It means that there are an algebra $\mathcal{A}^{QEC}(V, A)$, nominative set $\delta{\in}{}^{V}\!A$, and an interpretation J such that for any formula Φ: ${}_{\vdash}\Phi \in H \Rightarrow \Phi_J(\delta)\downarrow = T$; ${}_{\dashv}\Phi \in H \Rightarrow \Phi_J(\delta)\downarrow = F$; ${}_{\perp}\Phi \in H \Rightarrow \Phi_J(\delta)\uparrow$. Since ${}_{\vdash}\Gamma {}_{\perp}U {}_{\dashv}\Delta \subseteq H$, this holds for formulas of the sequent ${}_{\vdash}\Gamma {}_{\perp}U {}_{\dashv}\Delta$. Thus, for all $\Phi \in \Gamma$ we have $\Phi_J(\delta)\downarrow = T$; for all $\Phi \in U$ we have $\Phi_J(\delta)\uparrow$, for all $\Phi \in \Delta$ we have $\Phi_J(\delta)\downarrow = F$. This contradicts $U/\Gamma_A \models^{\perp}_{IR} \Delta$.□

5 Conclusion

Extensive usage of logic in Computer Science leads to new logics that more adequately represent applied domains. Logic of partial quasiary predicates is one of such logics oriented on proving properties of programs. In our previous papers we studied logics of propositional and renominative (quantifier-free) levels. In this paper we have generalized the obtained results for the first-order logic of partial quasiary predicates extended with the complement composition. For this logic a special consequence relation called irrefutability consequence relation under undefinedness conditions has been introduced. We have studied its properties, constructed a sequent calculus for it and proved soundness and completeness of this calculus.

The obtained results can be useful for software verification; some steps were made in [21].

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