Computer Science Journal of Moldova, vol.**27**, no.1(79), 2019

Graphs with Large Hop Roman Domination Number

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Abstract

A subset S of vertices of a graph G is a hop dominating set if every vertex outside S is at distance two from a vertex of S. A Roman dominating function on a graph $G = (V, E)$ is a function $f : V(G) \longrightarrow \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. A hop Roman dominating function (HRDF) of G is a function $f : V(G) \longrightarrow \{0, 1, 2\}$ having the property that for every vertex $v \in V$ with $f(v) = 0$ there is a vertex u with $f(u) = 2$ and $d(u, v) = 2$. The weight of a HRDF f is the sum $f(V) = \sum_{v\in V} f(v)$. The minimum weight of a HRDF on G is called the hop Roman domination number of G and is denoted by $\gamma_{hR}(G)$. In this paper we characterize all graphs G of order n with $\gamma_{hR}(G) = n$ or $\gamma_{hR}(G) = n - 1$.

Keywords: Domination, Roman domination, Hop Roman domination.

MSC 2010: 05C69.

1 Introduction

For notation and graph theory terminology not given here, we refer to [7]. Let $G = (V, E)$ be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The *order* of G is $n(G) = |V(G)|$. The *open neighborhood* of a vertex v is $N_G(v) = \{u \in V(G) \,|\, uv \in E(G)\}$. The *degree* of v, denoted by $\deg(v)$, is $|N_G(v)|$. The *open neighborhood* of a subset $S \subseteq V$, is $N_G(S) = \bigcup_{v\in S} N_G(v)$, and the *closed neighborhood* of S is the set $N_G[S] = N_G(S) \cup S$. The *distance* between two vertices u

and v in G, denoted by $d(u, v)$, is the minimum length of a (u, v)-path in G. The diameter, $diam(G)$, of G is the maximum distance among all pairs of vertices in G. If S is a subset of vertices in a graph G, then we denote by $G[S]$ the subgraph of G induced by S. For an integer $k \geq 1$, the set of all vertices at distance k from v is denoted by $N_k(v)$. Also we denote $N_k[v] = N_k(v) \cup \{v\}$.

A subset of vertices of a graph G is a dominating set of G if every vertex in $V(G) - S$ has a neighbor in S. The domination number, $\gamma(G)$, is the minimum cardinality of a dominating set of G. Ayyaswamy and Natarajan [4] introduced the concept of hop domination in graphs. A subset S of vertices of a graph G is a *hop dominating set* (HDS) if every vertex outside S is at distance two from a vertex of S. The *hop domination number*, $\gamma_h(G)$, of G is the minimum cardinality of a hop dominating set of G. A HDS of G of minimum cardinality is referred as a $\gamma_h(G)$-set. Farhadi et al. [6] generalized hope dominating sets and studied k-hop dominating sets for every integer $k \geq 2$. The concept of hop domination was further studied, for example, in [3], [8], [9]. We define the *hop-degree* of a vertex v in a graph G, denoted $\deg_h(v)$, to be the number of vertices at distance 2 from v in G. The maximum hop-degree among the vertices of G is denoted by $\Delta_h(G)$.

A function $f : V(G) \longrightarrow \{0, 1, 2\}$ having the property that for every vertex $v \in V$ with $f(v) = 0$, there exists a vertex $u \in N(v)$ with $f(u) = 2$, is called a *Roman dominating function* or just an RDF. The *weight* of an RDF f is the sum $f(V) = \sum_{v \in V} f(v)$. The minimum weight of an RDF on G is called the *Roman domination number* of G and is denoted by $\gamma_R(G)$. The mathematical concept of Roman domination was defined and discussed by Stewart [13], and ReVelle and Rosing [11], and subsequently developed by Cockayne et al. [5].

Roman dominating functions with further properties were considered by several authors, see for example, [1], [2], [10]. For any RDF f on a graph G, it is clear that $\{v \in V(G) : f(v) \neq 0\}$ is a dominating set for G. It is an interesting question to study those RDF f such that the set $\{v \in V(G) : f(v) \neq 0\}$ is a hop dominating set for G. Shabani [12] considered Roman dominating functions with the above property and introduced the concept of hop Roman dominating functions. A *hop Roman dominating function* (HRDF) on a graph G is a function $f : V(G) \longrightarrow \{0, 1, 2\}$ having the property that for every vertex $v \in V$ with $f(v) = 0$ there is a vertex u with $f(u) = 2$ and $d(u, v) = 2$. The weight of a HRDF f is the sum $f(V) = \sum_{v \in V} f(v)$. The minimum weight of a HRDF on G is called the *Hop Roman domination number* of G and is denoted $\gamma_{hR}(G)$. A HRDF with minimum weight is referred as a $\gamma_{hR}(G)$-function. For a HRDF f in a graph G, we denote by V_i (or V_i^f to refer to f) the set of all vertices of G with label i under f. Thus a HRDF f can be represented by a triple (V_0, V_1, V_2), and we can use the notation $f = (V_0, V_1, V_2)$. Among other results, Shabani et al. [12] obtained the following.

Proposition 1 (Shabani [12]) *For any graph G of order n and with $\Delta_h(G) \geq 1$, $\gamma_{hR}(G) \geq \dfrac{2n}{\Delta_h(G) + 1}$.*

Proposition 2 (Shabani [12]) *For any graph G of order n and with $\Delta_h(G) \geq 1$, $\gamma_{hR}(G) \leq n - \Delta_h(G) + 1$.*

In this paper, we characterize graphs with large hop Roman domination number. In Section 2, we characterize all graphs G of order n with $\gamma_{hR}(G) = n$. In Section 3, we characterize all graphs G of order n with $\gamma_{hR}(G) = n - 1$.

2 Graphs G with $\gamma_{hR}(G) = n$

In this section we characterize all graphs G of order n with $\gamma_{hR}(G) = n$. For this purpose, we define a family of graphs as follows. Let $\mathcal{G}$ be the family of graphs G such that G can be obtained from a sequence $G_0, G_1, ..., G_k$, $(k \geq 1)$, of graphs, where G_0 is a complete graph or $G_0 = \overline{K_2}$, $G = G_k$ and, if $k \geq 1$, then G_{i+1} can be obtained recursively from G_i by the following operation, for $i = 0, 1, ..., k - 1$.

Operation $\mathcal{O}$. Add two new vertices and join each new vertex to every vertex of G_i.

Theorem 3 *Let G be a connected graph of order n. Then $\gamma_{hR}(G) = n$ if and only if $G \in \mathcal{G} \cup \{P_4, K_n\}$.*

Proof. ($\Rightarrow$) The proof is by an induction on the order n of a graph G with $\gamma_{hR}(G) = n$. If $n \leq 2$ then clearly G is a complete graph. Thus assume that $n \geq 3$. Assume that the result is true for all graphs of order less than n and let G be a graph of order n with $\gamma_{hR}(G) = n$. From Proposition 2, we obtain that $\Delta_h(G) \leq 1$. Thus $diam(G) \leq 3$. If $diam(G) = 1$, then G is a complete graph, as desired.

Assume that $diam(G) = 2$. Let $v_1v_2v_3$ be a diametrical path in G. Assume that there exists a vertex $v \in N(v_1)$ such that $v \notin N(v_3)$. Since $diam(G) = 2$, we have $v \in N_2(v_3)$. Then $\deg_h(v_3) \geq 2$, a contradiction. Thus $N(v_1) \subseteq N(v_3)$, and similarly $N(v_3) \subseteq N(v_1)$. Consequently, $N(v_1) = N(v_3)$. If there are three vertices $a, b, c \in N(v_1)$ such that $a \notin N(b) \cup N(c)$, then $\deg_h(a) \geq 2$, a contradiction. Thus any vertex of $N(v_1)$ is adjacent to at least $|N(v_1)| - 2$ vertices of $N(v_1)$, and so $\delta(G[N(v_1)]) \geq |N(v_1)| - 2$. If $\delta(G[N(v_1)]) = |N(v_1)| - 1$, then $G[N(v_1)]$ is a complete graph. Thus G is obtained from $G[N(v_1)]$ by using Operation $\mathcal{O}$, and so $G \in \mathcal{G}$. Therefore assume that $\delta(G[N(v_1)]) = |N(v_1)| - 2$. Let $G' = G[N(v_1)]$. If $|V(G')| = 2$, then $G' = \overline{K_2}$, and so G is obtained from $\overline{K_2}$ by using Operation $\mathcal{O}$ and so $G \in \mathcal{G}$. Now, let $|V(G')| \geq 3$. Suppose that $\gamma_{hR}(G') \leq |V(G')| - 1$. Let f be a $\gamma_{hR}(G')$-function. Then $g = (V_0^f, V_1^f \cup \{v_1, v_3\}, V_2^f)$ is a HRDF of G with $g(V) \leq |V(G')| + 1$. Therefore, $\gamma_{hR}(G) \leq n - 1$, a contradiction. Therefore, $\gamma_{hR}(G') = |V(G')|$. By the inductive hypothesis, $G' \in \mathcal{G}$ or $G' \in \{P_4, K_n\}$. Since $\delta(G') = |V(G')| - 2$ we have $G' \in \mathcal{G}$. Therefore, G is obtained from $G' \in \mathcal{G}$ by using Operation $\mathcal{O}$ and so $G \in \mathcal{G}$.

It remains to assume that $diam(G) = 3$. Let $P : v_1v_2v_3v_4$ be a diametrical path in G. Suppose that $G \neq P_4$. Let $v \in V(G) - V(P)$. Clearly, v is not adjacent to both v_1 and v_4. Without loss of generality, assume that $v_1 \notin N(v)$. Since $diam(G) = 3$ and G is connected, we have $N(v) \cap \{v_2, v_3\} \neq \emptyset$. If $v_2 \in N(v)$, then $\deg_h(v_1) \geq 2$, a contradiction. Thus, $v_3 \in N(v)$, and so $\deg_h(v_2) \geq 2$, a contradiction. Thus $V(G) - V(P) = \emptyset$. Consequently, $G = P_4$.

($\Leftarrow$) Let $G \in \mathcal{G} \cup \{P_4, K_n\}$. We show that $\gamma_{hR}(G) = n$. If $G = K_n$ or $G = P_4$, then we can easily see that $\gamma_{hR}(G) = n$. Now suppose that $G \in \mathcal{G}$. There is a sequence of graphs $G_0, G_1, ...G_k$, $(k \geq 1)$, where G_0 is a complete graph or $G_0 = \overline{K_2}$, $G = G_k$ and, if $k \geq 1$, then G_{i+1} can be obtained recursively from G_i by using Operation $\mathcal{O}$ for $i = 0, ..., k-1$. We use an induction on k to show that $\delta(G) = |V(G)| - 2 = n - 2$. If $k = 1$, then $G = C_4$ or G is obtained from the complete graph K_{n-2} and we can easily see that $\delta(G) = n-2$. This establishes the basic step. Suppose now that $k \geq 2$ and the result is true for all graphs $G \in \mathcal{G}$ that can be constructed from a sequence of length at most $k-1$, and let $G^{'} = G_{k-1}$. By the induction hypothesis, $\delta(G^{'}) = |V(G^{'})| - 2$. By construction, G is obtained from $G^{'}$ by using Operation $\mathcal{O}$. Let x and y be two vertices joined to any vertex of $G^{'}$ according to the Operation $\mathcal{O}$. Then, clearly $d(x, y) = 2$ and $\deg(x) = \deg(y) = |V(G^{'})| = n - 2$. Now let $x^{'} \in V(G^{'})$ be a vertex with $\deg_{G^{'}}(x^{'}) = \delta(G^{'}) = |V(G^{'})| - 2$. Then $\deg_G(x^{'}) = \delta(G^{'}) + 2 = |V(G^{'})| = n - 2$. We conclude that $\delta(G) = n - 2$.

Now suppose, to the contrary, that $\gamma_{hR}(G) \leq n - 1$. From Proposition 2 we have $\Delta_h(G) \leq 2$. If $\Delta_h(G) = 2$, then there exists a vertex $w \in V(G)$ such that $\deg_h(w) = 2$ and so $\deg(w) \leq n - 3$, a contradiction with $\delta(G) = n - 2$. Thus $\Delta_h(G) = 1$. By Propositions 1 and 2 we have $\gamma_{hR}(G) = n$, a contradiction. We conclude $\gamma_{hR}(G) = n$. ■

3 Graphs G with $\gamma_{hR}(G) = n - 1$

In this section we characterize connected graphs G of order n with $\gamma_{hR}(G) = n - 1$. For this purpose, we define a family of graphs as follows. Let $\mathcal{G}$ be the family of graphs described in Section 2. Let $\mathcal{G}^*$ be the family of graphs G that can be obtained from a graph G', where G' is a complete graph or $G' \in \mathcal{G}$, by one of the following operations:

Operation $\mathcal{O}_1$: Add three vertices and join each of them to every vertex of G'.

Operation $\mathcal{O}_2$: Add a path $P_2 : v_1v_2$ and a vertex v, join v to every vertex of G' and join any vertex of $V(P_2)$ to at least $|V(G')| - 1$ vertices of G', such that $|V(G^{'})| \geq 2$ and if $G' \in \mathcal{G}$ and x and y are two vertices of G' with $d(x, y) = 2$, then the following conditions hold.

(i) $\{x, y\} \subseteq N(v_1) \cup N(v_2)$.

(ii) If $x \notin N(v_1) \cap N(v_2)$ or $y \notin N(v_1) \cap N(v_2)$, then v_1 and v_2 are adjacent to any vertex of $N_{G'}(x)$.

Lemma 4 *If $G \in \mathcal{G}^*$, then $\gamma_{hR}(G) = |V(G)| - 1$.*

Proof. Let $G \in \mathcal{G}^*$. We have the following cases.

Case 1. G is obtained from $G^{'}$ by using Operation $\mathcal{O}_1$. Let G be obtained from $G^{'}$ by joining three vertices x, y, z to any vertex of $G^{'}$. Clearly the distance between any pair of vertices of $\{x, y, z\}$ is two. Let $f^{'}$ be a $\gamma_{hR}(G^{'})-$function. By Theorem 3, $f^{'}(V) = |V(G^{'})|$. Then g defined by $g(x) = 2$, $g(y) = g(z) = 0$, and $g(u) = f^{'}(u)$ otherwise, is a HRDF for G of weight $f^{'}(V) + 2$. Thus $\gamma_{hR}(G) \leq f^{'}(V) + 2 = |V(G^{'})| + 2 = |V(G)| - 1$. On the other hand, let f be a $\gamma_{hR}(G)$-function. Then $f(x)+f(y)+f(z) \geq 2$ (otherwise, at least two vertices of $\{x, y, z\}$ are not hop Roman dominated by f, a contradiction). Then clearly $f|_{V(G')}$ is a HRDF for $G^{'}$. Then $\sum_{u \in V(G')} f(u) \geq \gamma_{hR}(G') = |V(G^{'})|$. Then $\gamma_{hR}(G) = w(f) \geq |V(G^{'})| + 2 = |V(G)| - 1$. Therefore, $\gamma_{hR}(G) = |V(G)| - 1$.

Case 2. G is obtained from $G^{'}$ by using Operation $\mathcal{O}_2$. Let v be the added vertex and v_1v_2 be the added P_2-path according to the Operation $\mathcal{O}_2$. Clearly the distance between any vertex of $V(P_2)$ and the vertex v is equal to two. Let $f^{'}$ be a $\gamma_{hR}(G^{'})-$function. By Theorem 3, $f^{'}(V) = |V(G^{'})|$. Then g defined by $g(v) = 2$, $g(v_1) = g(v_2) = 0$ and $g(z) = f^{'}(z)$ otherwise, is a HRDF for G with weight $f^{'}(V) + 2$. Thus $\gamma_{hR}(G) \leq f^{'}(V) + 2 = |V(G^{'})| + 2 = |V(G)| - 1$.

Claim 1. There is a $\gamma_{hR}(G)-$function, say as f, such that $f(v) + f(v_1) + f(v_2) \geq 2$ and also $\sum_{u \in V(G')} f(u) \geq |V(G^{'})|$.

Proof of Claim 1: Suppose that at least one vertex of $V(P_2)$ is adjacent to any vertex of $G^{'}$. Let g be a $\gamma_{hR}(G)-$function. Then $g(v) + g(v_1) + g(v_2) \geq 2$ (otherwise, at least one vertex of $\{v, v_1, v_2\}$ is not hop Roman dominated by g, a contradiction). If two vertices v_1 and v_2 are adjacent to any vertex of $G^{'}$, then clearly $g|_{V(G')}$ is a HRDF for $G^{'}$. Then $\sum_{u \in V(G')} g(u) \geq \gamma_{hR}(G') = |V(G^{'})|$. Thus, without loss of generality suppose that v_1 is adjacent to any vertex of $G^{'}$ and v_2 is adjacent to $|V(G^{'})| - 1$ vertices of $G^{'}$. Let a be a vertex of $G^{'}$ such that $a \notin N(v_2)$. Clearly $a \in N_2(v_2)$. If $g(a) \neq 0$, then clearly $g|_{V(G')}$ is a HRDF for $G^{'}$. Thus $\sum_{u \in V(G')} g(u) \geq \gamma_{hR}(G') = |V(G^{'})|$. Now suppose that $g(a) = 0$. Note that in this case either a is hop Roman dominate by a vertex of $V(G^{'}) - \{a\}$ or a is hop Roman dominate by v_2. If a is hop Roman dominate by a vertex of $V(G^{'}) - \{a\}$, then clearly $g|_{V(G')}$ is a HRDF for $G^{'}$. Thus $\sum_{u\in V(G')} g(u) \geq \gamma_{hR}(G') = |V(G^{'})|$. Next suppose that a is hop Roman dominate by v_2. Note that in this case $g(v_2) = 2$, $g(v) = 0$, $g(v_1) = 1$. Then we can change $g(v)$ to 2, $g(a)$ to 1, $g(v_1)$ and $g(v_2)$ to 0 to obtain a $\gamma_{hR}(G)-$function, $g^{'}$, such that $g^{'}(v) + g^{'}(v_1) + g^{'}(v_2) \geq 2$ and $g^{'}|_{V(G')}$ be a HRDF for $G^{'}$. Then $\sum_{u\in V(G')} g^{'}(u) \geq \gamma_{hR}(G') = |V(G^{'})|$.

Next suppose that any vertex of $V(P_2)$ is adjacent to exactly $|V(G^{'})| - 1$ vertices of $G^{'}$. First suppose that a be a vertex of $G^{'}$ such that $a \notin N(v_1) \cup N(v_2)$. Clearly $d(a, v_1) = d(a, v_2) = 2$. Then from Operation $\mathcal{O}_2$, we conclude that a is adjacent to any vertex of $V(G^{'}) - \{a\}$. Let g be a $\gamma_{hR}(G)-$function. If $g(v) + g(v_1) + g(v_2) \leq 1$, then clearly $g(v) = 1$ and two vertices v_1 and v_2 are hop Roman dominated by a. Thus, $g(a) = 2$. Then we can change $g(a)$ to 1 and $g(v)$ to 2 to obtain a $\gamma_{hR}(G)-$function, $g^{'}$, with $g^{'}(v) + g^{'}(v_1) + g^{'}(v_2) \geq 2$ such that $g^{'}|_{V(G')}$ be a HRDF for $G^{'}$. Then $\sum_{u\in V(G')} g^{'}(u) \geq \gamma_{hR}(G') = |V(G^{'})|$.

Now suppose that a, b be two vertices of $V(G^{'})$ such that $a \neq b$ and $a \notin N(v_1)$, $b \notin N(v_2)$. Clearly, $d(a, v_1) = d(b, v_2) = 2$. Note that in this case, from Operation $\mathcal{O}_2$, we conclude that a and b are adjacent to any vertex of $V(G^{'}) - \{a, b\}$ and note that in this case $1 \leq d(a, b) \leq 2$. Let g be a $\gamma_{hR}(G)-$function. If $g(v) + g(v_1) + g(v_2) \leq 1$, then clearly $g(v) = 1$ and $g(a) = g(b) = 2$. Then we can change $g(a)$ and $g(b)$ to 1 and $g(v)$ to 2, to obtain a HRDF for G with weight less than g, a contradiction. Thus $g(v) + g(v_1) + g(v_2) \geq 2$. Now suppose that $\sum_{u\in V(G')} g(u) < |V(G^{'})|$. Then, there is $u \in V(G^{'})$, such that $g(u) = 0$ and u is hop Roman dominated just by one vertex of $\{v_1, v_2\}$. Note that in this case $u = a$ or $u = b$. If $g(a) = g(b) = 0$, then $g(v_1) = g(v_2) = 2$ and clearly $g(v) = 0$. Then we can change $g(v)$ to 2, $g(v_1)$ and $g(v_2)$ to 0, and also $g(a)$ and $g(b)$ to 1, to obtain a $\gamma_{hR}(G)-$function, $g^{'}$, such that $g^{'}(v) + g^{'}(v_1) + g^{'}(v_2) \geq 2$ and also $g^{'}|_{V(G')}$ be a HRDF for $G^{'}$. Then $\sum_{u\in V(G')} g^{'}(u) \geq \gamma_{hR}(G') = |V(G^{'})|$. Now suppose that $g(a) = 0$ and $g(b) \neq 0$. Then, clearly $g(v_1) = 2$ and $g(v) = 0$. Suppose that $g(v_2) = 0$, then $g(b) = 2$. Thus we can change $g(v)$ to 2, $g(v_1)$ and $g(v_2)$ to 0, and also $g(a)$ and $g(b)$ to 1, to obtain a $\gamma_{hR}(G)-$function, $g^{'}$, such that $g^{'}(v) + g^{'}(v_1) + g^{'}(v_2) \geq 2$ and also $g^{'}|_{V(G^{'})}$ be a HRDF for $G^{'}$. Then $\sum_{u \in V(G^{'})} g^{'}(u) \geq \gamma_{hR}(G') = |V(G^{'})|$. Next suppose that $g(v_2) \neq 0$. Note that in this case $g(v_2) = 1$. Then we can change $g(v)$ to 2, $g(v_1)$ and $g(v_2)$ to 0, and also $g(a)$ to 1, to obtain a $\gamma_{hR}(G)-$function, $g^{'}$, such that $g^{'}(v) + g^{'}(v_1) + g^{'}(v_2) \geq 2$ and also $g^{'}|_{V(G^{'})}$ be a HRDF for $G^{'}$. Then $\sum_{u \in V(G^{'})} g^{'}(u) \geq \gamma_{hR}(G') = |V(G^{'})|$. ♦

Now from Claim 1 we conclude that $|V(G)| - 1 = 2 + |V(G^{'})| \leq \gamma_{hR}(G)$. Therefore, $\gamma_{hR}(G) = |V(G)| - 1$. ■

Let $\mathcal{F}$ be the family of graphs illustrated in Figure 1. We are now ready to state the main result of this section.

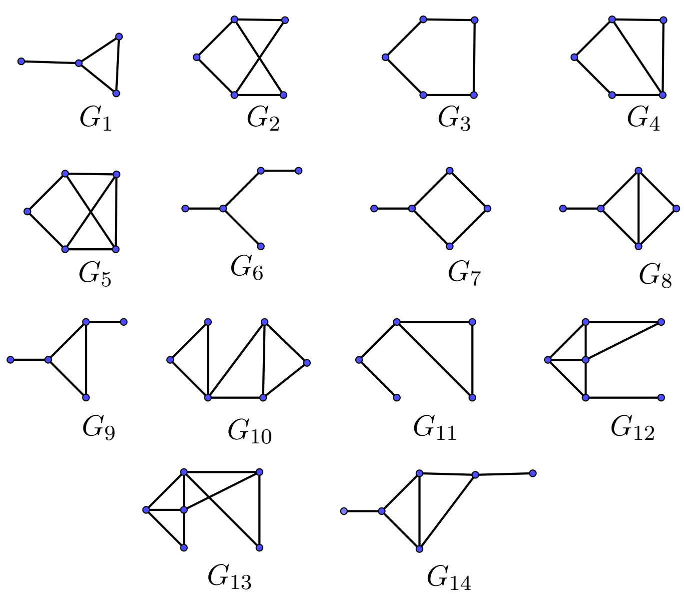

Figure 1. All graphs in the family $\mathcal{F}$

Theorem 5 *If G is a connected graph of order n, then $\gamma_{hR}(G) = n-1$ if and only if $G \in \mathcal{G}^* \cup \mathcal{F} \cup \{P_5\}$.*

Proof. ($\Rightarrow$) Let G be a connected graph of order n with $\gamma_{hR}(G) = n-1$. If $\Delta_h(G) \geq 3$, then Proposition 2 leads to $\gamma_{hR}(G) \leq n-2$, a contradiction. Thus $\Delta_h(G) \leq 2$. If $\Delta_h(G) = 1$, then Propositions 1 and 2 imply that $\gamma_{hR}(G) = n$, a contradiction. Thus $\Delta_h(G) = 2$. Let v be a vertex of G with $\deg_h(v) = \Delta_h(G) = 2$ and $N_2(v) = \{v_1, v_2\}$ and let $G' = G[N(v)]$. We proceed with two claims namely Claim 1 and Claim 2.

Claim 1. If $\deg_h(z) = \deg_h(w) = 2$ for two distinct vertices z and w of G, then $N_2(z) \cap N_2(w) \neq \emptyset$.

Proof of Claim 1: Assume that z, w be two distinct vertices of G such that $\deg_h(z) = \deg_h(w) = 2$. Suppose to the contrary that $N_2(z) \cap N_2(w) = \emptyset$. Then f defined by $f(z) = f(w) = 2$, $f(x) = 0$ for $x \in N_2(z) \cup N_2(w)$, and $f(u) = 1$ otherwise, is a HRDF for G with weight $n-2$. Thus $\gamma_{hR}(G) \leq n-2$, a contradiction. ♦

Claim 2. $\delta(G') \geq |V(G')| - 2$.

Proof of Claim 2: If there are three vertices $a, b, c \in V(G')$ such that $a \notin N(b) \cup N(c)$, then $\deg_h(a) \geq 2$. Thus $\deg_h(a) = 2$ and $N_2(a) = \{b, c\}$. Since $\deg_h(a) = \deg_h(v) = 2$ and $N_2(a) \cap N_2(v) = \emptyset$, then Claim 1 leads to a contradiction. Therefore, any vertex of G' is adjacent to at least $|V(G')| - 2$ vertices of G'. Therefore, $\delta(G') \geq |V(G')| - 2$. ♦

Note that $diam(G) \geq 2$, since $\gamma_{hR}(G) = n-1$. Suppose that $diam(G) \geq 5$. Let $P : u_1u_2u_3u_4u_5u_6...u_d$ be diametrical path in G. Then $u_1, u_5 \in N_2(u_3)$ and $u_2, u_6 \in N_2(u_4)$. Since $\Delta_h(G) = 2$, thus $\deg_h(u_3) = \deg_h(u_4) = 2$. This contradicts Claim 1. Thus $diam(G) \leq 4$. Therefore, we conclude that $2 \leq diam(G) \leq 4$. We consider the following cases.

Case 1. $diam(G) = 2$. Clearly $V(G) - \big(N_2(v) \cup N[v]\big) = \emptyset$. According to the Claim 2, we consider two following subcases.

Subcase 1.1. $\delta(G') = |V(G')| - 1$. Then G' is a complete graph. Assume that $v_1v_2 \notin E(G)$. Note that $d(v_1, v_2) = 2$. If $|V(G^{'})| = 1$, then any vertex of $N_2(v)$ is adjacent to the vertex of $G^{'}$, and so G is obtained from $G^{'}$ by Operation $\mathcal{O}_1$. Thus assume that $|V(G^{'})| \geq 2$. If at least a vertex of $N_2(v)$ is adjacent to at most $|V(G')|-1$ vertices of $G^{'}$, then $\Delta_h(G)=2$ leads to a contradiction. Thus both vertices v_1 and v_2 are adjacent to any vertex of $V(G')$. Hence G is obtained from $G^{'}$ by Operation $\mathcal{O}_1$. Next assume that $v_1v_2 \in E(G)$. If $|V(G')|=1$, then v_1 and v_2 are adjacent to the vertex of $G^{'}$. So $G=G_1 \in \mathcal{F}$. Thus assume that $|V(G')| \geq 2$. Then we can see that there exists at most one vertex in $N(v)$, such as x, such that $d(v_1,x)=2$. Also there exists at most one vertex in $N(v)$, such as y, such that $d(v_2,y)=2$. Thus v_1 and v_2 are adjacent to at least $|V(G')|-1$ vertices of G'. Therefore, G is obtained from G' by Operation $\mathcal{O}_2$. Therefore, $G \in \mathcal{G}^*$.

Subcase 1.2. $\delta(G')=|V(G')|-2$. Clearly $|V(G')| \geq 2$. Suppose that $|V(G')| \geq 3$. First we show that $G' \in \mathcal{G}$. Suppose to the contrary, that $\gamma_{hR}(G') \leq |V(G')|-1$. Let f be a $\gamma_{hR}(G')$-function. Then we can define g with $g(v)=2$, $g(v_1)=g(v_2)=0$ and $g(z)=f(z)$ otherwise, to obtain a HRDF for graph G with weight $g(V)=f(V)+2$. Thus $\gamma_{hR}(G) \leq f(V)+2 \leq |V(G')|-1+2=|V(G')|+1$. Since $|V(G)|=|V(G')|+3$, therefore we have $\gamma_{hR}(G) \leq |V(G)|-2=n-2$, a contradiction. Therefore, $\gamma_{hR}(G')=|V(G')|$ and by Theorem 3, $G' \in \mathcal{G} \cup \{P_4, K_{n-3}\}$. Since $\delta(G')=|V(G')|-2$, thus $G' \neq P_4$ and $G' \neq K_{n-3}$. Therefore, $G' \in \mathcal{G}$. Assume that $v_1v_2 \notin E(G)$. Then there exists at most one vertex in $N(v) \cup \{v_2\}$ at distance two from v_1 (since otherwise $\deg_h(v_1)>2$, a contradiction). Since $diam(G)=2$, we have $d(v_1,v_2)=2$. Then we conclude that v_1 is adjacent to any vertex of $G^{'}$. Similarly v_2 is adjacent to any vertex of $G^{'}$. Therefore, G is obtained from $G^{'}$ by using Operation $\mathcal{O}_1$. Therefore, $G \in \mathcal{G}^*$. Now suppose that $v_1v_2 \in E(G)$. Note that in this case, if there is $z \in V(G^{'})$, such that $z \notin N(v_1)$, then $d(z,v_1)=2$. Similarly if $z \notin N(v_2)$, then $d(z,v_2)=2$. Thus v_1 and v_2 are adjacent to at least $|V(G^{'})|-1$ vertices of $G^{'}$. Let $u \in V(G')$ be a vertex with $\deg_{G'}(u)=\delta(G')=|V(G')|-2$ and let w be the vertex of G' such that $w \notin N(u)$. Clearly $d_{G'}(u,w)=2$. Then $u,w \in N(v_1) \cup N(v_2)$ (since in the otherwise, if $u \notin N(v_1) \cup N(v_2)$ or $w \notin N(v_1) \cup N(v_2)$, then $\deg_h(u)>2$ or $\deg_h(w)>2$, a contradiction). Now suppose that $u \notin N(v_1) \cap N(v_2)$. Then since $u \in N(v_1) \cup N(v_2)$, so either $u \in N(v_1)$ or $u \in N(v_2)$. Without loss of generality, assume that $u \in N(v_1)$ and $u \notin N(v_2)$. Note that in this case v_1 is adjacent to any vertex of $N_{G'}(u)$ (since otherwise, $\deg_h(v_1) = \deg_h(u) = 2$, $N_2(v_1) \cap N_2(u) = \emptyset$, and Claim 1 leads to contradiction). Also v_2 is adjacent to any vertex of $N_{G'}(u)$, since otherwise $\deg_h(v_2) > 2$, a contradiction. Therefore, G is obtained from $G^{'} \in \mathcal{G}$ by operation $\mathcal{O}_2$. Therefore, $G \in \mathcal{G}^*$.

Now suppose that $|V(G')| = 2$. Then $G^{'} = \overline{K_2}$. If $v_1v_2 \notin E(G)$, then v_1 and v_2 are adjacent to any vertex of G', since otherwise there exist two vertices at distance three or four from each other, a contradiction with $diam(G) = 2$. Thus $G = G_2 \in \mathcal{F}$. Thus assume that $v_1v_2 \in E(G)$. Note that in this case any vertex of $V(G')$ is adjacent to at least one vertex of $N_2(v)$, since otherwise there exist two vertices in G at distance three from each other, a contradiction with $diam(G) = 2$. On the other hand any vertex of $N_2(v)$ is adjacent to at least one vertex of $V(G^{'})$. We conclude that $G \in \{G_3, G_4, G_5\}$. Consequently, $G \in \mathcal{F}$.

Case 2. $diam(G) = 3$. According to the Claim 2, we consider two following subcases.

Subcase 2.1. $\delta(G') = |V(G')| - 1$. Then G' is a complete graph. Assume that $|V(G')| \geq 3$. We can easily see that v_1 and v_2 are adjacent to at least $|V(G')| - 1$ vertices of G' and $N(v_1) \cap N(v_2) \neq \emptyset$. Let $v_1v_2 \notin E(G)$, then $d(v_1, v_2) = 2$. If there is $z \in N(v)$ such that $z \notin N(v_1)$, then $z \in N_2(v_1)$. Thus $\deg_h(v_1) \geq 3$, a contradiction. Therefore, v_1 is adjacent to any vertex of $N(v)$. Similarly we can show that v_2 is adjacent to any vertex of $N(v)$. In this case, distance between any two vertices of $N[v] \cup N_2(v)$ is at most two. Since $diam(G) = 3$, thus we may suppose that $V(G) - (N[v] \cup N_2(v)) \neq \emptyset$. Note that in this case there is a vertex z in $V(G) - (N[v] \cup N_2(v))$ such that $z \in N(v_1) \cup N(v_2)$. Then clearly $\deg_h(z) \geq 3$, a contradiction. Next suppose that $v_1v_2 \in E(G)$. Note that in this case v_1 and v_2 are adjacent to at least $|V(G')| - 1$ vertices of G'. Thus distance between any two vertices of $N[v] \cup N_2(v)$ is at most two. Since $diam(G) = 3$, thus we may suppose that $V(G) - (N[v] \cup N_2(v)) \neq \emptyset$. Note that in this case there is a vertex z in $V(G) - (N[v] \cup N_2(v))$ such that $z \in N(v_1) \cup N(v_2)$. Then clearly by Claim 1 or this fact that $\Delta_h(G) = 2$, we have a contradiction. Hence, we conclude that $|V(G')| \leq 2$.

Suppose that $|V(G')| = 1$. Let $V(G') = \{w\}$. Then v_1 and v_2

are adjacent to w and distance between any two vertices of $N[v] \cup N_2(v)$ is at most two. Since $diam(G) = 3$, thus we may suppose that $V(G) - (N[v] \cup N_2(v)) \neq \emptyset$. Let z be a vertex of $V(G) - (N[v] \cup N_2(v))$ such that $z \in N(v_1) \cup N(v_2)$. Then suppose that there is a vertex $z^{'} \in V(G) - (N[v] \cup N_2(v))$ such that $z^{'} \neq z$. If $z^{'} \in N(v_1) \cup N(v_2)$, then $z, z^{'} \in N_2(w)$ and by this fact that $\Delta_h(G) = 2$ we have $\deg_h(w) = 2$. Then Claim 1 leads to a contradiction. Hence, $z^{'} \in N(z) - \{v_1, v_2\}$. Then clearly $d(v, z^{'}) = 4$, a contradiction. Thus $|V(G) - (N[v] \cup N_2(v))| \leq 1$. Since $diam(G) = 3$, we conclude that $|V(G) - (N[v] \cup N_2(v))| = 1$. Now suppose that $v_1v_2 \notin E(G)$, then we can easily see that $G \in \{G_6, G_7\}$ and if $v_1v_2 \in E(G)$, then we can easily see that $G \in \{G_8, G_9\}$. Therefore, $G \in \mathcal{F}$.

Next suppose that $|V(G')| = 2$. Then $G' = K_2$. Suppose that $v_1v_2 \notin E(G)$. If a vertex of $N_2(v)$ is adjacent to both vertices of $G^{'}$, then the other vertex of $N_2(v)$ is also adjacent to both vertices of $G^{'}$ (otherwise this fact that $\Delta_h(G) = 2$ leads to contradiction). Note that in this case for any pair $a, b \in N[v] \cup N_2(v)$, $d(a, b) \leq 2$. Since $diam(G) = 3$, thus we may suppose that $V(G) - (N[v] \cup N_2(v)) \neq \emptyset$. Let z be a vertex of $V(G) - (N[v] \cup N_2(v))$ such that $z \in N(v_1) \cup N(v_2)$. Then $\deg_h(z) \geq 2$. Since $\Delta_h(G) = 2$, we have $\deg_h(z) = 2$. Note that $N_2(z) = N(v)$, that Claim 1 leads to a contradiction. Hence any vertex of $N_2(v)$ is adjacent to one vertex of $G^{'}$. If $N_{G'}(v_1) \cap N_{G'}(v_2) \neq \emptyset$, then $\deg_h(v_1) = \deg_h(v_2) > 2$, a contradiction. Thus $N_{G'}(v_1) \cap N_{G'}(v_2) = \emptyset$ and $d(v_1, v_2) = 3$. If $V(G) - (N[v] \cup N_2(v)) \neq \emptyset$, then Claim 1 or this fact that $diam(G) = 3$, leads to contradiction. Therefore, we conclude that $V(G) - (N[v] \cup N_2(v)) = \emptyset$. Therefore, $G = G_9 \in \mathcal{F}$.

Now suppose that $v_1v_2 \in E(G)$. If at least one vertex of $N_2(v)$ is adjacent to both vertices of $N(v)$, then for any pair $a, b \in N[v] \cup N_2(v)$, $d(a, b) \leq 2$. Since $diam(G) = 3$, thus we may suppose that $V(G) - (N[v] \cup N_2(v)) \neq \emptyset$. Let z be a vertex of $V(G) - (N[v] \cup N_2(v))$ such that $z \in N(v_1) \cup N(v_2)$. Then $\deg_h(z) \geq 2$. Since $\Delta_h(G) = 2$, we have $\deg_h(z) = 2$. Note that in this case, Claim 1 leads to contradiction. Hence each vertex of $N_2(v)$ is adjacent to one vertex of $V(G^{'})$. If $N_{G'}(v_1) \cap N_{G'}(v_2) = \emptyset$, then for any pair $a, b \in N[v] \cup N_2(v)$, $d(a, b) \leq 2$. Thus we may suppose that $V(G) - (N[v] \cup N_2(v)) \neq \emptyset$. Let z be a vertex of $V(G) - (N[v] \cup N_2(v))$ such that $z \in N(v_1) \cup N(v_2)$. Then clearly $\deg_h(z) = 2$ and we can see that Claim 1 leads to a contradiction. Hence $N_{G'}(v_1) \cap N_{G'}(v_2) \neq \emptyset$. Let $\{x\} = N_{G'}(v_1) \cap N_{G'}(v_2)$. Clearly for any pair $a, b \in N[v] \cup N_2(v)$, $d(a, b) \leq 2$. Since $diam(G) = 3$, thus we may suppose that $V(G) - (N[v] \cup N_2(v)) \neq \emptyset$. Let z be a vertex of $V(G) - (N[v] \cup N_2(v))$ such that $z \in N(v_1) \cup N(v_2)$. Suppose that there is $z^{'} \in V(G) - (N[v] \cup N_2(v))$, such that $z \neq z^{'}$. If $z^{'} \in N(v_1) \cup N(v_2)$, then $\deg_h(x) = 2$ and $N_2(x) \cap N_2(v) = \emptyset$, and so Claim 1 leads to contradiction. Hence we shall have $z^{'} \in N(z) - \{v_1, v_2\}$, that in this case clearly $d(z^{'}, v) = 4$, a contradiction. Thus $|V(G) - (N[v] \cup N_2(v))| \leq 1$. If z is adjacent to only one vertex of $N_2(v)$, without loss of generality, suppose that z be only adjacent to v_1, then $\deg_h(v_2) = 3$, a contradiction. Hence $z \in N(v_1) \cap N(v_2)$ and $G = G_{10} \in \mathcal{F}$.

Subcase 2.2. $\delta(G') = |V(G')| - 2$. Clearly $|V(G')| \geq 2$. Suppose that $|V(G')| \geq 4$. Let x be a vertex of $G^{'}$ such that $\deg_{G'}(x) = |V(G^{'})| - 2$ and let y be the other vertex of $G^{'}$ such that $y \notin N_{G'}(x)$. We can see that $d(x, y) = 2$ and $N_{G'}(x) = N_{G'}(y)$. We show that if there is a vertex $a \in V(G^{'})$ such that $a \notin N(v_1)$, then $d(a, v_1) = 2$. If $a = x$, then at least one vertex of $N_{G'}(x)$ is adjacent to the vertex v_1, since in the otherwise, $\deg_h(v_1) > 2$, a contradiction. Thus we deduce that $d(a, v_1) = 2$. Next suppose that $a \in N_{G'}(x)$. If $\{x, y\} \cap N(v_1) = \emptyset$, then $\deg_h(v_1) > 2$, a contradiction. Thus $\{x, y\} \cap N(v_1) \neq \emptyset$, and so $d(a, v_1) = 2$. If there are two vertices $a, b \in V(G^{'})$ such that $a, b \notin N(v_1)$. Then, $\deg_h(v_1) > 2$, a contradiction. Therefore, we conclude that v_1 and v_2 are adjacent to at least $|V(G')| - 1$ vertices of G' and clearly $N_{G'}(v_1) \cap N_{G'}(v_2) \neq \emptyset$. Note that, we have $d(v_1, v_2) \leq 2$. Then clearly distance between any two vertices of $N[v] \cup N_2(v)$ is at most two. Since $diam(G) = 3$, thus we may suppose that $V(G) - (N[v] \cup N_2(v)) \neq \emptyset$. Let z be a vertex of $V(G) - (N[v] \cup N_2(v))$ such that $z \in N(v_1) \cup N(v_2)$. Then clearly $\deg_h(z) > 2$, a contradiction. Consequently, $|V(G')| \leq 3$.

Now suppose that $|V(G')| = 2$. Then $G' = \overline{K_2}$. Let $V(G') = \{x, y\}$. Let $v_1 v_2 \notin E(G)$. If v_1 and v_2 are adjacent to both vertices of $V(G')$, then distance between any two vertices of $N[v] \cup N_2(v)$ is at most two. Since $diam(G) = 3$, thus we may suppose that $V(G) - (N[v] \cup N_2(v)) \neq$

$\emptyset$. Let z be a vertex of $V(G) - (N[v] \cup N_2(v))$ such that $z \in N(v_1) \cup N(v_2)$. Then clearly $\deg_h(z) = 2$ and $N_2(z) \cap N_2(v) = \emptyset$. By Claim 1 we have a contradiction. Thus there exists at least one vertex in $N_2(v)$ such that is adjacent to only one vertex of G'. If v_1 is adjacent to both vertices of $V(G')$ and v_2 be adjacent to only one vertex of $V(G')$, then $d(v_2, x) = 3$ or $d(v_2, y) = 3$. We show that $V(G) - (N[v] \cup N_2(v)) = \emptyset$. Suppose to contrary that there is $z \in V(G) - (N[v] \cup N_2(v))$ such that $z \in N(v_1) \cup N(v_2)$. Then we can see that Claim 1 or $diam(G) = 3$ leads to contradiction. Therefore, we conclude that $G = G_7 \in \mathcal{F}$. Now suppose that each vertex of $N_2(v)$ is adjacent to one vertex of G'. If v_1 is adjacent to x and v_2 be adjacent to y, then $d(v_1, v_2) = 4$, a contradiction. Hence, both vertices of $N_2(v)$ are adjacent to precisely one vertex of G', without loss of generality, suppose that v_1 and v_2 are adjacent to x. Clearly $d(y, v_1) = d(y, v_2) = 3$. Then we can easily see that $V(G) - (N[v] \cup N_2(v)) = \emptyset$. Thus we deduce that $G = G_6 \in \mathcal{F}$.

Now suppose that $v_1v_2 \in E(G)$. If there is at least a vertex in $N_2(v)$ such that is adjacent to both vertices of G', then distance between any two vertices of $N[v] \cup N_2(v)$ is at most two. Since $diam(G) = 3$, we may suppose that $V(G) - (N[v] \cup N_2(v)) \neq \emptyset$. Let z be a vertex of $V(G) - (N[v] \cup N_2(v))$ such that $z \in N(v_1) \cup N(v_2)$. Then clearly $\deg_h(z) = 2$ and Claim 1 leads to contradiction. Therefore, we conclude that each vertex of $N_2(v)$ is adjacent to one vertex of G'. If v_1 is adjacent to x and v_2 be adjacent to y, then distance between any two vertices of $N[v] \cup N_2(v)$ is at most two and we can easily see that $V(G) - (N[v] \cup N_2(v)) = \emptyset$. Thus both v_1 and v_2 are adjacent to precisely one vertex of G'. In this case, there exist two vertices in $N[v] \cup N_2(v))$ such that distance between them is equal to three and also we can easily see that $V(G) - (N[v] \cup N_2(v)) = \emptyset$. Therefore $G = G_{11} \in \mathcal{F}$.

Next Suppose that $|V(G')| = 3$. Then G' is a path of length two. Let G' be the path $P : xyu$. Assume that $v_1v_2 \notin E(G)$. If one of the vertices $N_2(v)$ is adjacent to all vertices of $G^{'}$, then clearly the other vertex of $N_2(v)$ is also adjacent to all vertices of $G^{'}$. Now suppose that both vertices of $N_2(v)$ are adjacent to any vertex of $V(G^{'})$, then distance between any two vertices of $N[v] \cup N_2(v)$ is at most two. Since

$diam(G) = 3$, thus we may suppose that $V(G) - (N[v] \cup N_2(v)) \neq \emptyset$. Let z be a vertex of $V(G)-(N[v]\cup N_2(v))$ such that $z \in N(v_1)\cup N(v_2)$. Then clearly $\deg_h(z) > 2$, a contradiction. Thus each vertex of $N_2(v)$ is adjacent to at most two vertices of G'. If each vertex of $N_2(v)$ is adjacent to two vertices of G', then clearly $N_{G'}(v_1) \cap N_{G'}(v_2) \neq \emptyset$ and so $d(v_1, v_2) = 2$. Thus hop-degree of each vertex of $N_2(v)$ is more than two, a contradiction. Thus at least one vertex of $N_2(v)$ is adjacent to only one vertex of G'. Suppose that both vertices of $N_2(v)$ are adjacent to one vertex of G', then we can easily see that $y \notin N(v_1) \cup N(v_2)$, and also $N_{G'}(v_1) \cap N_{G'}(v_2) = \emptyset$. Thus if v_1 is adjacent to x, then v_2 is adjacent to u. Then $d(v_1, v_2) = 4$, a contradiction with $diam(G) = 3$. Thus one vertex of $N_2(v)$ is adjacent to one vertex of G' and the other vertex is adjacent to two vertices of G'. Without loss of generality suppose that v_1 is adjacent to two vertices of G' and v_2 is adjacent to only one vertex of G'. Then clearly $N_{G'}(v_1) \cap N_{G'}(v_2) = \emptyset$, and as before, we observe that v_2 is not adjacent to y. Note that in this case $d(v_1, v_2) = 3$ and we can easily see that $V(G) - (N[v] \cup N_2(v)) = \emptyset$. Therefore, we conclude that $G = G_{12} \in \mathcal{F}$.

Now assume that $v_1v_2 \in E(G)$. If any vertex of $N_2(v)$ is adjacent to at least two vertices of G', then distance between any two vertices of $N[v] \cup N_2(v)$ is at most two. Since $diam(G) = 3$, thus we may suppose that $V(G) - (N[v] \cup N_2(v)) \neq \emptyset$. Let z be a vertex of $V(G) - (N[v]\cup N_2(v))$ such that $z \in N(v_1)\cup N(v_2)$. Then $\deg_h(z) > 2$, that is a contradiction. Hence at least one vertex of $N_2(v)$ is adjacent to only one vertex of G'. Note that in this case, any vertex of $N_2(v)$ is adjacent to at most two vertices of G'. If both vertices v_1 and v_2 are adjacent to one vertex of G', then $y \notin N(v_1)\cup N(v_2)$, since in the otherwise $\deg_h(v_1) \geq 3$ and $\deg_h(v_2) \geq 3$, a contradiction. Also $N_{G'}(v_1)\cap N_{G'}(v_2) \neq \emptyset$, in the otherwise $\deg_h(v_1) > 2$ and $\deg_h(v_2) > 2$. Hence v_1 and v_2 are adjacent to precisely one vertex of $\{x, u\}$ and so $G = G_{10} \in \mathcal{F}$. Now suppose that a vertex of $N_2(v)$ be adjacent to both vertices of G' and the other vertex of $N_2(v)$ is adjacent to one vertex of G'. Without loss of generality suppose that v_2 is adjacent to one vertex of G'. Then $y \notin N(v_2)$, since in the otherwise $\deg_h(v_2) \geq 3$, a contradiction. Without loss of generality suppose that v_2 is adjacent to u. If $u \notin N(v_1)$, then $\{x, y\} =$

$N_{G'}(v_1)$ and we can see that $\deg_h(v_2) \geq 3$, a contradiction. Therefore, $u \in N(v_1)$. Next suppose that $x \in N(v_1)$, then we can easily see that $\deg_h(v_2) > 2$, a contradiction. Hence, we have $N_{G'}(v_1) = \{y, u\}$ and $N_{G'}(v_2) = \{u\}$. Note that in this case $d(x, v_2) = 3$ and we can see that $V(G) - (N[v] \cup N_2(v)) = \emptyset$. Consequently, we have $G = G_{13} \in \mathcal{F}$.

Case 3. $diam(G) = 4$. According to the Claim 2, we consider two following subcases.

Subcase 3.1. $\delta(G') = |V(G')| - 1$. Then G' is a complete graph. Suppose that $|V(G')| \geq 3$. Clearly each vertex of $N_2(v)$ is adjacent to at least $|V(G')| - 1$ vertices of G'. Also v_1 is adjacent to at least $|V(G') \cup \{v_2\}| - 1$ vertices of $V(G') \cup \{v_2\}$, since in the otherwise if there exist two vertices $a, b \in V(G') \cup \{v_2\}$ such that $a, b \notin N(v_1)$, then $\deg_h(v_1) > 2$, a contradiction. Similarly, we can see that v_2 is adjacent to at least $|V(G') \cup \{v_1\}| - 1$ vertices of $V(G') \cup \{v_1\}$. Clearly, in this case distance between any two vertices of $N[v] \cup N_2(v)$ is at most two. Since $diam(G) = 4$, we conclude that $V(G) - (N[v] \cup N_2(v)) \neq \emptyset$. Then there is the vertex $z \in V(G) - (N[v] \cup N_2(v))$ such that $z \in N(v_1) \cup N(v_2)$. Note that in this case $\deg_h(z) \geq 2$ and the fact that $\Delta_h(G) = 2$ or Claim 1, lead to contradiction. Therefore, we may assume that $|V(G')| \leq 2$.

We first assume that $|V(G')| = 2$, then we can easily see that distance between any two vertices of $N[v] \cup N_2(v)$ is at most three. Since $diam(G) = 4$, we may suppose that $V(G) - (N[v] \cup N_2(v)) \neq \emptyset$. Note that in this case from the fact that $\Delta_h(G) = 2$ or Claim 1 we have a contradiction.

Hence we can assume that $|V(G')| = 1$. Let $V(G') = \{z\}$. Then v_1 and v_2 are adjacent to z and distance between every two vertices in $N[v] \cup N_2(v)$ is at most two. Since $diam(G) = 4$, we conclude that there is a vertex $y \in V(G) - (N[v] \cup N_2(v))$, such that $d(y, v_1) = 2$ or $d(y, v_2) = 2$. In this case, if $v_1v_2 \notin E(G)$, then hop-degree of at least one vertex of $N_2(v)$ is more than two, a contradiction. Thus $v_1v_2 \in E(G)$. Now suppose that $y \in V(G) - (N[v] \cup N_2(v))$ is a vertex at distance two from v_1 and let v_1xy be the path between v_1 and y. If $x \notin N(v_2)$, then $\deg_h(x) = \deg_h(v_1) = 2$ and $N_2(x) \cap N_2(v_1) = \emptyset$, that is a contradiction with Claim 1. Thus $x \in N(v_2)$. If there exists a vertex $x' \in V(G) - (N[v] \cup N_2(v))$ such that $x' \neq x$ and $x' \in N(v_2) \cup N(v_1)$, then $\deg_h(z) = 2$, and $N_2(z) \cap N_2(v) = \emptyset$, thus Claim 1 leads to contradiction. Since $y \in V(G) - (N[v] \cup N_2(v))$ and $y \neq x$, so we conclude that $y \notin N(v_2)$ and since $diam(G) = 4$, so $N(y) - \{x\} = \emptyset$ and y is a leaf. On the other hand $N(x) - N_2(v) = \{y\}$, since in the otherwise, $\deg_h(v_1) \geq 3$ and also $\deg_h(v_2) \geq 3$, a contradiction. Thus we conclude that $G = G_{14} \in \mathcal{F}$. Similarly, we can show that if $y \in V(G) - (N[v] \cup N_2(v))$ is a vertex at distance two from v_2, then $G = G_{14} \in \mathcal{F}$.

Subcase 3.2. $\delta(G') = |V(G')| - 2$. Clearly $|V(G')| \geq 2$. Suppose that $|V(G')| \geq 4$. Let x be a vertex of $V(G^{'})$ such that $\deg_{G'}(x) = |V(G^{'})| - 2$ and let y be a vertex of $G^{'}$ such that $y \notin N_{G'}(x)$. We can see that $d(x, y) = 2$ and $N_{G'}(x) = N_{G'}(y)$. We show that if there is a vertex $a \in V(G^{'})$ such that $a \notin N(v_1)$, then $d(a, v_1) = 2$. If $a = x$, then at least one vertex of $N_{G'}(x)$ is adjacent to the vertex v_1, since in the otherwise, $\deg_h(v_1) > 2$, a contradiction. Thus we deduce that $d(a, v_1) = 2$. Next suppose that $a \in N_{G'}(x)$. If $\{x, y\} \cap N(v_1) = \emptyset$, then $\deg_h(v_1) > 2$, a contradiction. Thus $\{x, y\} \cap N(v_1) \neq \emptyset$, and so $d(a, v_1) = 2$. If there are two vertices $a, b \in V(G^{'})$ such that $a, b \notin N(v_1)$, then $\deg_h(v_1) > 2$, a contradiction. Therefore, we conclude that v_1 is adjacent to at least $|V(G')| - 1$ vertices of G'. With a similar argument we can show that v_2 is adjacent to least $|V(G')| - 1$ vertices of G'. Clearly $N_{G'}(v_1) \cap N_{G'}(v_2) \neq \emptyset$. Then the distance between any two vertices of $N[v] \cup N_2(v)$ is at most 2. We may suppose that $V(G) - (N[v] \cup N_2(v)) \neq \emptyset$, then there is the vertex $z \in V(G) - (N[v] \cup N_2(v))$ such that $z \in N(v_1) \cup N(v_2)$. Note that in this case clearly $\deg_h(z) > 2$, a contradiction. Hence, we conclude that $V(G) - (N[v] \cup N_2(v)) = \emptyset$ and so we may assume that $|V(G')| \leq 3$.

We first assume that $|V(G')| = 3$. Then G' is a path of length two. Let G' be the path $P : xyu$. Assume that at least one vertex of $N_2(v)$ is adjacent to at least two vertices of G'. Without loss of generality, suppose that v_1 is adjacent to at least two vertices of G'. Then clearly the distance between any two vertices of $N[v] \cup N_2(v)$ is at most 3. Thus we may suppose that there is at least a vertex $z \in V(G) - (N[v] \cup N_2(v))$ that is adjacent to v_1 or v_2. If z is adjacent to v_1, then $\deg_h(z) \geq 2$ and Claim 1 or the fact that $\Delta_h(G) = 2$

leads to a contradiction. Thus z is adjacent to v_2. Note that in this case v_2 is adjacent to at least one vertex of G'. Then we can easily see that Claim 1 or the fact that $\Delta_h(G) = 2$ leads to a contradiction. Therefore, each vertex of $N_2(v)$ is adjacent to one vertex of G'. We can see that $y \notin N(v_1) \cup N(v_2)$. If $N_{G'}(v_1) \cap N_{G'}(v_2) \neq \emptyset$, then the distance between any two vertices of $N[v] \cup N_2(v)$ is at most 3. Thus we may suppose that there is a vertex $z \in V(G) - (N[v] \cup N_2(v))$ that is adjacent to v_1 or v_2. Let $x \in N_{G'}(v_1) \cap N_{G'}(v_2)$. Then $\deg_h(x) = 2$ and $N_2(x) = \{u, z\}$. Thus $N_2(x) \cap N_2(v) = \emptyset$ and so Claim 1 leads to a contradiction. Thus $N_{G'}(v_1) \cap N_{G'}(v_2) = \emptyset$. If $v_1v_2 \in E(G)$, then clearly $\deg_h(v_1) > 2$ and $\deg_h(v_2) > 2$, a contradiction. Thus $v_1v_2 \notin E(G)$. Then $d(v_1, v_2) = 4$ and clearly $V(G)-(N[v]\cup N_2(v)) = \emptyset$, and so we conclude that $G = G_{14} \in \mathcal{F}$.

Next we assume that $|V(G')| = 2$, then $G^{'} = \overline{K_2}$. Let $V(G') = \{x, y\}$. Clearly the vertices v_1 and v_2 are adjacent to at least one vertex of G'. If each vertex of $N_2(v)$ is adjacent to both vertices of G', then clearly the distance between any two vertices in $N[v]\cup N_2(v)$ is at most two. Since $diam(G) = 4$, we may suppose that $V(G)-(N[v]\cup N_2(v)) \neq \emptyset$. Then there is the vertex $z \in V(G) - (N[v] \cup N_2(v))$ such that $z \in N(v_1) \cup N(v_2)$. Note that in this case $\deg_h(z) \geq 2$ and Claim 1 or the fact that $\Delta_h(G) = 2$ lead to a contradiction. Thus there is at least one vertex in $N_2(v)$ such that it is adjacent to only one vertex of G'. Suppose that one vertex of $N_2(v)$ is adjacent to both vertices of $V(G')$ and the other vertex is adjacent to only one vertex of G'. Then the distance between any two vertices of $N[v]\cup N_2(v)$ is at most two. Since $diam(G) = 4$, we may suppose that $V(G) - (N[v] \cup N_2(v)) \neq \emptyset$. Then there is the vertex $z \in V(G) - (N[v] \cup N_2(v))$ such that $z \in N(v_1) \cup N(v_2)$. Note that in this case $\deg_h(z) \geq 2$ and Claim 1 or the fact that $\Delta_h(G) = 2$ lead to a contradiction. Therefore, any vertex of $N_2(v)$ is adjacent to only one vertex of G'. Assume that $v_1v_2 \in E(G)$. Note that in this case the distance between any two vertices of $N[v] \cup N_2(v)$ is at most 3. We may suppose that $V(G) - (N[v] \cup N_2(v)) \neq \emptyset$. Then there is the vertex $z \in V(G) - (N[v] \cup N_2(v))$ such that $z \in N(v_1) \cup N(v_2)$. Then we can easily see that Claim 1 or the fact that $\Delta_h(G) = 2$ lead to a contradiction. Hence $v_1v_2 \notin E(G)$. Assume $N_{G'}(v_1) \cap N_{G'}(v_2) \neq \emptyset$, then clearly the distance between any two vertices of $N[v] \cup N_2(v)$ is at most 3. Thus we may suppose that $V(G) - (N[v] \cup N_2(v)) \neq \emptyset$. Then there is the vertex $z \in V(G) - (N[v] \cup N_2(v))$ such that $z \in N(v_1) \cup N(v_2)$. Note that in this case we can easily see that Claim 1 leads to a contradiction. Therefore, $N_{G'}(v_1) \cap N_{G'}(v_2) = \emptyset$, and $d(v_1, v_2) = 4$. Thus we conclude that $V(G) - (N[v] \cup N_2(v)) = \emptyset$ and so $G = P_5$.

($\Leftarrow$) Suppose that $G \in \mathcal{G}^* \cup \mathcal{F} \cup \{P_5\}$. It is obvious that if $G = P_5$ or $G \in \mathcal{F}$, then $\gamma_{hR}(G) = n - 1$. Next suppose that $G \in \mathcal{G}^*$. Then by Lemma 4, we have $\gamma_{hR}(G) = n - 1$. ■

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Received September 17, 2018
Revised December 17, 2018

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