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Connected Dominating Sets and a New Graph Invariant

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Abstract

Based on concept of connected dominating sets of a simple graph G we introduce a new invariant $\eta(G)$ which does not exceed the number of Hadwiger. The Nordhaus-Gaddum inequalities are: $\eta(G)\eta(\overline{G}) \geq n(G)$ and $\eta(G) + \eta(\overline{G}) \leq 6n(G)/5$. For values of chromatic number $\chi(G) \leq 4$ we prove $\eta(G) \geq \chi(G)$. We put forward the hypothesis: the last inequality holds for all simple graphs G.

Keywords: dominating set, number of Hadwiger, chromatic number, Nordhaus-Gaddum inequalities.

1 Introduction

All graphs G considered in this paper are undirected, simple and finite with vertex set $V(G)$. We denote $|V(G)|$ by $n(G)$. For $X \subseteq V(G)$, we denote by $G[X]$ the subgraph of G induced by X, further, $G - X = G[V(G) - X]$. For the subgraph H of graph G, $G-H = G[V(G)-V(H)]$. We shall write $v \sim u$ ($v \nsim u$) when vertices v and u are (are not) adjacent. If every pair of vertices in X are adjacent, then $G[X]$ is a complete subgraph or a clique K. The clique number $\omega(G)$ of a graph G is the number of vertices in a maximum clique in G. The degree of a vertex is the number of edges incident to the vertex. The number of Hadwiger $h(G)$ is the largest number of connected subgraphs of G, pairwise without common vertices and connected with at least one edge. Contracting the edges within each of these subgraphs so that each subgraph collapses to a single vertex produces a maximum complete graph $K_{h(G)}$ on $h(G)$ vertices. From definition of $h(G)$ it follows that $h(G) = max_{1 \leq i \leq k} h(G_i)$, where $G_1, G_2, \ldots, G_k$ are components of disconnected graph G.

If $D \subseteq V(G)$, $G[D]$ is connected and every vertex not in D has a neighbor in D, then it's clear that $h(G-D) \leq h(G)-1$. This property allows us to introduce new invariants of the graph, in the definitions of which we use the concept of connected dominating sets, and these invariants do not exceed the number of Hadwiger.

2 Domination and New Invariants

Definition 1 *A connected dominating set D is a subset of vertices of a graph G such that every vertex is in D or adjacent to at least one vertex in D, and $G[D]$ is connected in all connected components (subgraphs) of G.*

Further, unless otherwise specified, dominating set means connected dominating set. The edge $e = vu$ is dominating if the set $\{v, u\}$ is dominating.

Definition 2 *Let V_1 be a dominating set in a graph G, let V_2 be a dominating set in $G - V_1$, V_3 – dominating set in $G - V_1 - V_2$ and so on. $\eta_0(G)$ is the maximum length of the sequence of dominating sets $V_1, V_2, V_3, \ldots$.*

If $V_1, V_2, \ldots, V_{\eta_0}$ is a maximum sequence of dominating sets, then it's obvious that $|V_{\eta_0}| = 1$ or the set V_{η_0} is independent, i.e. no two of its vertices are adjacent.

Theorem 1 *For any graph G*

(i) $\eta_0(G) \leq h(G)$,

(ii) If D is any dominating set of G, then $\eta_0(G) \geq \eta_0(G-D)+1$,

(iii) $\eta_0(G) = max_{1 \leq j \leq k} \eta_0(G_j)$, where $G_1, G_2, \ldots, G_k$ are connected components of G.

Proof. Let $\eta_0(G) = \eta_0$.

(i) If we contract all edges in induced subgraphs $G(V_1), G(V_2), \ldots, G(V_{\eta_0})$), we obtain a graph G^c with clique number $\omega(G^c) = \eta_0$. From definition of number of Hadwiger, $h(G) \geq \omega(G^c)$.

(ii) If $V_1, V_2, \ldots, V_l$ is a maximum sequence of dominating sets of $G - D$, then $D, V_1, V_2, \ldots, V_l$ is a sequence of dominating sets of G.

(iii) Let $\eta_{0j} = \eta_0(G_j)$ for all $j (1 \leq j \leq k)$ and $V_1^j, V_2^j, \ldots, V_{\eta_{0j}}^j$ be a maximum sequence of dominating sets in G_j. Let $max_{1 \leq j \leq k} \eta_{0j} = \eta_{01}$. If for some j, $\eta_{0j} < \eta_{01}$, then we put $V_i^j = \varnothing$ for $\eta_{0j} + 1 \leq i \leq \eta_{01}$. The sequence $V_1, V_2, \ldots, V_{\eta_{01}}$, where $V_i = \cup_{j=1}^k V_i^j$, is the sequence of dominating sets in G. So, $\eta_0(G) \geq max_{1 \leq j \leq k} \eta_0(G_j)$.

Let $V_1, V_2, \ldots, V_{\eta_0}$ be a maximum sequence of dominating sets in graph G. For all $1 \leq i \leq \eta_0$, $V_i = \cup_{j=1}^k V_i^j$, where V_i^j is a corresponding dominating set in G_j or $V_i^j = \varnothing$. If for $i = i_0$, $V_{i_0}^j \neq \varnothing$, then $V_i^j \neq \varnothing$ for all $1 \leq i \leq i_0$. Since $V_{\eta_0} \neq \varnothing$, there exists at least one j for which the sequence $V_1^j, V_2^j, \ldots, V_{\eta_0}^j$ is the sequence of dominating sets in G_j. So, $\eta_0(G) \leq max_{1 \leq j \leq k} \eta_0(G_j)$.

Therefore, $\eta_0(G) = max_{1 \leq j \leq k} \eta_0(G_j)$. ■

The independence number $\alpha(G)$ of a graph G is the size of the largest independent set of its vertices. Duchet and Meyniel [2] showed that for every connected graph G with independence number $\alpha(G)$, there exists a dominating set with at most $2\alpha(G) - 1$ vertices, and therefore $h(G)(2\alpha(G) - 1) \geq n(G)$.

Theorem 2 *Let G be any graph with k connected components. Then*

$$\eta_0(G)(2\alpha(G) - k) \geq n(G).$$

Proof. The proof is by induction on $n = n(G)$. For $n \leq 3$, the result is true by inspection. Suppose $n \geq 4$ and suppose the result is true for all graphs with fewer than n vertices, and let G be a graph with $n(G) = n$ and its components are $G_1, G_2, \ldots, G_k$. In each component G_i $(1 \leq i \leq k)$ there exists dominating set D_i and $|D_i| \leq 2\alpha(G_i) - 1$. If $D = \cup_{i=1}^k D_i$, then $|D| = \sum_{i=1}^k |D_i| \leq \sum_{i=1}^k (2\alpha(G_i) - 1) = 2\alpha(G) - k$. The induction hypotheses implies that $\eta_0(G)(2\alpha(G) - k) =$

$\eta_0(G)\sum_{i=1}^{k}(2\alpha(G_i)-1) \geq \sum_{i=1}^{k}\eta_0(G_i)(2\alpha(G_i)-1) \geq \sum_{i=1}^{k}(\eta_0(G_i - D_i)+1)(2\alpha(G_i)-1) = \sum_{i=1}^{k}\eta_0(G_i-D_i)(2\alpha(G_i)-1)+\sum_{i=1}^{k}(2\alpha(G_i) - 1) \geq \sum_{i=1}^{k}\eta_0(G_i-D_i)(2\alpha(G_i-D_i)-1)+2\alpha(G)-k \geq \sum_{i=1}^{k}n(G_i-D_i)+ 2\alpha(G)-k = \sum_{i=1}^{k}n(G_i)-\sum_{i=1}^{k}|D_i|+2\alpha(G)-k \geq \sum_{i=1}^{k}n(G_i) = n(G)$.
■

Let $C_l = \{v_1, v_2, \ldots, v_l\}$ be a chordless cycle. A Mycielsky graph M_k of order k is a triangle-free graph ($\omega(M_k) = 2$) with chromatic number $\chi(M_k) = k$ (see [5]). M_2 contains two connected vertices, $M_3 = C_5$. If $V(M_k) = v_1, v_2, \ldots, v_n$, the graph M_{k+1} contains M_k itself as a subgraph, together with $n+1$ vertices: $u_1, u_2, \ldots, u_n, w$. Each vertex u_i is connected to w, and for each edge v_iv_j of M_k, graph M_{k+1} includes two edges, u_iv_j and u_jv_i. The set $\{u_1, u_2, \ldots, u_n, w\}$ is dominating in M_{k+1} and $\eta_0(M_{k+1}) \geq \eta_0(M_k)+1$. Because $\eta_0(M_2) = 2$, $\eta_0(M_3) = 3$, for any order k: $\eta_0(M_k) \geq \chi(M_k)$. So, the difference $\eta_0(G) - \omega(G)$ can be arbitrarily large. However, if G is obtained by adding some adjacent vertices with degree one to each vertex of any graph H, then $\eta_0(G) = 2$.

Definition 3 *$\eta(G) = max\ \eta_0(G')$, where the maximum is taken over all induced subgraphs G' of the graph G.*

Theorem 3 *For any graph G*
(i) $\omega(G) \leq \eta(G) \leq h(G)$,
(ii) If D is any dominating set in G then $\eta(G) \geq \eta(G-D)+1$,
(iii) If $\triangle(G)$ the maximum degree, then $\eta(G) \leq \triangle(G)+1$.

Proof. *(i)* Let $\omega = \omega(G)$. For any maximum clique K_ω, $\eta_0(K_\omega) = \omega$, so $\eta(G) \geq \omega$. If G' is one of subgraphs for which $\eta_0(G') = \eta(G)$, then, by Theorem 2, $\eta_0(G') \leq h(G')$, and therefore, $\eta(G) \leq h(G)$.

(ii) D is dominating set for all subgraphs of G containing D. The statement follows from Theorem 2.

(iii) If $\eta = \eta(G)$ and $V_1, V_2, \ldots, V_\eta$ is one of the longest sequence of dominating sets, then V_η is an independent set. Degrees of all vertices in this set are at least $\eta - 1$. ■

If we take n cycles C_{n-1}, join every two different cycles with a single edge using different vertices in each cycle, then for obtained graph G with $\triangle(G) = 3$, $h(G) \geq n$, $\eta(G) \leq 4$. So, the difference $h(G) - \eta(G)$ can be arbitrarily large.

The new invariant $\eta(G)$ also can be defined as a maximum length of a sequence of subsets of vertices $U_1, U_2, U_3, \ldots$, where $U_i \cap U_j = \varnothing$ $(i \neq j)$, every vertex from $U_1 \cup U_2 \cup \ldots \cup U_{k-1}$ is adjacent to at least one vertex in U_k, and subgraph induced by U_k $(k \geq 2)$ is connected.

In the proof of some lower bounds for number of Hadwiger we can just replace h by η. Below we give a proof of one theorem (see [6]) with this replacement.

Theorem 4 *Let G be any graph with $\alpha(G) = 2$. Let $n = n(G)$, $\omega = \omega(G)$, then $\eta(G) \geq (n + \omega)/3$.*

Proof. We proceed to prove by induction on n. Let $K = K_\omega$ be a maximum clique of G and let $H = G - V(K)$. By a 2-path of H we mean an induced subpath of length 2. The vertex set of any 2-path is dominating in G. If H contains a 2-path P, then $\omega(G-P) = \omega(G) = \omega$ and the induction hypothesis implies that $\eta(G) \geq \eta(G - P) + 1 \geq \frac{(n-3+\omega)}{3} + 1 = (n + \omega)/3$. If H does not contain any 2-path, then H is either a complete graph or disjoint union of two complete graphs. In both cases we claim that $\omega \geq n/2$. In the first case, this is evident. In the second case, H is a disjoint union of two complete graphs, say H_1 and H_2 and, because of $\alpha(G) = 2$, every vertex of the complete subgraph K is either joined to all vertices of H_1 or to all vertices of H_2. This implies the claim. Consequently, $\eta(G) \geq \omega \geq (n + \omega)/3$. Thus Theorem 4 is proved. ■

3 The Nordhaus-Gaddum inequalities for $\eta(G)$

Nordhaus and Gaddum studied the chromatic number in a graph G and its complement $\overline{G}$ together. They proved lower and upper bounds of the sum and of the product of chromatic numbers of G and $\overline{G}$ in terms of $n(G)$. Since then, any bound of the sum and/or the product of an invariant in a graph and its complement is called a Nordhaus-Gaddum type inequality. We prove these inequalities for $\eta(G)$.

The diameter of a connected graph G, denoted $d(G)$, is the maximum distance between two vertices. If graph is not connected, then the diameter is defined as infinite. Clearly, if $d(G) \geq 3$, then in its complement $\overline{G}$ exists dominating edge. Let G be connected. We call the induced subgraph G_0 a cut-subgraph if $G - G_0$ is disconnected. If $d(G) = 2$, then the vertex set of any connected cut-subgraph is dominating in G.

Lemma 1 *If G is connected, not complete and does not have a complete cut-subgraph, then $h(G) \geq \omega(G) + 1$.*

Proof. Let K_ω be a maximum clique in G and subgraph $H = G - K_\omega$ is connected. If we assume that H is not joined to some vertex $v \in V(K_\omega)$, then $K_\omega - v$ is complete cut-subgraph, hence $V(H)$ is dominating in G. ∎

Lemma 2 *If K is a complete subgraph of G and $G - K$ is connected, then in G exists dominating set D such that $|D| \leq 2\alpha(G) - 1$ and $|D \cap V(K)| \leq 1$.*

Proof. Let $H = G - K$ and $v \sim u$, where $v \in V(K)$ and $u \in V(H)$. By the result of Duchet and Meyniel (see [2]), the subgraph H has the dominating set D_1 with $p + q$ vertices; p vertices form independent set (independent vertices), and $q \leq p - 1$. If $p = \alpha(G)$, then $D = D_1$. If $p \leq \alpha(G) - 1$ and D_1 is not dominating in G, then $D = D_1 \cup \{v, u\}$. In this case $|D| = |D_1| + 2 = p + q + 2 \leq 2p - 1 + 2 \leq 2(\alpha(G) - 1) + 1 = 2\alpha(G) - 1$. ∎

Theorem 5 *$\eta(G) \cdot \eta(\overline{G}) \geq n(G)$.*

Proof. We proceed by induction on $n = n(G)$. For $n \leq 5$, the result is clear. Suppose $n \geq 6$, and statement holds for all graphs with fewer than n vertices. Clearly, if $max\{d(G), d(\overline{G})\} \geq 3$, then Theorem 5 holds. Let $d(G) = d(\overline{G}) = 2$.

Case 1: A complete cut-subgraph K exists in graph G (or in $\overline{G}$) and $H_1, H_2, \ldots, H_k$ ($k \geq 2$) are connected components of $G - K$. In complement $\overline{G}$ (or in G) each edge $e = v_i v_j$ ($v_i \in V(H_i)$, $v_j \in V(H_j)$, $i \neq j$) is dominating in subgraph induced in $\overline{G}$ by the set of vertices $V(G) - V(K)$. Let $G' = G - K - \{v_i, v_j\}$. We have: $\eta(G) \cdot \eta(\overline{G}) \geq \eta(G)(\eta(\overline{G'}) + 1) = \eta(G)\eta(\overline{G'}) + \eta(G) \geq (\eta(G') + 1)\eta(\overline{G'}) + \eta(G) = \eta(G')\eta(\overline{G'}) + \eta(\overline{G'}) + \eta(G) \geq n(G') + \eta(\overline{G'}) + \eta(G) = n(G) - n(K) - 2 + \eta(\overline{G'}) + \eta(G) \geq n(G) + \eta(\overline{G'}) - 2$. If $\eta(\overline{G'}) \leq 1$, then $\omega(G) \geq (n-2)/2$, hence Theorem 5 holds in this case.

Case 2: Neither G nor $\overline{G}$ does not contain a complete cut-subgraph. Without loss of generality we may assume $\omega = \omega(G) \geq \alpha(G) = \alpha$. Let K_ω be a maximum clique of G. By Lemma 2, G has a dominating set D with $n(D) \leq 2\alpha(G) - 1$ and $|D \cap V(K_\omega)| \leq 1$.

Case 2.1: D is a dominating set in G, $|D| = p + 1$. The set D consists of p independent vertices and one vertex adjacent to all p vertices. Let $G' = G - D$. Thus $\eta(G)\eta(\overline{G}) \geq (\eta(G') + 1)\eta(\overline{G}) = \eta(G')\eta(\overline{G}) + \eta(\overline{G}) \geq \eta(G')\eta(\overline{G'}) + \eta(\overline{G}) \geq n - p - 1 + \eta(\overline{G})$. By Lemma 1, $\eta(\overline{G}) \geq \omega(\overline{G}) + 1 \geq p + 1$, hence Theorem 5 holds in this case.

Case 2.2: D is a dominating set in G, $|D| = p + q$, p vertices are independent and $2 \leq q \leq p - 1$. None of the vertices of G is joined to all p independent vertices. In this case these p vertices are dominating in $\overline{G}$. Let $G' = G - D$. We have: $\eta(G)\eta(\overline{G}) \geq (\eta(G') + 1)\eta(\overline{G}) = \eta(G')\eta(\overline{G}) + \eta(\overline{G}) \geq \eta(G')(\eta(\overline{G'}) + 1) + \eta(\overline{G}) = \eta(G')\eta(\overline{G'}) + \eta(G') + \eta(\overline{G}) \geq n(G') + \eta(G') + \eta(\overline{G}) = n(G) - p - q + \eta(G') + \eta(\overline{G}) \geq n(G) - 2\alpha + 1 + \eta(G') + \eta(\overline{G}) \geq n(G) - 2\alpha + 1 + \omega - 1 + \alpha \geq n(G) - \alpha + \omega \geq n(G)$.

∎

Now we prove the upper bound of a sum.

Kostochka proved (see [4]) that for number of Hadwiger, $h(G) + h(\overline{G}) \leq 6n(G)/5$. We show that there are graphs G for which $\eta(G) + \eta(\overline{G}) = 6n(G)/5$.

Given a graph G, we say that graph $H = inf(G)$ is an inflation of G if each vertex $v \in V(G)$ is replaced by a complete graph K^v. If $u \in V(G)$ and $v \sim u$, then in H every vertex of K^v is joined to every vertex of K^u. We call the complete graphs K^v, K^u the atoms of inflation.

Let $H = inf(C_5)$ and for corresponding atoms $n(K^1) = n(K^2) = n(K^3) = n(K^4) = n(K^5) = k$. In complete K^2 and K^4 we remove all edges and the resulting graph we denote by G. In the graph G, subgraph induced by the set of vertices $V(K^1) \cup V(K^5)$ is complete. In the complement $\overline{G}$, subgraph induced by the set of vertices $V(K^2) \cup V(K^4)$ is complete, hence $\omega(G) = \omega(\overline{G}) = 2k$. There are k disjoint dominating sets $\{v_2, v_3, v_4\}$ in G : $v_2 \in V(K^2)$, $v_3 \in V(K^3)$, $v_4 \in V(K^4)$, and k disjoint dominating sets $\{v_1, v_3, v_5\}$ in $\overline{G}$: $v_1 \in V(K^1)$, $v_3 \in V(K^3)$, $v_5 \in V(K^5)$. So $\eta(G) \geq 3k$, $\eta(\overline{G}) \geq 3k$, and $\eta(G)+\eta(\overline{G}) \geq 6k = 6n(G)/5$.

4 New Invariant and Chromatic Number

Let G be any graph and chromatic number $\chi(G) = \chi$. In definition of chromatic number, for independent sets of vertices $V_1, V_2, , V_\chi$, $\cup_{i=1}^{\chi} V_i = V(G)$, $V_i \cap V_j = \varnothing$ $(i \neq j)$, we can always assume that V_1 is a dominating set in graph G, V_2 is a dominating set in $G - V_1$, V_3 is a dominating set in $G - V_1 - V_2$ and so on.

Now we begin to study the relationship between the chromatic number and $\eta(G)$.

Let $\delta(G)$ be the minimum degree.

Lemma 3 *If $\delta(G) \geq 3$, then in G exists a cycle and connected subgraph joined to all vertices of this cycle.*

Proof. We define the distance between two cycles C^1 and C^2 as the shortest distance between all pairs of vertices v and u, where $v \in C^1$ and $u \in C^2$. If two cycles have common vertices, then the distance between them is zero. Among the pairs of cycles with maximum distance ρ, choose one C_l with minimum length l. If $G - C_l$ is connected, then

C_l and $G - C_l$ are desired. Let $G - C_l$ be not connected. If $\rho = 0$, then all components of $G - C_l$ do not have cycles. If $\rho > 0$, then only one component has cycles (otherwise ρ is not the maximum). Let T be any component without cycles. Consider the graph $H = C_l \cup T$. First, prove that $l \leq 4$.

An arbitrary terminal vertex $t \in V(T)$ is joined to at least two vertices of C_l. These two vertices divide C_l into two simple paths. If $l \geq 5$, then the length of one path is at least 3, and then the length of the cycle induced by vertices of another path and t is less than l.

If T is joined to all vertices of C_l, then C_l and T are desired. Assume that vertex $v_1 \in C_l$ is not joined to T. From the minimality of l it follows that the number of terminal vertices in T is at least two.

Case 1. $l = 3$ and $C_3 = \{v_1, v_2, v_3\}$. In the graph T all terminal vertices $t_1, t_2, \ldots, t_s$ $(s \geq 2)$ are joined to v_2 and to v_3. The cycle $C = \{t_1, v_2, v_3\}$ and the graph $T - t_1$ are desired.

Case 2. $l = 4$ and $C_4 = \{v_1, v_2, v_3, v_4\}$. All terminal vertices $t_1, t_2, \ldots, t_s$ $(s \geq 2)$ are adjacent to v_2 and to v_4. Assume that $s = 2$, i.e. T is the chain and the number of vertices $n(T) \geq 3$. If in this chain $t_1 \sim t$, then $t \sim v_3$. The cycle $C = \{t_1, v_2, t_2, v_4\}$ and the induced subgraph with the set of vertices $V(T) \cup \{v_3\} - \{t_1, t_2\}$ are desired. Now assume that $s \geq 3$. The cycle C is the same as above and the connected graph is induced by vertices $V(T) - \{t_1, t_2\}$.

■

Theorem 6 *If* $\chi(G) \leq 4$, *then* $\eta(G) \geq \chi(G)$.

Proof. The cases where $\chi(G) = 1, 2$ are trivial. The case $\chi(G) = 3$ is also easy: the graph requiring three colors has an odd cycle $C_l = \{v_1, v_2, \ldots, v_l\}$. The set of vertices $\{v_3, v_4, \ldots, v_l\}$ is joined to connected v_1 and v_2. If $\chi(G) = 4$, then without loss of generality, we may assume that G is critical, i.e. for each vertex v: $\chi(G - v) = 3$. For such graphs (see [1]) $\delta(G) \geq 3$, and by Lemma 3, $\eta(G) \geq 4$. ■

Let $\chi(k) = max\{\chi(G) \mid \eta(G) \leq k\}$.

Theorem 7 $\chi(k) \leq 3 \cdot 2^{k-3}$ *for* $k \geq 3$.

Proof. We proceed by induction on k. As it is shown above, $\chi(3) = 3$. So suppose $k \geq 4$ and suppose the result is true for all graphs with $\eta < k$ and let $G = (V, E)$ be a graph with $\eta(G) = k$. Let $v_0 \in V$ and D_i be subgraph induced by all vertices at distance i from v_0. Subgraph $G[\{v_0\} \cup D_1 \cup D_2 \cup \ldots \cup D_{i-1}]$ is connected, so $\eta(D_i) \leq k-1$ for all i. Clearly, that $\chi(G) \leq max\{\chi(D_{i-1}) + \chi(D_i)\}$ $(i = 1, 2, 3, \ldots)$, where $D_0 = \{v_0\}$. So, $\chi(G) \leq 2 \cdot \chi(k-1) \leq 2 \cdot 3 \cdot 2^{k-1-3} = 3 \cdot 2^{k-3}$. ∎

Conjecture: For all graphs G, $\chi(G) \leq \eta(G)$.

5 Concluding Remarks

The Hadwigers conjecture (HC) can be stated in form: For all graphs G, $\chi(G) \leq h(G)$. Therefore, a new conjecture is a strengthening of HC. If HC is false, then a counterexample might possibly be obtained among counterexamples to the new conjecture.

A good place to look for a counterexample to our conjecture are graphs G with independence number $\alpha(G) = 2$. For such graphs $\chi(G) \geq n(G)/2$. In [3] Kim has proved that there is a constant $c > 0$ such that there exist graphs G on n vertices, with $\alpha(G) = 2$ and clique number $\omega(G) \leq c\sqrt{n \cdot \log n}$. For these graphs to have $\chi(G) \leq \eta(G)$ we need to find a sequence with at least $\chi(G) - c\sqrt{n \cdot \log n}$ dominating sets with at least two vertices.

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