

## Closeness centrality in some splitting networks

Vecdi Aytacı      Tufan Turacı

### Abstract

A central issue in the analysis of complex networks is the assessment of their robustness and vulnerability. A variety of measures have been proposed in the literature to quantify the robustness of networks, and a number of graph-theoretic parameters have been used to derive formulas for calculating network reliability. *Centrality* parameters play an important role in the field of network analysis. Numerous studies have proposed and analyzed several *centrality* measures. We consider *closeness centrality* which is defined as the total graph-theoretic distance to all other vertices in the graph. In this paper, closeness centrality of some splitting graphs is calculated, and exact values are obtained.

**Keywords:** Graph theory, network design and communication, complex networks, closeness centrality, splitting graphs.

## 1 Introduction

Most of the communication systems of the real world can be represented as complex networks, in which the nodes are the elementary components of the system and the links connect a pair of nodes that mutually interact exchanging information. A central issue in the analysis of complex networks is the assessment of their robustness and vulnerability. One of the major concerns of network analysis is the definition of the concept of *centrality*. This concept measures the importance of a node's position in a network. In social, biological, communication, and transportation networks, among others, it is important to know the relative structural prominence of nodes to identify the key elements in the network.

There are several *centrality* measures like *degree centrality*, *closeness centrality*, *residual closeness*, *vertex-betweenness centrality*, *edge-betweenness centrality*, etc. [1]–[9]. *Centrality* is a complex notion that requires a clear definition. For example, a node has high centrality if it can communicate directly with many other nodes, or if it serves as an intermediary point in communication among other nodes. *Degree centrality* is defined as the number of links incident upon a node. It is a straightforward and efficient metric; however, it is less relevant since a node having a few high influential neighbors may have much higher influence than a node having a larger number of less influential neighbors. A node with larger degree is likely to have higher influence than a node with smaller degree. However, in some cases, this method fails to identify influential nodes since it considers only very limited information. For example, as it is shown in Figure 1, although node 1 has the largest degree among all 15 nodes, the information, if it originates at node 1, may not spread the fastest or the most broadly since all neighbors of node 1 have a very low degree. In contrast, node 15 may be of higher influence although it has lower degree comparing with node 1.

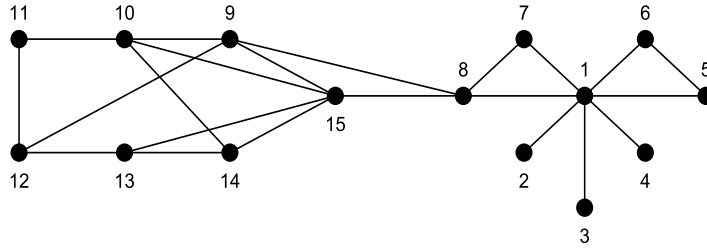


Figure 1. A graph with 15-vertices and 21-edges

Although some well-known global metrics such as *betweenness* and *closeness* centralities can give better results, due to the very high computational complexity, they are not easy to manage very large-scale online social webs [2], [5], [8]. For a network  $G = (V, E)$  with  $n = |V|$  nodes and  $m = |E|$  edges, time complexities of betweenness and close-

ness centralities are  $O(nm)$  and  $O(n^3)$ , respectively [10]. Closeness can be considered as a measure of how long it will spread information from a given node to other reachable nodes in the network. Calculation of closeness for simple graph types is important because if one can break a more complex network into smaller networks, then under some conditions the solutions for the optimization problem on the smaller networks can be combined to a solution for the optimization problem on the larger network and by calculating the closeness centrality for some real networks very good practical results can be achieved. Thus, we want to find exact values, upper bounds or lower bounds for metrics which are difficult to compute.

Graph theory has become one of the most powerful mathematical tools in the analysis and study of the architecture of a network. As usual, a network is described by an undirected simple graph. For example, if we want to occur a computer network with graph topologies the following correspondences are used: vertices represent computers and edges represent connections between computers.

We consider simple finite undirected graphs without loops and multiple edges in this paper. Let  $G = (V; E)$  be a graph with vertex set  $V$  and edge set  $E$ . We also specify the vertex set of  $G$  by  $V(G)$  and the edge set by  $E(G)$  instead of  $V$  and  $E$ , respectively. For vertices  $u$  and  $x$  of a graph  $G$ , the *open neighborhood* of  $u$  is  $N(u) = \{v \in V(G) | uv \in E(G)\}$  and  $N_x(u) = \{v \in V(G-x) | uv \in E(G-x)\}$ , the  $G-x$  graph corresponds to the graph from which the  $x$  vertex is removed from  $G$ . The *closed neighborhood* of  $u$  is  $N[u] = N(u) \cup \{u\}$ . For a set  $S \subseteq V(G)$ , its open neighborhood is the set  $N(S) = \bigcup_{v \in S} N(v)$ , and its closed neighborhood is the set  $N[S] = N(S) \cup S$ . A set  $S \subseteq V(G)$  is a dominating set of  $G$ , if  $N[S] = V(G)$ , that is a set  $S \subseteq V(G)$  is a *dominating set* if every vertex in  $V(G) - S$  is adjacent to at least one vertex in  $S$ . The *domination number*  $\gamma(G)$  is defined as the minimum cardinality of a dominating set of  $G$ . The *diameter* of  $G$ , denoted by  $diam(G)$  is the largest distance between two vertices in  $V(G)$ . The *degree*  $\deg_G(v)$  of a vertex  $v \in V(G)$  is the number of edges incident to  $v$ . The *maximum degree* of  $G$  is  $\Delta(G) = \max \{\deg_G(v) | v \in V(G)\}$  and the *minimum*

*degree* of  $G$  is  $\delta(G) = \min \{ \deg_G(v) \mid v \in V(G) \}$ . The *distance*  $d(u, v)$  between two vertices  $u$  and  $v$  in  $G$  is the length of a shortest path between them. If  $u$  and  $v$  are not connected, then  $d(u, v) = \infty$ , and for  $u = u$ ,  $d(u, u) = 0$  [11].

Our aim in this paper is to consider the computing of the closeness centrality of some splitting graphs. In Section 2, some definitions and results for closeness centrality are given. In Section 3, closeness centralities of some splitting graphs are determined.

## 2 Closeness centrality

In a network, even nodes with the same available resources, vary in their importance due to their different locations. It is reasonable to choose, firstly, the substrate nodes with the same available resources in a more important location. Furthermore, the importance of a node is more complex when the network is dynamically changing. The current states of all the elements in the global network determine the importance of a node. *Centrality* analysis provides effective methods for measuring the importance of nodes in a complex network and it has been widely used in complex network analysis [5].

*Closeness centrality* can be considered as a measure of how long it will spread information from a given node to other reachable nodes in the network. *Closeness centrality* of node  $v$  is defined as the reciprocal of the sum of geodesic distances to all other nodes of  $V(G)$ . This definition was modified by Dangalchev in [3], [4]. The *closeness centrality* of a graph is defined as

$$C(G) = \sum_{v \in V(G)} C(v),$$

where  $C(v)$  is the *closeness centrality* of a vertex  $v$ , and  $C(v) = \sum_{t \in (V-v)} \frac{1}{2d(v, t)}$  [4], [11]. Next, we give some results for closeness centrality.

**Theorem 1.** [1], [3], [4], [9] *The closeness centrality of*

- a) *the complete graph  $K_n$  with  $n$  vertices is  $C(K_n) = (n(n-1))/2$  ;*

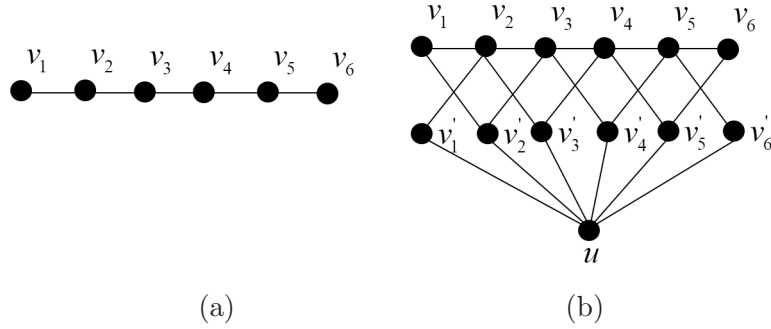


Figure 2. a) The graph  $P_6$  (b) The graph  $\mu(P_6)$

b) the star graph  $S_n$  with  $n$  vertices is  $C(S_n) = (n - 1)(n + 2)/4$ ;

c) the path  $P_n$  with  $n$  vertices is  $C(P_n) = 2n - 4 + 1/2^{n-2}$ ;

d) the cycle  $C_n$  with  $n$  vertices

$$C(C_n) = \begin{cases} n(2 - 3/2^{n/2}) & , \text{if } n \text{ is even;} \\ 2n(1 - 1/2^{(n-1)/2}) & , \text{if } n \text{ is odd;} \end{cases}$$

e) the wheel  $W_n$  with  $n + 1$  vertices is  $C(W_n) = n(n + 5)/4$ ;

f) the gear graph  $G_n$  with  $2n + 1$  vertices is  $C(G_n) = n(9n + 49)/16$ ;

g) the friendship graph  $f_n$  with  $2n + 1$  vertices is  $C(f_n) = n(n + 2)$ ;

h) the fan  $F_n$  with  $n$  vertices is  $C(F_n) = (n^2 + 3n - 6)/4$ .

**Definition 1.** [12] Let  $G$  be a graph with vertex set  $V(G) = \{v_1, v_2, \dots, v_n\}$ . The Mycielski graph of  $G$ , denoted  $\mu(G)$ , has for its vertex set  $V(\mu(G)) = \{v_1, v_2, \dots, v_n, v'_1, v'_2, \dots, v'_n, u\}$ . As for adjacency,  $v_i$  is adjacent with  $v_j$  in  $\mu(G)$  if and only if  $v_i$  is adjacent with  $v_j$  in  $G$ ,  $v_i$  is adjacent with  $v'_j$  in  $\mu(G)$  if and only if  $v_i$  is adjacent with  $v_j$  in  $G$ , and  $v'_i$  is adjacent with  $u$  in  $\mu(G)$  for all  $i = \overline{1, n}$ . We display the Mycielski graph  $\mu(P_6)$  in Figure 2.

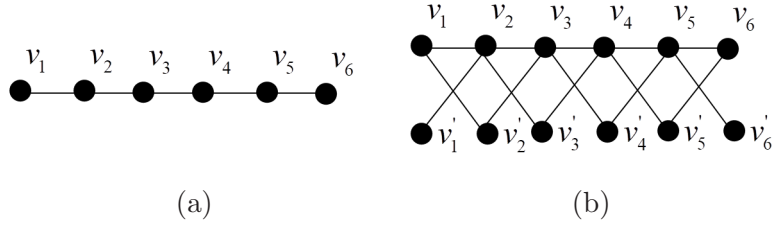


Figure 3. (a) The graph  $P_6$  (b) The graph  $S'(P_6)$

**Theorem 2.** [6] *The closeness centrality of Mycielski graphs of*

- a) *the cycle  $C_n$  with  $n$  vertices for  $n \geq 8$  is  $C(\mu(C_n)) = (9n^2 + 77n)/16$ ;*
- b) *the wheel  $K_n$  with  $n$  vertices for  $n \geq 3$  is  $C(\mu(K_n)) = (7n^2 + n)/4$ ;*
- c) *the star graph  $S_n$  with  $n$  vertices for  $4$  is  $C(\mu(S_n)) = (2n^2 + 5n - 3)/2$ ;*

**Definition 2.** [13] *For a graph  $G$  on vertices  $V(G) = \{v_1, v_2, \dots, v_n\}$  and edges  $E(G)$ , let splitting graph  $S'(G)$  be the graph on vertices and edges  $V(G) \cup V'(G) = \{v_1, v_2, \dots, v_n, v'_1, v'_2, \dots, v'_n\}$  and  $E(G) \cup \{v_i v'_j \mid v_i v_j \in E(G)\}$ , respectively. We display the Splitting graph  $S'(P_6)$  in Figure 3.*

In this paper, the following notations will be used throughout the article to make the proof of the given theorems understandable. Let the vertex-set of graph  $S'(G)$  be  $V(S'(G)) = V_1 \cup V_2$ , where:

$V_1$ : The set contains the vertices of the graph  $G$ , that is,  $V_1 = \{v_i \in V(G), 1 \leq i \leq n\}$ .

$V_2$ : The set contains the new vertices which are obtained by definition of splitting graph, that is,  $V_2 = \{v'_i \in V(G'), 1 \leq i \leq n\}$

**Lemma 1.** [11] *Let  $G$  be any connected graph of order  $n$  and size  $m$ . Then,*

$$\sum_{i=1}^n \deg_G(v_i) = 2m.$$

**Theorem 3.** *Let  $G$  be any connected graph of order  $n$  and size  $m$ . If  $\gamma(G) = 1$  and  $\delta(G) \geq 2$ , then  $C(S'(G)) = (2n^2 + 3m - n)/2$ .*

*Proof.* Since the structure of the splitting graph  $S'(G)$  and definition of the domination number, the distance between two vertices is at most 2. Let  $v_i \in V(G)$ . Clearly, the distance from the vertex  $v_i$  to  $2\deg_G(v_i)$ -vertices is 1, similarly the distance from the vertex  $v_i$  to  $((2n - 1) - 2\deg_G(v_i))$ -vertices is 2. Thus we get

$$C(v_i) = 2\deg_G(v_i) 2^{-1} + (2n - 1 - 2\deg_G(v_i)) 2^{-2}$$

for every vertices of  $v_i \in V_1$ , where  $i = \overline{1, n}$ .

Similarly, we get

$$C(v'_i) = \deg_G(v'_i) 2^{-1} + (2n - 1 - \deg_G(v'_i)) 2^{-2}$$

for every vertices of  $v'_i \in V_2$ , where  $i = \overline{1, n}$ .

Thus,

$$\begin{aligned} C(S'(G)) &= \sum_{i=1}^n C(v_i) + \sum_{i=1}^n C(v'_i) \\ &= \sum_{i=1}^n \deg_G(v_i) + \frac{1}{4} \sum_{i=1}^n (2n - 1 - 2\deg_G(v_i)) + \frac{1}{2} \sum_{i=1}^n \deg_G(v'_i) \\ &\quad + \frac{1}{4} \sum_{i=1}^n (2n - 1 - \deg_G(v'_i)). \end{aligned}$$

By Lemma 1, we have

$$C(S'(G)) = 2m + \frac{2n^2 - n}{2} - \frac{2m}{4} = \frac{2n^2 + 3m - n}{2}.$$

□

**Corollary 1.** *Let  $G$  be any connected graph of order  $n$  and size  $m$ . If  $\gamma(G) = 1$ , then  $C(S'(G)) \geq C(S'(S_n))$ .*

*Proof.* Star graph  $S_n$  with  $n$  vertices provides the requirement of the theorem.  $\gamma(S_n) = 1$  and star graph  $S_n$  has  $(n - 1)$ -vertices with degree 1. Thus, we get lower bound of the  $C(S'(G))$  of any graph  $G$  whose domination number is 1. As a result,  $C(S'(G)) \geq C(S'(S_n))$  is obtained. □

**Corollary 2.** *Let  $G$  be any connected graph of order  $n$ . Then,  $C(\mu(G) - \{u\}) = C(S'(G))$ , where the vertex  $u$  is the vertex  $u$  of the definition of the Mycielski graph  $\mu(G)$ .*

*Proof.* It is clear. □

### 3 Calculation of closeness centrality of some splitting graphs

In this section, we consider the *closeness centrality* of the splitting graphs  $S'(G)$  when  $G$  is a specified family of graphs.

**Theorem 4.** *The closeness centrality of  $S'(C_n)$  is*

$$C(S'(C_n)) = \begin{cases} n(31/4 - 12/2^{n/2}) & , \text{if } n \text{ is even} \\ n(31/4 - 8/2^{(n-1)/2}) & , \text{if } n \text{ is odd.} \end{cases}$$

*Proof.* We have two cases depending on the vertices of  $S'(C_n)$ :

**Case1.** If  $n$  is even, then we have also two cases:

**Case1.1.** For any vertex of  $v_i$  ( $i = \overline{1, n}$ ) of  $V_1$  in the graph  $S'(C_n)$ . Hence, the closeness centrality of subgraph  $S'(C_n) \setminus V_2$  is

$$C(v_i) = \sum_{\substack{v_j \neq v_i \\ v_j \in V_1}} 2^{-d(v_i, v_j)} + \sum_{\substack{v'_j \neq v_i \\ v'_j \in V_2}} 2^{-d(v_i, v'_j)}, \quad (1)$$



where the first and second sums are denoted by  $C_1$  and  $C_2$ , respectively, that is  $C(v_i) = C_1 + C_2$ . Furthermore, let  $S_1$  be  $nC(v_i)$ . It is clear that  $C_1$  is equal to the closeness centrality of the path  $C_n$  with  $n$  vertices for  $n$  even by Theorem 2(a).

To calculate  $C_2$ , the closeness centrality of each vertex  $v_i$  of  $V_1$  is determined by the sum of the minimum distances from  $v_i$  to all the other vertices  $v'_i \in V_2$  in the graph  $S'(C_n)$ . Hence, we obtain

$$\begin{aligned} C_2 &= 2/2^1 + 3/2^2 + 2/2^3 + \dots + 1/2^{\text{diam}(C_n)} \\ &= 2/2^1 + 3/2^2 + 2/2^3 + \dots + 2/2^{(n/2)-1} + 1/2^{n/2} \\ &= 2 \sum_{j=1}^{\frac{n}{2}-1} 1/2^j + 1/2^2 + 1/2^{n/2}. \end{aligned} \quad (2)$$

To calculate the sum (1), we use the formula of the finite geometric series. As a result, we get,  $C_2 = 9/4 - 3/2^{n/2}$ .

Consequently, we get the following result:

$$\begin{aligned} S_1 &= n \left( 2 - 3/2^{n/2} \right) + n \left( 9/4 - 3/2^{n/2} \right) \\ &= 17n/4 - 6n/2^{n/2}. \end{aligned} \quad (3)$$

**Case1.2.** For any vertex of  $v'_i (i = \overline{1, n})$  of  $V_2$  in the graph  $S'(C_n)$ . Hence, the closeness centrality of subgraph  $S'(C_n) \setminus V_1$  is

$$C(v'_i) = \sum_{\substack{v_j \neq v'_i \\ v_j \in V_1}} 2^{-d(v'_i, v'_j)} + \sum_{\substack{v'_j \neq v'_i \\ v'_j \in V_2}} 2^{-d(v'_i, v'_j)}, \quad (4)$$

where the first and second sums are denoted by  $C'_1$  and  $C'_2$ , respectively, that is  $C(v'_i) = C'_1 + C'_2$ . Let  $S_2$  be  $nC(v'_i)$ . It is clear that  $C'_1$  is  $9/4 - 3/2^{n/2}$  by the Case 1.1. To calculate  $C'_2$ , we consider only the distance  $d(v'_i, v'_j)$  between  $v'_i$  and  $v'_j$  in the graph  $S'(C_n)$ . Then we get  $C'_2$

$$\begin{aligned}
 C'_2 &= 2/2^2 + 4/2^3 + 2/2^4 + 2/2^5 + \dots \\
 &\quad + 2/2^{\text{diam}(C_n)-1} + 1/2^{\text{diam}(C_n)} \\
 &= 2/2^2 (1 + 1/2^1 + 1/2^2 + \dots \\
 &\quad + 1/2^{(n/2)-3}) + 2/2^3 + 1/2^{n/2}. \quad (5)
 \end{aligned}$$

To calculate the sum (5), we use the formula of the finite geometric series. So, we get  $C'_2 = 5/4 - 3/2^{n/2}$ . Consequently, we get the following result:

$$\begin{aligned}
 S_2 &= n \left( 9/4 - 3/2^{n/2} \right) + n \left( 5/4 - 3/2^{n/2} \right) \\
 &= 7n/2 - 6n/2^{n/2}. \quad (6)
 \end{aligned}$$

Clearly,  $C(S'(C_n)) = S_1 + S_2$ . By summing (3) and (6), we obtain the closeness centrality of  $S'(C_n)$  for  $n$  is even as follows:

$$\begin{aligned}
 C(S'(C_n)) &= 17n/4 - 6n/2^{n/2} + 7n/2 - 6n/2^{n/2} = \\
 &= n \left( 31/4 - 12/2^{n/2} \right).
 \end{aligned}$$

**Case2.** If  $n$  is odd, then the proof is similar to that of  $n$  is even. Hence, by using the Theorem 1.(d), the closeness centrality for each vertex  $v_i$  ( $i = \overline{1, n}$ ) of  $V_1$  is  $C(v_i) = 4 \left( 1 - 1/2^{(n-1)/2} \right) + 1/4$ , and also the closeness centrality for each vertex  $v'_i$  ( $i = \overline{1, n}$ ) of  $V_2$  is  $C(v'_i) = 7/2 - 4/2^{(n-1)/2}$ . Consequently, the closeness centrality of  $S'(C_n)$  for  $n$  is odd,

$$C(S'(C_n)) = nC(v_i) + nC(v'_i) = n \left( 31/4 - 8/2^{(n-1)/2} \right).$$

□

**Theorem 5.** *The closeness centrality of  $S'(P_n)$  is*

$$C(S'(P_n)) = 4(2n - 4 + 1/2^{n-2}) - (n - 3)/4.$$

*Proof.* The proof is similar to that of Theorem 4. Hence, similarly we have two cases depending on the vertices of  $S'(P_n)$ :

**Case1.** For any vertex of  $v_i (i = \overline{1, n})$  of  $V_1$  in the graph  $S'(P_n)$ . Hence, the closeness centrality of subgraph  $S'(P_n) \setminus V_2$  is as like the equation (1), where the first and second sums are denoted by  $C_1$  and  $C_2$ , respectively, that is  $C(v_i) = C_1 + C_2$ . Furthermore, let  $S_1$  be the sum of the all  $C(v_i)$ , where  $i = \overline{1, n}$ .

The value of  $C_2$ , due to the structure of the  $S'(P_n)$  graph, is  $C_2 = \sum_{\substack{v_j \neq v_i \\ v_j \in V_1}} 2^{-d(v_i, v_j)} + n/2^2$ . Thus, the closeness centrality of each vertex  $v_i$  in  $S'(P_n)$  is

$$C(v_i) = 2 \sum_{\substack{v_j \neq v_i \\ v_j \in V_1}} 2^{-d(v_i, v_j)} + n/2^2. \quad (7)$$

It is clear that the equation (7) is equal to the closeness centrality of the path  $P_n$  with  $n$  vertices by Theorem 1.(c). Consequently, we get the following result:

$$S_1 = 2(2n - 4 + 1/2^{n-2}) + n/2^2. \quad (8)$$

**Case2.** For any vertex of  $v'_i (i = \overline{1, n})$  of  $V_2$  in the graph  $S'(P_n)$ . Hence, the closeness centrality of subgraph  $S'(P_n) \setminus V_1$  is as like the equation (4), where the first and second sums are denoted by  $C'_1$  and  $C'_2$ , respectively, that is  $C(v'_i) = C'_1 + C'_2$ . Furthermore, let  $S_2$  be the sum of the all  $C(v'_i)$ , where  $i = \overline{1, n}$ . By the Case 1, we have

$$C'_1 = (9n - 16)/4 + 1/2^{n-2}. \quad (9)$$

To calculate  $C'_2$ , we get

$$C'_2 = 2(n-2)/2^2 + 2(n-3)/2^3 + \dots + 2.2/2^{n-2} + 2.1/2^{n-1} + 2(n-2)/2^3 + 2/2^3.$$

By using Theorem 1.(c), we get

$$\begin{aligned} C'_2 &= (2n - 4 + 1/2^{n-2}) - 2(n - 1)/2^1 + (n - 2)/2^2 + 1/2^2 \\ &= (2n - 4 + 1/2^{n-2}) - 3(n - 1)/4. \end{aligned} \quad (10)$$

Consequently, we get the following result:

$$\begin{aligned} S_2 &= (9n - 16)/4 + 1/2^{n-2} + (2n - 4 + 1/2^{n-2}) - 3(n - 1)/4 \\ &= 2(2n - 4 + 1/2^{n-2}) + (3 - 2n)/4. \end{aligned} \quad (11)$$

Clearly,  $C(S'(P_n)) = S_1 + S_2$ . By summing (8) and (11), we obtain the closeness centrality of  $C(S'(P_n))$  as follows:

$$C(S'(P_n)) = 4(2n - 4 + 1/2^{n-2}) + (n - 3)/4.$$

□

**Theorem 6.** *The closeness centrality of  $S'(W_n)$  is*

$$C(S'(W_n)) = (2n^2 + 9n + 1)/2.$$

*Proof.* The proof is similar to that of Theorem 4. The wheel  $W_n$  with  $n + 1$  vertices contains an  $n$ -cycle and a central vertex  $c$  that is adjacent to all vertices of the cycle. Then, we have  $\deg_G(v_c) = n$ . Hence, we have two cases depending on the vertices of  $S'(W_n)$ :

**Case1.** For any vertex of  $v_i (i = \overline{1, n + 1})$  of  $V_1$  in the graph  $S'(W_n)$ . Clearly we have  $|N(v_i)| = 6$ , except the central vertex  $v_c$  of  $V_1$ . Thus, the closeness centrality of subgraph  $S'(W_n) \setminus V_2$  is as like the equation (1), where the first and second sums are denoted by  $C_1$  and  $C_2$ , respectively, that is  $C(v_i) = C_1 + C_2$ . Furthermore, let  $S_1$  be the sum of the all  $C(v_i)$ , where  $i = \overline{1, n + 1}$ . It is clear that  $C_1$  is equal to the closeness centrality of the path  $W_n$  by Theorem 2(b). To calculate  $C_2$ , we have two cases depending on the vertices of survival subgraph  $S'(W_n) \setminus V_2$ :

**Case1.1.** Let  $v_i$  be a vertex of an  $n$ -cycle. Since  $N(v_i) = \{v_m, v_t, v_c, v'_m, v'_t, v'_c\}$ , where  $v_m, v_t$  are the vertices of an  $n$ -cycle and  $v_c$  is the central vertex of  $S'(W_n) \setminus V_2 = W_n$ , also  $v_i$  is adjacent to  $v'_m, v'_t$  and  $v'_c$  in  $V_2$ . Each of these vertices,  $v'_m, v'_t$  and  $v'_c$ , is the vertex in the copy  $S'(W_n)$ . Since  $d(v_i, v'_m) = d(v_i, v'_t) = d(v_i, v'_c) = 1$ ,  $v_i$  is at distance 2 to other  $n - 2$  vertices of  $V_2$ . Thus,

$$\begin{aligned} C(v_i) &= \sum_{\substack{v'_j \neq v_i \\ v'_j \in V_2}} 2^{-d(v_i, v'_j)} = 3/2^1 + (n - 2)/2^2 \\ &= (n + 4)/4. \end{aligned} \tag{12}$$

**Case1.2.** If  $v_c$  is the central vertex of  $S'(W_n) \setminus V_2 = W_n$ , then  $|N(v_c)| = n$ .  $N_{v_1}(c) = \{v'_1, v'_2, \dots, v'_n\}$ , where  $v'_i \in V_2$ , ( $i = \overline{1, n}$ ). So, there remains only one vertex  $v'_c$  in  $G'$  is at distance 2 from  $v_c$ . Thus, we have

$$\begin{aligned} C(v_c) &= \sum_{\substack{v'_j \neq v_c \\ v'_j \in V_2}} 2^{-d(v_c, v'_j)} = n/2^1 + 1/2^2 = n/2 + 1/4 \\ &= (2n + 1)/4. \end{aligned} \tag{13}$$

By Summing (12) and (13), we have the value of  $C_2$ , that is  $C_2 = n((n + 4)/4) + (2n + 1)/4$ . Consequently, we get the following result:

$$\begin{aligned} S_1 &= n(n + 5) + n(n + 4)/4 + (2n + 1)/4 \\ &= (2n^2 + 11n + 1)/4. \end{aligned} \tag{14}$$

**Case2.** For any vertex of  $v'_i \in V_2$  ( $i = \overline{1, n + 1}$ ) in the graph  $S'(W_n)$ . Clearly the vertices of  $V_2$  except  $v'_c$ ,  $|N(v'_i)| = 3$ , ( $i = \overline{1, n}$ )

and  $|N(v'_c)| = n$ . Hence, the closeness centrality of subgraph  $S'(W_n) \setminus V_1$  is as like the equation (4), where the first and second sums are denoted by  $C'_1$  and  $C'_2$ , respectively, that is  $C(v'_i) = C'_1 + C'_2$ . Furthermore, let  $S_2$  be the sum of the all  $C(v'_i)$ , where  $i = \overline{1, n+1}$ .

It is clear that the value of  $C'_1$  is  $C_2$  in Case1, that is  $C'_1 = n((n+4)/4) + (2n+1)/4$ . So, we must calculate  $C'_2$ . Since each of the vertices in  $V_2$  is at distance 2 from  $v'_i$ , we get  $n/2^2$ . Consequently, we get the following result:

$$\begin{aligned} S_2 &= n(n+4)/4 + (2n+1)/4 + (n+1)(n/4) \\ &= (n^2 + 4n + 2n + 1 + n^2 + n)/4 \\ &= (2n^2 + 7n + 1)/4. \end{aligned} \tag{15}$$

Clearly,  $C(S'(W_n)) = S_1 + S_2$ . By summing (14) and (15), we obtain the closeness centrality of  $C(S'(W_n))$  as follows:

$$\begin{aligned} C(S'(W_n)) &= (2n^2 + 11n + 1)/4 + (2n^2 + 7n + 1)/4 = \\ &= (2n^2 + 9n + 1)/2. \end{aligned}$$

□

**Theorem 7.** *The closeness centrality of  $S'(S_n)$  is  $C(S'(S_n)) = (4n^2 + 11n + 2)/4$ .*

*Proof.* We have two cases depending on the vertices  $S'(S_n)$ :

**Case1.** For any vertex of  $v_i \in V_1$  in the graph  $S'(S_n)$ . The vertices of  $V_1$  are of two kinds:  $v_c$  and  $v_i (i = \overline{1, n})$ . The vertex  $v_c$  will be referred to as central vertex and the vertex  $v_i$  – as minor vertex. For the central vertex  $v_c$ , clearly it is exactly adjacent to  $2n$  vertices except  $v'_c$  of  $S'(S_n)$ . Thus  $|N(v_c)| = 2n$ . The vertex  $v_c$  is at distance 2 to other remaining one vertex  $v'_c$  of  $S'(S_n)$ . Then, the closeness centrality for the vertex  $v_c$  is

$$\begin{aligned} C(v_c) &= \sum_{v_i \neq v_c} 2^{-d(v_i, v_c)} = 2n/2^1 + 1/2^2 \\ &= (4n + 1)/4. \end{aligned} \tag{16}$$

For minor vertex  $v_i$  of  $V_1$   $i = \overline{1, n}$ .  $v_i$  is exactly adjacent to 2 vertices of  $S'(S_n)$ , named  $v_c$  and  $v'_c$ . Thus  $|N(v_i)| = 2$ . The other remaining  $n + (n - 1)$  vertices are at distance 2 from  $v_i$ . Hence, we get:

$$C(v_i) = 2/2^1 + (2n - 1)/2^2 = (2n + 3)/4. \quad (17)$$

Let  $S_1$  be the sum of the all  $C(v_i)$ , where  $i = \overline{1, n+1}$ . Then, we get the following:

$$\begin{aligned} S_1 &= (4n + 1)/4 + n((2n + 3)/4) \\ &= (2n^2 + 7n + 1)/4. \end{aligned} \quad (18)$$

**Case2.** For any vertex of  $v'_i \in V_2$  in the graph  $S'(S_n)$ . For the central vertex  $v'_c$  of  $V_2$ ,  $v'_c$  is adjacent to  $n$  minor vertices of  $V_1$  in  $S'(S_n)$ . Thus,  $|N(v'_c)| = n$ . The other remaining  $n$  minor vertices of  $V_2$  and central vertex  $v_c$  of  $V_1$  are at distance 3 and 2, respectively, from  $v'_c$ . Hence, we get:

$$C(v'_c) = n/2^1 + n/2^3 + 1/2^2 = (5n + 2)/8. \quad (19)$$

For minor vertex  $v'_i$  of  $V_2$   $i = \overline{1, n}$ .  $v'_i$  is adjacent to one central vertex  $v_c$  of  $V_1$  in  $S'(S_n)$ . Thus  $|N(v'_i)| = 1$ . The other remaining  $2n - 1$  vertices of  $S'(S_n)$  and central vertex  $v'_c$  of  $V_2$  are at distance 2 and 3, respectively from  $v'_i$ . Hence, we get:

$$C(v'_i) = 1/2^1 + (2n - 1)/2^2 + 1/2^3 = (4n + 3)/8. \quad (20)$$

Let  $S_2$  be the sum of the all  $C(v'_i)$ , where  $i = \overline{1, n+1}$ . Then, we get the following:

$$\begin{aligned} S_2 &= (5n + 2)/8 + n((4n + 3)/8) \\ &= (4n^2 + 8n + 2)/8. \end{aligned} \quad (21)$$

Clearly,  $C(S'(S_n)) = S_1 + S_2$ . By summing (18) and (21), we obtain the closeness centrality of  $C(S'(S_n))$  as follows:

$$\begin{aligned} C(S'(S_n)) &= (2n^2 + 7n + 1)/4 + (4n^2 + 8n + 2)/8 \\ &= (4n^2 + 11n + 2)/4. \end{aligned}$$

□

**Theorem 8.** *The closeness centrality of  $S'(K_n)$  is  $C(S'(K_n)) = (7n^2 - 5n)/4$ .*

*Proof.* We have two cases depending on the vertices  $S'(K_n)$ :

**Case1.** For any vertex of  $v_i \in V_1$  in the graph  $S'(K_n)$ . Then  $|N(v_i)| = 2(n-1)$ . The vertex  $v'_i$  is a copy of the vertex  $v_i$  in  $S'(K_n)$ . Since the vertex  $v_i$  is adjacent to every vertex except  $v'_i$  in  $S'(K_n)$ , the vertex  $v'_i$  is at distance 2 from  $v_i$ . Thus, the closeness centrality for the vertex  $v_i$  is

$$C(v_i) = \sum_{v_j \neq v_i} 2^{-d(v_i, v_j)} = 2(n-1)/2^1 + 1/2^2 = (4n-3)/4.$$

Let  $S_1$  be the sum of the all  $C(v_i)$ , where  $i = \overline{1, n}$ . Then, we get the following:

$$S_1 = n((4n-3)/4) = (4n^2 - 3n)/4. \quad (22)$$

**Case2.** For any vertex of  $v'_i \in V_2$  in the graph  $S'(K_n)$ . The vertex  $v'_i$  is at distance 1 to the  $n-1$  vertices in  $V_1 \setminus \{v_i\}$ . Therefore, we have  $|N(v'_i)| = n-1$ . The vertices of  $V_2$  and the vertex  $v_i$  are at distance 2 from  $v'_i$ . Thus, the closeness centrality for the vertex  $v'_i$  is

$$C(v'_i) = (n-1)/2 + (n-1+1)/2^2 = (n-1)/2 + n/4.$$

Let  $S_2$  be the sum of the all  $C(v'_i)$ , where  $i = \overline{1, n}$ . Then, we get the following:

$$S_2 = n((n-1)/2 + n/4) = (3n^2 - 2n)/4. \quad (23)$$

Clearly,  $C(S'(K_n)) = S_1 + S_2$ . By summing (22) and (23), we obtain the closeness centrality of  $C(S'(K_n))$  as follows:

$$\begin{aligned} C(S'(K_n)) &= (4n^2 - 3n)/4 + (3n^2 - 2n)/4 \\ &= (7n^2 - 5n)/4. \end{aligned}$$

□



## 4 Conclusion

In this article, we have studied the closeness centrality of various networks as a centrality measure. Closeness centrality helps us to know how long it will spread information from a given node to other reachable nodes in the network. We present comparisons between popular interconnection networks and splitting graphs of these below. These networks are complete graph  $K_{20}$ , path graph  $P_{20}$ , cycle graph  $C_{20}$ , star graph  $S_{20}$  and wheel graph  $W_{20}$ . The splitting graphs of these networks were considered to measure how far information can spread throughout the graph. The closeness centrality values of the above graphs are shown in Table 1.

Table 1. The closeness centrality values of some graphs.

Graph $G$	$C(G)$	Splitting graph $S'(G)$	$C(S'(G))$
$K_{20}$	190	$S'(K_{20})$	675
$P_{20}$	36	$S'(P_{20})$	139,75
$C_{20}$	39,94	$S'(C_{20})$	155,77
$S_{20}$	104,5	$S'(S_{20})$	455,5
$W_{20}$	125	$S'(W_{20})$	490,5

By using Table 1, we say that the graphs  $S'(G)$  are better than the graph  $G$ . Calculation of *closeness centrality* for graph types considered in the article is important because if one can break a more complex network into smaller networks, then under some conditions the solutions for the optimization problem on the smaller networks can be combined to a solution for the optimization problem on the larger network. Therefore, designers for choosing the appropriate networks can use these results.

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Vecdi Aytac, Tufan Turaci

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Vecdi Aytac  
Ege University, Engineering Faculty, Computer Engineering Dept.  
Bornova-IZMIR-TURKEY  
Phone:+90 232 311 53 24  
E-mail: [vecdi.aytac@ege.edu.tr](mailto:vecdi.aytac@ege.edu.tr)

Tufan Turaci  
Karabük University, Science Faculty, Mathematics Dept.  
Karabük-TURKEY  
Phone:+90 370 433 83 74 / 1229  
E-mail: [tufanturaci@karabuk.edu.tr](mailto:tufanturaci@karabuk.edu.tr)