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Degree subtraction eigenvalues and energy of graphs

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Abstract

The degree subtraction matrix $DS(G)$ of a graph G is introduced, whose (j,k)-th entry is $d_G(v_j) - d_G(v_k)$, where $d_G(v_j)$ is the degree of a vertex v_j in G. If G is a non-regular graph, then $DS(G)$ has exactly two nonzero eigenvalues which are purely imaginary. Eigenvalues of the degree subtraction matrices of a graph and of its complement are the same. The degree subtraction energy of G is defined as the sum of absolute values of eigenvalues of $DS(G)$ and we express it in terms of the first Zagreb index.

Keywords: Degree of a vertex, degree subtraction matrix, eigenvalues, energy, first Zagreb index.

1 Introduction

In the study of spectral graph theory, we use the spectrum of certain matrices associated with the graph, such as the adjacency matrix, Laplacian matrix and other related matrices. Some useful information about the graph can be obtained from the spectrum of these various matrices [4], [5].

The ordinary energy of a graph G is defined as the sum of the absolute values of the eigenvalues of its adjacency matrix [10]. It is closely related with the total π-electron energy of molecules [13]. This motivates the researchers to introduce different matrices associated with the graph and study the various energies. Several graph energies, such as, Laplacian energy [15], distance energy [16], Randić energy [8], [17], skew energy [1], [20], incidence energy [8], degree sum energy [21], distance-based energies [9], [19], [22] etc. have been introduced to study the properties of graphs.

In this paper we introduce the degree subtraction matrix of a graph and study the eigenvalues and energy, related to this matrix.

Let G be a simple graph without loops and multiple edges on n vertices and m edges. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ be the vertex set and $E(G)$ be the edge set of G. The edge between the vertices u and v is denoted by uv. The *degree* of a vertex v_j in G is the number of edges incident to it and is denoted by $d_j = d_G(v_j)$. If the degrees of all vertices of a graph are the same, then the graph is called a *regular graph*. The *degree subtraction matrix* (DS-matrix) of a graph G is a square matrix of order n, defined as $DS(G) = [d_{jk}]$, where

$$d_{jk} = \begin{cases} d_G(v_j) - d_G(v_k) & \text{if } j \neq k \\ 0 & \text{if } j = k. \end{cases}$$

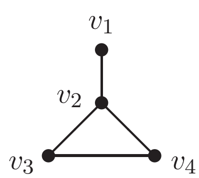
Then *DS-polynomial* of a graph G is the characteristic polynomial of degree subtraction matrix of G and is donoted by $\phi(G : \eta)$. That is $\phi(G : \eta) = \det(\eta I_n - DS(G))$, where I_n is an identity matrix of order n. The roots of the equation $\phi(G : \eta) = 0$ are called the *DS-eigenvalues* of G and they are labeled as $\eta_1, \eta_2, \ldots, \eta_n$. Since $DS(G)$ is a skew symmetric matrix, its eigenvalues are purely imaginary or zero. Two graphs are said to be *DS-cospectral* if they have the same DS-eigenvalues. The *DS-energy* of a graph G, denoted by $E_{DS}(G)$ is difined as

$$E_{DS}(G) = \sum_{j=1}^{n} |\eta_j|. \tag{1}$$

The Eq. (1) is in full analogy with the *ordinary graph energy* defined as [10]

$$E_{\pi}(G) = \sum_{j=1}^{n} |\lambda_j|,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of the adjacency matrix of G. Details on graph energies can be found in the books [12], [18] and the references cited therein.



$$DS(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & -2 & -1 & -1 \\ 2 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \end{array}$$

Figure 1. Graph and its DS-matrix

The DS-polynomial of a graph given in Fig. 1 is $\phi(G : \eta) = \eta^4 + 8\eta^2$ and the DS-eigenvalues are $\mathbf{i}2\sqrt{2}$, 0, 0, $-\mathbf{i}2\sqrt{2}$, where $\mathbf{i} = \sqrt{-1}$. Therefore, $E_{DS}(G) = 4\sqrt{2}$.

The *first Zagreb index* is defined as [14]:

$$M_1 = M_1(G) = \sum_{u \in V(G)} [d_G(u)]^2 = \sum_{uv \in E(G)} [d_G(u) + d_G(v)] \,.$$

The first Zagreb index is one of the most studied degree-based topological index. For details, see the recent surveys [2], [3] and the references cited therein.

2 DS-eigenvalues

We need the following Lemma.

Lemma 1. [5] *If Q is a nonsingular square matrix, then*

$$\begin{vmatrix} M & N \\ P & Q \end{vmatrix} = |Q||M - NQ^{-1}P|.$$

Theorem 1. *Let G be a graph having n vertices, m edges and first Zagreb index $M_1(G)$. Then the DS-polynomial of G is*

$$\phi(G : \eta) = \eta^n + (nM_1(G) - 4m^2)\eta^{n-2}. \tag{2}$$

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of G and let $d_G(v_j) = d_j$ be the degree of a vertex v_j in G, $j = 1, 2, \ldots, n$. Then the characteristic polynomial of DS-matrix of G is

$$\begin{aligned}\phi(G:\eta) &= \det(\eta I - DS(G)) \\ &= \begin{vmatrix} \eta & -d_1+d_2 & -d_1+d_3 & \cdots & -d_1+d_n \\ -d_2+d_1 & \eta & -d_2+d_3 & \cdots & -d_2+d_n \\ -d_3+d_1 & -d_3+d_2 & \eta & \cdots & -d_3+d_n \\ \vdots & & & \vdots & \\ -d_n+d_1 & -d_n+d_2 & -d_n+d_3 & \cdots & \eta \end{vmatrix}. \quad (3)\end{aligned}$$

Subtract the first row from the rows $2, 3, \ldots, n$ of (3) to obtain (4).

$$\begin{vmatrix} \eta & -d_1+d_2 & -d_1+d_3 & \cdots & -d_1+d_n \\ -d_2+d_1-\eta & \eta+d_1-d_2 & -d_2+d_1 & \cdots & -d_2+d_1 \\ -d_3+d_1-\eta & -d_3+d_1 & \eta+d_1-d_3 & \cdots & -d_3+d_1 \\ \vdots & & & \vdots & \\ -d_n+d_1-\eta & -d_n+d_1 & -d_n+d_1 & \cdots & \eta+d_1-d_n \end{vmatrix}. \quad (4)$$

Subtract the first column from columns $2, 3, \ldots, n$ of (4) to obtain (5).

$$\begin{vmatrix} \eta & -d_1+d_2-\eta & -d_1+d_3-\eta & \cdots & -d_1+d_n-\eta \\ -d_2+d_1-\eta & 2\eta & \eta & \cdots & \eta \\ -d_3+d_1-\eta & \eta & 2\eta & \cdots & \eta \\ \vdots & & & \vdots & \\ -d_n+d_1-\eta & \eta & \eta & \cdots & 2\eta \end{vmatrix}. \quad (5)$$

Subtract the second column from columns $3, 4, \ldots, n$ in (5) to obtain (6).

$$\begin{vmatrix} \eta & -d_1+d_2-\eta & d_3-d_2 & \cdots & d_n-d_2 \\ -d_2+d_1-\eta & 2\eta & -\eta & \cdots & -\eta \\ -d_3+d_1-\eta & \eta & \eta & \cdots & 0 \\ \vdots & & & \vdots & \\ -d_n+d_1-\eta & \eta & 0 & \cdots & \eta \end{vmatrix}. \quad (6)$$

Add rows $3, 4, \ldots, n$ to the second row in (6) to obtain (7).

$$\begin{vmatrix} \eta & -d_1+d_2-\eta & d_3-d_2 & \cdots & d_n-d_2 \\ -2m+nd_1-(n-1)\eta & n\eta & 0 & \cdots & 0 \\ -d_3+d_1-\eta & \eta & \eta & \cdots & 0 \\ \vdots & & & \vdots & \\ -d_n+d_1-\eta & \eta & 0 & \cdots & \eta \end{vmatrix} \tag{7}$$

$$= \begin{vmatrix} M & N \\ P & Q \end{vmatrix},$$

where

$$M = \begin{bmatrix} \eta & -d_1+d_2-\eta \\ -2m+nd_1-(n-1)\eta & n\eta \end{bmatrix}_{2\times 2},$$

$$N = \begin{bmatrix} d_3-d_2 & d_4-d_2 & \cdots & d_n-d_2 \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{2\times(n-2)},$$

$$P = \begin{bmatrix} -d_3+d_1-\eta & \eta \\ -d_4+d_1-\eta & \eta \\ \vdots & \vdots \\ -d_n+d_1-\eta & \eta \end{bmatrix}_{(n-2)\times 2} \quad \text{and}$$

$$Q = \begin{bmatrix} \eta & 0 & \cdots & 0 \\ 0 & \eta & \cdots & 0 \\ \vdots & & \vdots & \\ 0 & 0 & \cdots & \eta \end{bmatrix}_{(n-2)\times(n-2)}.$$

By Lemma 1, and taking into account that $\sum\limits_{j=1}^{n} d_j = 2m$ and noting that

$X = -M_1(G) + 2m(d_1+d_2) - nd_1d_2 - 2m\eta + d_1\eta + d_2\eta + (n-2)d_2\eta$

and $Y = (2m - d_1 - d_2)\eta - (n-2)d_2\eta$, the Eq. (7) reduces to

$$
\begin{aligned}
& \phi(G:\eta) \\
= \; & \eta^{n-2}\left|\begin{bmatrix} \eta & -d_1+d_2-\eta \\ -2m+nd_1-(n-1)\eta & n\eta \end{bmatrix} - N\frac{1}{\eta}I_{n-2}P\right| \\
= \; & \eta^{n-2}\left|\begin{bmatrix} \eta & -d_1+d_2-\eta \\ -2m+nd_1-(n-1)\eta & n\eta \end{bmatrix} - \frac{1}{\eta}\begin{bmatrix} X & Y \\ 0 & 0 \end{bmatrix}\right| \\
= \; & \eta^n + (nM_1(G)-4m^2)\eta^{n-2}.
\end{aligned}
$$

□

Corollary 1. *Let G be a regular graph on n vertices. Then the DS-polynomial of G is*

$$\phi(G:\eta) = \eta^n.$$

By Theorem 1 we have the following result.

Theorem 2. *Let G be a graph having n vertices, m edges and first Zagreb index $M_1(G)$. Then the DS-eigenvalues of G are 0 ($n-2$ times) and $\pm\mathbf{i}\sqrt{nM_1(G)-4m^2}$, where $\mathbf{i}=\sqrt{-1}$.*

By Theorem 2 we observe that, if G is a non-regular graph, then it has exactly two non-zero DS-eigenvalues.

In the following, $\overline{G}$ denotes the complement graph of G.

Theorem 3. *If η_j, $j=1,2,\ldots,n$ are the DS-eigenvalues of G, then $-\eta_j$, $j=1,2,\ldots,n$ are the DS-eigenvalues of $\overline{G}$.*

Proof. For any vertex u of a graph G of order n, $d_{\overline{G}}(u) = n-1-d_G(u)$. This implies that $DS(\overline{G}) = -DS(G)$. Consequently, if the DS-eigenvalues of a graph G are η_j, $j=1,2,\ldots,n$, then the DS-eigenvalues of $\overline{G}$ are $-\eta_j$, $j=1,2,\ldots,n$. □

Theorem 4. *For any graph G, $\phi(\overline{G}:\eta) = \phi(G:\eta)$.*

Proof. *Case 1:* If G is regular, then $\overline{G}$ is also regular. Therefore by the Corollary 1, $\phi(\overline{G} : \eta) = \phi(G : \eta)$.
Case 2: If G is non-regular, then $\overline{G}$ also non-regular. Therefore by Theorem 2, both G and $\overline{G}$ have exactly two non-zero DS-eigenvalues. Further by Theorem 3, if η_j, $j = 1, 2, \ldots, n$ are the DS-eigenvalues of G, then $-\eta_j$, $j = 1, 2, \ldots, n$ are the DS-eigenvalues of $\overline{G}$. Also the sum of all DS-eigenvalues is zero. This implies that $\phi(\overline{G} : \eta) = \phi(G : \eta)$. □

Theorem 5. *Let G be a graph on n vertices and m edges. Let δ and Δ be the minimum and maximum vertex degrees of G respectively. Then for $j = 1, 2, \ldots, n$,*

$$n\delta - 2m \leq \eta_j \leq n\Delta - 2m.$$

Proof. If the vertices of G are labeled as $v_1, v_2, \ldots, v_n$, then the sum of the elements of j-th row in DS-matrix is $nd_j - 2m$, where $d_j = d_G(v_j)$. It is well known that the eigenvalues of any matrix lie between the minimum row sum and maximum row sum. Hence

$$\min\{nd_j - 2m\} \leq \eta_j \leq \max\{nd_j - 2m\}.$$

This implies

$$n\delta - 2m \leq \eta_j \leq n\Delta - 2m.$$

□

By Theorem 4, G and $\overline{G}$ are DS-cospectral graphs. By Theorem 2, if G_1 and G_2 are two different graphs having the same number of vertices and the same number of edges, and if $M_1(G_1) = M_1(G_2)$, then G_1 and G_2 are DS-cospectral.

3 DS-Energy

From Theorem 2 and by the definition of DS-energy via Eq. (1), we have the following theorem.

Theorem 6. *Let G be a graph having n vertices, m edges and first Zagreb index $M_1(G)$. Then*

$$E_{DS}(G) = 2\sqrt{nM_1(G) - 4m^2}. \tag{8}$$

By Theorem 3, we have the following corollary.

Corollary 2. *For any graph G, $E_{DS}(G) = E_{DS}(\overline{G})$.*

If G is a regular graph, then by the Corollary 1, $E_{DS}(G) = 0$.
For fixed n and m, the Eq. (8) depends on the nature of $M_1(G)$.

Theorem 7. *Let G_1 and G_2 be two different graphs having equal number of vertices and equal number of edges.*
(i) *If $M_1(G_1) > M_1(G_2)$, then $E_{DS}(G_1) > E_{DS}(G_2)$.*
(ii) *If $M_1(G_1) = M_1(G_2)$, then $E_{DS}(G_1) = E_{DS}(G_2)$.*

Let $p = \lfloor 2m/n \rfloor$. Then the first Zagreb index $M_1(G)$ satisfies the inequalities:

$$2(2p+1)m - p(p+1)n \leq M_1(G) \leq m\left(\frac{2m}{n-1} + n - 2\right). \tag{9}$$

The right hand side of the Eq. (9) is due to de Caen [7], whereas the left hand side inequality is due to Das [6].
Using Eqs. (8) and (9) we have the following result.

Theorem 8. *Let G be a graph with n vertices and m edges. Let $p = \lfloor 2m/n \rfloor$. Then*

$$\begin{aligned} & 2\sqrt{2(2p+1)mn - p(p+1)n^2 - 4m^2} \\ \leq\ & E_{DS}(G) \\ \leq\ & 2\sqrt{mn\left(\frac{2m}{n-1} + n - 2\right) - 4m^2}. \end{aligned}$$

Let S_n be the star and P_n be the path on n vertices. Among all n-vertex trees, the star S_n has maximum value and the path P_n has minimum value of the first Zagreb index [11]. If T_n is an n-vertex tree, different from the star and path, then $M_1(S_n) > M_1(T_n) > M_1(P_n)$. Using this result and Eq. (8) we have the following result.

Theorem 9. *If T_n is an n-vertex tree, different from the star S_n and path P_n, then $E_{DS}(S_n) > E_{DS}(T_n) > E_{DS}(P_n)$.*

If u and v are the adjacent vertices of G, then $G - uv$ is the graph obtained from G by removing the edge uv. If u and v are non-adjacent vertices of G, then the graph $G + uv$ is obtained from G by adding an edge uv.

Theorem 10. *Let G be a graph having n vertices, m edges and first Zagreb index $M_1(G)$. Let u, v, w be three distinct vertices of G such that u is adjacent to v and u is not adjacent to w. Let $H = G - uv + uw$. Then*

$$E_{DS}(H) = 2\sqrt{nM_1(G) - 4m^2 + 2n[d_G(w) - d_G(v) + 1]}. \tag{10}$$

Further,
(i) *if $d_G(w) - d_G(v) + 1 = 0$, then $E_{DS}(H) = E_{DS}(G)$*
(ii) *if $d_G(w) - d_G(v) + 1 > 0$, then $E_{DS}(H) > E_{DS}(G)$*
(iii) *if $d_G(w) - d_G(v) + 1 < 0$, then $E_{DS}(H) < E_{DS}(G)$.*

Proof. Let $d_1, d_2, \ldots, d_n$ be the degrees of the vertices of G. Without loss of generality, let $d_G(u) = d_1$, $d_G(v) = d_2$ and $d_G(w) = d_3$. Therefore, $d_H(u) = d_1$, $d_H(v) = d_2 - 1$ and $d_H(w) = d_3 + 1$. Hence by Theorem 6,

$$\begin{aligned} E_{DS}(H) &= 2\sqrt{nM_1(H) - 4m^2} \\ &= 2\sqrt{n\left[d_1^2 + (d_2-1)^2 + (d_3+1)^2 + \sum_{j=4}^{n}(d_G(v_j))^2\right] - 4m^2} \\ &= 2\sqrt{nM_1(G) - 4m^2 + 2n(d_3 - d_2 + 1)} \\ &= 2\sqrt{nM_1(G) - 4m^2 + 2n[d_G(w) - d_G(v) + 1]}. \end{aligned}$$

The results (i), (ii), and (iii) follow from Eqs. (8) and (10). ☐

Proof of the Theorem 11 is analogous to that of Theorem 10.

Theorem 11. *Let G be a graph having n vertices, m edges and first Zagreb index $M_1(G)$. Let u, v, w, x be the four distinct vertices of G such that u is adjacent to v and w is not adjacent to x. Let $H = G - uv + wx$. Then*

$$E_{DS}(H) = 2\sqrt{nM_1(G) - 4m^2 + 2n[d_G(w) + d_G(x) - d_G(u) - d_G(v) + 2]}. \tag{11}$$

Further,
(i) *if $d_G(w) + d_G(x) - d_G(u) - d_G(v) + 2 = 0$, then $E_{DS}(H) = E_{DS}(G)$*
(ii) *if $d_G(w) + d_G(x) - d_G(u) - d_G(v) + 2 > 0$, then $E_{DS}(H) > E_{DS}(G)$*
(iii) *if $d_G(w) + d_G(x) - d_G(u) - d_G(v) + 2 < 0$, then $E_{DS}(H) < E_{DS}(G)$.*

Theorem 12. *Let G be a graph having n vertices $v_1, v_2, \ldots, v_n$ and m edges and first Zagreb index $M_1(G)$. Let G' be the subgraph of G on $k \geq 1$ vertices $v_1, v_2, \ldots, v_k$, and m' edges. Let H be the graph obtained from G by removing the edges of G'. Then*

$$E_{DS}(H) = 2\sqrt{\begin{array}{l} nM_1(G) - 4m^2 - 2n\sum\limits_{j=1}^{k}[d_G(v_j)d_{G'}(v_j)] \\ +nM_1(G') + 4m'(2m - m') \end{array}}. \tag{12}$$

Further,
(i) *if $-2n\sum\limits_{j=1}^{k}[(d_G(v_j))(d_{G'}(v_j))] + nM_1(G') + 4m'(2m - m') = 0$, then $E_{DS}(H) = E_{DS}(G)$*
(ii) *if $-2n\sum\limits_{j=1}^{k}[(d_G(v_j))(d_{G'}(v_j))] + nM_1(G') + 4m'(2m - m') > 0$, then $E_{DS}(H) > E_{DS}(G)$*
(iii) *if $-2n\sum\limits_{j=1}^{k}[(d_G(v_j))(d_{G'}(v_j))] + nM_1(G') + 4m'(2m - m') < 0$, then $E_{DS}(H) < E_{DS}(G)$.*

Proof. Let $v_1, v_2, \ldots, v_k$ be the vertices of a subgraph G' of G, $k \geq 1$. Therefore $d_H(v_j) = d_G(v_j) - d_{G'}(v_j)$, for $j = 1, 2, \ldots, k$ and $d_H(v_j) = d_G(v_j)$, for $j = k+1, k+2, \ldots, n$. Also, if m' is the number of edges of G', then H has $m - m'$ edges. By Eq. (8)

$$
\begin{aligned}
E_{DS}(H) &= 2\sqrt{nM_1(H) - 4(m-m')^2} \\
&= 2\sqrt{\begin{array}{l} n\left[\sum\limits_{j=1}^{k}(d_G(v_j) - d_{G'}(v_j))^2 + \sum\limits_{j=k+1}^{n}[d_G(v_j)]^2\right] \\ -4(m-m')^2 \end{array}} \\
&= 2\sqrt{\begin{array}{l} n\left[\sum\limits_{j=1}^{n}[d_G(v_j)]^2 - 2\sum\limits_{j=1}^{k}[d_G(v_j)d_{G'}(v_j)]\right. \\ \left. + \sum\limits_{j=1}^{k}[d_{G'}(v_j)]^2\right] - 4(m^2 - 2mm' + m'^2) \end{array}} \\
&= 2\sqrt{\begin{array}{l} nM_1(G) - 4m^2 - 2n\sum\limits_{j=1}^{k}[(d_G(v_j))(d_{G'}(v_j))] \\ +nM_1(G') + 4m'(2m-m') \end{array}}\,.
\end{aligned}
$$

The results (i), (ii), and (iii) follow from the Eqs. (8) and (12). □

Corollary 3. *Let G be a graph having n vertices $v_1, v_2, \ldots, v_n$ and m edges and first Zagreb index $M_1(G)$. Let $e_1, e_2, \ldots, e_k$ be the k independent edges of G, $1 \le k \le \lfloor n/2 \rfloor$, where $e_j = v_{2j-1}v_{2j}$, $j = 1, 2, \ldots, k$. Let H be the graph obtained from G by removing its k independent edges e_j, $j = 1, 2, \ldots, k$. Then*

$$E_{DS}(H) = 2\sqrt{nM_1(G) - 4m^2 + 2n\left(k - \sum_{j=1}^{2k} d_G(v_j)\right) + 4k(2m-k)}.$$

Further,

(i) *if* $2n\left(k - \sum\limits_{j=1}^{2k} d_G(v_j)\right) + 4k(2m-k) = 0$, *then* $E_{DS}(H) = E_{DS}(G)$

(ii) *if* $2n\left(k - \sum\limits_{j=1}^{2k} d_G(v_j)\right) + 4k(2m-k) > 0$, *then* $E_{DS}(H) > E_{DS}(G)$

(iii) *if* $2n\left(k-\sum_{j=1}^{2k} d_G(v_j)\right)+4k(2m-k)<0$, *then* $E_{DS}(H)<E_{DS}(G)$
(iv) *if* $n=2k$, *then* $E_{DS}(H)=E_{DS}(G)$.

Proof. Follows from Theorem 12 by taking $G'=kK_2$, a k-matching. □

Corollary 4. *Let G be a graph having n vertices $v_1, v_2, \ldots, v_n$ and m edges and first Zagreb index $M_1(G)$. Let $V_k=\{v_1, v_2, \ldots, v_k\}$ be a k-element subset of the vertex set of the graph G, $k\geq 2$ such that every pair of vertices of V_k is adjacent in G. Let H be the graph obtained from G by deleting all the edges connecting pairs of vertices from V_k. Then*

$$E_{DS}(H)=2\sqrt{\begin{array}{l} nM_1(G)-4m^2+2(k-1)\left[2mk-n\sum_{j=1}^{k} d_G(v_j)\right] \\ +k(n-k)(k-1)^2 \end{array}}.$$

Further,
(i) *if* $2(k-1)\left[2mk-n\sum_{j=1}^{k} d_G(v_j)\right]+k(n-k)(k-1)^2=0$, *then* $E_{DS}(H)=E_{DS}(G)$
(ii) *if* $2(k-1)\left[2mk-n\sum_{j=1}^{k} d_G(v_j)\right]+k(n-k)(k-1)^2>0$, *then* $E_{DS}(H)>E_{DS}(G)$
(iii) *if* $2(k-1)\left[2mk-n\sum_{j=1}^{k} d_G(v_j)\right]+k(n-k)(k-1)^2<0$, *then* $E_{DS}(H)<E_{DS}(G)$
(iv) *if* $n=k$, *then* $E_{DS}(H)=E_{DS}(G)$.

Proof. Follows from Theorem 12 by taking $G'=K_k$, a complete graph on k vertices. □

Corollary 5. *Let G be a graph having n vertices $v_1, v_2, \ldots, v_n$ and m edges and first Zagreb index $M_1(G)$. Let C_k be the cycle of G, where* *the vertices $v_1, v_2, \ldots, v_k$ are on C_k, $k \geq 1$. Let H be the graph obtained from G by removing the edges of C_k. Then*

$$E_{DS}(H) = 2\sqrt{nM_1(G) - 4m^2 + 4n\left[k - \sum_{j=1}^{k} d_G(v_j)\right] + 4k(2m-k)}.$$

Further,

(i) *if* $4n\left[k - \sum_{j=1}^{k} d_G(v_j)\right] + 4k(2m-k) = 0$, *then* $E_{DS}(H) = E_{DS}(G)$

(ii) *if* $4n\left[k - \sum_{j=1}^{k} d_G(v_j)\right] + 4k(2m-k) > 0$, *then* $E_{DS}(H) > E_{DS}(G)$

(iii) *if* $4n\left[k - \sum_{j=1}^{k} d_G(v_j)\right] + 4k(2m-k) < 0$, *then* $E_{DS}(H) < E_{DS}(G)$

(iv) *if* $k = n$, *then* $E_{DS}(H) = E_{DS}(G)$.

Proof. Follows from Theorem 12 by taking $G' = C_k$, a cycle on k vertices. □

Algorithm: Computation of DS-energy using adjacency matrix.

1. **Start**
2. **Declare:** $\mathbf{A}[n][n]$, d$[n]$, r, s, n, m, $N = 0$, $S = 0$ as integers.
3. **Declare:** *Result* as floating point.
   Read n, $\mathbf{A}[r][s]$
4. Compute the degree of each vertex
   for $r = 1$ to n increment by 1
   d$[r] \longleftarrow 0$
   for $s = 1$ to n increment by 1
   d$[r] \longleftarrow$ d$[r]$ + $\mathbf{A}[r][s]$
   Display: Degree of vertex d$[r]$
   The square of degree of a vertex, d$[r]$ * d$[r]$

5. **Compute:** The sum of each row
   $S \longleftarrow S + \mathrm{d}[r]$.
   Sum of squares of each row sum as $N = N + \mathrm{d}[r] * \mathrm{d}[r]$
   and number of edges $m = S/2$.
6. **Display:** Sum of squares of each row sum N and number of edges m.
7. **Compute the Result**
   $Result = 2 * \mathrm{sqrt}(n * N - 4 * m * m)$
8. **Display:** the *Result*.
9. **Stop**

Terms:

n - Total number of vertices in a given graph.

m - Total number of edges in a given graph.

A - Adjacency Matrix.

d - Degree of a vertex.

In the above algorithm, the outer loop iterates r times and the inner loop iterates s times. Hence the statements inside the inner loop will be executed rs times. This means that the outer and inner loops are dependent on problem size n. Hence the time complexity of the algorithm is $O(n^2)$.

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