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Vertex weighted Laplacian Energy of union of graphs

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Abstract

The vertex weighted Laplacian energy with respect to the vertex weight w of a graph G with n vertices is defined as $LE_w(G) = \sum_{i=1}^{n} |\mu_i - \bar{w}|$, where $\mu_1, \mu_2, ..., \mu_n$ are the Laplacian eigenvalues of G and $\bar{w}$ is the average value of the weight w. In this paper, we derive upper and lower bounds of weighted Laplacian energy of union of k-number of connected disjoint graphs G_1, G_2,...,G_k and hence consider some particular cases.

Keywords: Eigenvalue, Energy (of graph), Laplacian energy, Topological index.

AMS Subject Classification: 05C05

1 Introduction

Let G be a non empty graph with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$. The degree of a vertex $v_i \in V(G)$ is the number of vertices adjacent with that vertex and is denoted by $d_G(v_i)$ for $i = 1, 2, ..., n$. Let $A = [a_{ij}]$ be the adjacency matrix of G. Let the eigenvalues of A be denoted by $\lambda_1, \lambda_2, ..., \lambda_n$ which are the eigenvalues of the graph G. Ivan Gutman in 1978 [1] introduced the energy of a graph which is defined as the sum of the absolute values of its eigenvalues and is denoted by E(G), so that

$$E(G) = \sum_{i=1}^{n} |\lambda_i|.$$

Let $D(G) = [d_{ij}]$ be the diagonal matrix associated with the graph G, where $d_{ii} = d_G(v_i)$ and $d_{ij} = 0$ if $i \neq j$. Define $L(G) = D(G) - A(G)$, where $L(G)$ is called the Laplacian matrix of G. Let $\mu'_1, \mu'_2, ..., \mu'_n$ be the Laplacian eigenvalues of G. Then the Laplacian energy of G is defined as [2]

$$LE(G) = \sum_{i=1}^{n} |\mu'_i - \frac{2m}{n}|.$$

Till date a very extensive study on graph energy and Laplacian graph energy can be found in literature. The interested reader can refer to the survey [3], [4], recent papers [5]–[11], and references cited therein. Sharafdini et al. in [12] introduced vertex weighted Laplacian energy of a graph with respect to a vertex weight w. For example, a vertex weight of a graph can be considered as degree of the vertices or eccentricity of the vertices. A graph is called w-regular if for any $u, v \in V(G)$, $w(v) = w(u)$. Let us consider a diagonal matrix of order n with respect to the weight w, $D_w(G) = diag\{w(v_1), w(v_2), ..., w(v_n)\}$. The adjacency matrix of G is denoted by $A(G) = [a_{ij}]$, where $a_{ij} = 1$ if and only if the vertices v_i and v_j are adjacent. Then, the matrix $L_w(G) = D_w(G) - A(G)$ is the weighted Laplacian matrix of G with respect to the vertex weight w. It is clear that, if the vertex weight is considered as degree of the vertices of G, then the matrix $L_w(G)$ is called Laplacian matrix of G. Similarly, if the vertex weight is equal to the eccentricity of the vertices, then we get the eccentricity version of the Laplacian energy introduced by the present author in [13]. Let μ_1 μ_2,..., μ_n be the eigenvalues of the weighted Laplacian matrix $L_w(G)$ with respect to some arbitrary vertex weight w. Then, the weighted Laplacian energy $LE_w(G)$ of G with respect to the vertex weight w is defined as

$$LE_w(G) = \sum_{i=1}^{n} |\mu_i - \bar{w}|,$$

where $\bar{w} = \frac{1}{n}\sum_{i=i}^{n} w(v_i)$. Clearly, $\sum_{i=1}^{n} \mu_i = n\bar{w}$.

Ramane et. al in [14] derived the bounds of Laplacian energy of union of two graphs where they generalized the result derived in [2]. In this paper, we further generalized the result derived in [14] by calculating upper and lower bounds of weighted Laplacian energy of union of k-number of connected disjoint graphs with respect to some particular vertex weight w.

2 Main Results

Let G_1, G_2,...,G_k be k-number of connected and disjoint graphs. Now let the vertex and edge sets of G_i for $i = 1, 2, ..., k$ be respectively denoted by V_i and E_i. Also, let $|V_i| = n_i$ and $|E_i| = m_i$ for $i = 1, 2, ..., k$. Then the union of k-number of graphs G_1, G_2,...,G_k denoted by $G_1 \bigcup G_2 \bigcup ... \bigcup G_k$ is a graph with vertex set $V_1 \bigcup V_2 \bigcup ... \bigcup V_k$ and edge set $E_1 \bigcup E_2 \bigcup ... \bigcup E_k$. Thus $G_1 \bigcup G_2 \bigcup ... \bigcup G_k$ has in total $n_1 + n_2 + ... + n_k$ vertices and $m_1 + m_2 + ... + m_k$ number of edges. Let us denote $G_1 \bigcup G_2 \bigcup ... \bigcup G_k$ by $\bigcup\limits_{i=1}^{k} G_i$.

Theorem 1. *Let G_1, G_2,...,G_k be k-connected disjoint graphs, then*

$$\begin{aligned}
& \sum_{i=1}^{k} LE_w(G_i) - \frac{\sum\limits_{i=1}^{k} n_i \sum\limits_{j=1, j\neq 1}^{k} n_j |\bar{w}_{G_i} - \bar{w}_{G_j}|}{\sum\limits_{i=1}^{k} n_i} \\
\leq\ & LE_w(\bigcup_{i=1}^{k} G_i) \\
\leq\ & \sum_{i=1}^{k} LE_w(G_i) + \frac{\sum\limits_{i=1}^{k} n_i \sum\limits_{j=1, j\neq 1}^{k} n_j |\bar{w}_{G_i} - \bar{w}_{G_j}|}{\sum\limits_{i=1}^{k} n_i}.
\end{aligned}$$

Proof. From definition, the vertex weighted Laplacian energy of union of k-number of graphs G_1, G_2,...,G_k is given by

$$LE_w(\bigcup_{i=1}^{k} G_i) = \sum_{i=1}^{\sum\limits_{i=1}^{k} n_i} |\mu_i(\bigcup_{i=1}^{k} G_i) - \bar{w}_{\bigcup\limits_{i=1}^{k} G_i}|,$$

where the average of the vertex weight of $\bigcup\limits_{i=1}^{k} G_i$ is given by

$$\begin{aligned}
\bar{w}_{\bigcup\limits_{i=1}^{k} G_i} &= \frac{1}{\sum\limits_{i=1}^{k} n_i}\Big[\sum_{i=1}^{n_1} w_{G_1}(v_i) + \sum_{i=1}^{n_2} w_{G_2}(v_i) + ... + \sum_{i=1}^{n_k} w_{G_k}(v_i)\Big] \\
&= \frac{1}{\sum\limits_{i=1}^{k} n_i}[n_1\bar{w}_{G_1} + n_2\bar{w}_{G_2} + ... + n_k\bar{w}_{G_k}] \\
&= \frac{\sum\limits_{i=1}^{k} n_i\bar{w}_{G_i}}{\sum\limits_{i=1}^{k} n_i}.
\end{aligned}$$

Now, since the weighted Laplacian spectrum of $\bigcup\limits_{i=1}^{k} G_i$ is the union of the weighted Laplacian spectra of G_1, G_2,...,G_k, we have

$$LE_w(\bigcup_{i=1}^{k} G_i) = \sum_{i=1}^{\sum\limits_{i=1}^{k} n_i} |\mu_i(\bigcup_{i=1}^{k} G_i) - \frac{\sum\limits_{i=1}^{k} n_i\bar{w}_{G_i}}{\sum\limits_{i=1}^{k} n_i}|$$

$$= \sum_{i=1}^{n_1} |\mu_i(\bigcup_{i=1}^{k} G_i) - \frac{\sum\limits_{i=1}^{k} n_i\bar{w}_{G_i}}{\sum\limits_{i=1}^{k} n_i}| + \sum_{i=1+n_1}^{n_1+n_2} |\mu_i(\bigcup_{i=1}^{k} G_i) - \frac{\sum\limits_{i=1}^{k} n_i\bar{w}_{G_i}}{\sum\limits_{i=1}^{k} n_i}|$$

$$+ ... + \sum_{i=n_i+1}^{\sum\limits_{i=1}^{k} n_i} |\mu_i(\bigcup_{i=1}^{k} G_i) - \frac{\sum\limits_{i=1}^{k} n_i\bar{w}_{G_i}}{\sum\limits_{i=1}^{k} n_i}|$$

$$
\begin{aligned}
&= \sum_{i=1}^{n_1} |\mu_i(G_1) - \frac{\sum\limits_{i=1}^{k} n_i \bar{w}_{G_i}}{\sum\limits_{i=1}^{k} n_i}| + \sum_{i=1}^{n_2} |\mu_i(G_2) - \frac{\sum\limits_{i=1}^{k} n_i \bar{w}_{G_i}}{\sum\limits_{i=1}^{k} n_i}| \\
&\quad +...+ \sum_{i=1}^{n_k} |\mu_i(G_k) - \frac{\sum\limits_{i=1}^{k} n_i \bar{w}_{G_i}}{\sum\limits_{i=1}^{k} n_i}| \\
&= \sum_{i=1}^{n_1} |\mu_i(G_1) - \bar{w}_{G_1} + \bar{w}_{G_1} - \frac{\sum\limits_{i=1}^{k} n_i \bar{w}_{G_i}}{\sum\limits_{i=1}^{k} n_i}| \\
&\quad + \sum_{i=1}^{n_2} |\mu_i(G_2) - \bar{w}_{G_2} + \bar{w}_{G_2} - \frac{\sum\limits_{i=1}^{k} n_i \bar{w}_{G_i}}{\sum\limits_{i=1}^{k} n_i}| \\
&\quad +...+ \sum_{i=1}^{n_k} |\mu_i(G_{n_k}) - \bar{w}_{G_{n_k}} + \bar{w}_{G_{n_k}} - \frac{\sum\limits_{i=1}^{k} n_i \bar{w}_{G_i}}{\sum\limits_{i=1}^{k} n_i}|. \qquad (1)
\end{aligned}
$$

To find upper bound, we can write from (1)

$$
\begin{aligned}
LE_w(\bigcup_{i=1}^{k} G_i) \quad &\leq \quad \sum_{i=1}^{n_1} |\mu_i(G_1) - \bar{w}_{G_1}| + n_1 |\bar{w}_{G_1} - \frac{\sum\limits_{i=1}^{k} n_i \bar{w}_{G_i}}{\sum\limits_{i=1}^{k} n_i}| \\
&\quad + \sum_{i=1}^{n_2} |\mu_i(G_2) - \bar{w}_{G_2}| + n_2 |\bar{w}_{G_2} - \frac{\sum\limits_{i=1}^{k} n_i \bar{w}_{G_i}}{\sum\limits_{i=1}^{k} n_i}|
\end{aligned}
$$

$$+ ... + \sum_{i=1}^{n_k} |\mu_i(G_k) - \bar{w}_{G_k}| + n_k|\bar{w}_{G_k} - \frac{\sum\limits_{i=1}^{k} n_i \bar{w}_{G_i}}{\sum\limits_{i=1}^{k} n_i}|$$

$$= \sum_{i=1}^{k} LE_w(G_i) + \frac{\sum\limits_{i=1}^{k} n_i \sum\limits_{j=1, j\neq i}^{k} n_j |\bar{w}_{G_i} - \bar{w}_{G_j}|}{\sum\limits_{i=1}^{k} n_i}. \qquad (2)$$

Similarly, to obtain lower bound, from (1) using the similar arguments, we have

$$LE_w(\bigcup_{i=1}^{k} G_i) \geq \sum_{i=1}^{n_1} |\mu_i(G_1) - \bar{w}_{G_1}| - n_1|\bar{w}_{G_1} - \frac{\sum\limits_{i=1}^{k} n_i \bar{w}_{G_i}}{\sum\limits_{i=1}^{k} n_i}|$$

$$+ \sum_{i=1}^{n_2} |\mu_i(G_2) - \bar{w}_{G_2}| - n_2|\bar{w}_{G_2} - \frac{\sum\limits_{i=1}^{k} n_i \bar{w}_{G_i}}{\sum\limits_{i=1}^{k} n_i}|$$

$$+ ... + \sum_{i=1}^{n_k} |\mu_i(G_k) - \bar{w}_{G_k}| - n_k|\bar{w}_{G_k} - \frac{\sum\limits_{i=1}^{k} n_i \bar{w}_{G_i}}{\sum\limits_{i=1}^{k} n_i}|$$

$$= \sum_{i=1}^{k} LE_w(G_i) - \frac{\sum\limits_{i=1}^{k} n_i \sum\limits_{j=1}^{k} n_j |\bar{w}_{G_i} - \bar{w}_{G_j}|}{\sum\limits_{i=1}^{k} n_i}. \qquad (3)$$

Combining, (2) and (3) we get the desired result as in Theorem 1. □

Next, as a special case we derive some additional relations for weighted Laplacian energy union of two graphs.

If G_i (i=1,2,...,k) is w_i-regular graph with respect to the parameter w, then $w_{G_i}(v_j) = w_i$ for $i = 1, 2, ..., k$ and $j = 1, 2, ..., n_i$. Thus we can write,

$$\bar{w}_{G_i} = \frac{1}{n_i}\sum_{j=1} n_i w_{G_i}(v_j) = \frac{1}{n_i} n_i w_i = w_i, \ for\ i = 1, 2, ..., k.$$

Therefore, using Theorem 1, we have the following result:

Corollary 1. *Let, G_i (i=1,2,...,k) be w_i-regular graph with respect to the parameter w, then $w_{G_i}(v_j) = w_i$ for $i = 1, 2, ..., k$ and $j = 1, 2, ..., n_i$, then*

$$\begin{aligned}
&\sum_{i=1}^{k} LE_w(G_i) - \frac{\sum\limits_{i=1}^{k} n_i |\sum\limits_{j=1}^{k} n_j(w_i - w_j)|}{\sum\limits_{i=1}^{k} n_i} \\
\leq\ & LE_w(\bigcup_{i=1}^{k} G_i) \\
\leq\ & \sum_{i=1}^{k} LE_w(G_i) + \frac{\sum\limits_{i=1}^{k} n_i |\sum\limits_{j=1}^{k} n_j(w_i - w_j)|}{\sum\limits_{i=1}^{k} n_i}.
\end{aligned}$$

Corollary 2. *Let, G_i (i=1,2,...,k) be w_i regular graph with respect to the parameter w, so that $w_1 = w_2 = ... = w_k$, then*

$$LE_w(\bigcup_{i=1}^{k} G_i) = \sum_{i=1}^{k} LE_w(G_i)$$

The above result can be considered for two regular graphs G_1 and G_2 as follows:

Corollary 3. *Let G_1 and G_2 be w_1 and w_2 -regular graph with respect to parameter w, then*

$$\begin{aligned} & LE_w(G_1) + LE_w(G_2) - \frac{2n_1n_2}{n_1+n_2}|w_1 - w_2| \\ \leq \; & LE_w(G_1 \cup G_2) \\ \leq \; & LE_w(G_1) + LE_w(G_2) + \frac{2n_1n_2}{n_1+n_2}|w_1 - w_2|. \end{aligned}$$

Next, if the vertex weight is considered as degree of vertices, then we get the similar results as in [14] from the above result.

Corollary 4. *Let G_1 and G_2 be two connected graphs with n_1, n_2 number of vertices and m_1 and m_2 number of edges, then if $\frac{2m_1}{n_1} > \frac{2m_2}{n_2}$, we have*

$$\begin{aligned} & LE(G_1) + LE(G_2) - \frac{4(m_1n_2 - m_2n_1)}{n_1+n_2} \\ \leq \; & LE_w(G_1 \cup G_2) \\ \leq \; & LE(G_1) + LE(G_2) + \frac{4(m_1n_2 - m_2n_1)}{n_1+n_2}. \end{aligned}$$

We know that, the eccentricity of a vertex in G is the largest distance from that vertex to any other vertex of G. Let, $\zeta(G)$ denote the sum of eccentricities of all the vertices of G, so that $\frac{\zeta(G)}{n}$ is the average vertex eccentricity of G. Then the eccentricity version of Laplacian energy of G is given by [13].

$$LE_\varepsilon(G) = \sum_{i=1}^{n} |\mu_i - \frac{\zeta(G)}{n}|.$$

In the following, we derive eccentricity version of Laplacian energy of union of two connected graphs G_1 and G_2 using Theorem 1.

Corollary 5. *Let G_1 and G_2 be two connected graphs with n_1, n_2 number of vertices and m_1 and m_2 number of edges, then if $\frac{\zeta(G_1)}{n_1} >$*

$\frac{\zeta(G_2)}{n_2}$, *we have*

$$\begin{aligned} & LE_\varepsilon(G_1)+LE(G_2)-\frac{4(n_2\zeta(G_1)-n_1\zeta(G_2))}{n_1+n_2} \\ \leq\ & LE_\varepsilon(G_1\cup G_2) \\ \leq\ & LE_\varepsilon(G_1)+LE_\varepsilon(G_2)+\frac{4(n_2\zeta(G_1)-n_1\zeta(G_2))}{n_1+n_2}. \end{aligned}$$

3 Conclusion

In this paper, we study weighted Laplacian energy of union of k-number of connected graphs G_1, G_2, ... G_k to find upper and lower bounds of these. This is a generalization of Laplacian energy of union of graphs when the degree of a vertex is considered as vertex weight. Different other bounds and results are also derived from the general results as a special case.

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