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Bounds for the Independence Number in k-Step Hamiltonian Graphs

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Abstract

For a given integer k, a graph G of order n is called k-step Hamiltonian if there is a labeling $v_1, v_2, ..., v_n$ of vertices of G such that $d(v_1, v_n) = d(v_i, v_{i+1}) = k$ for $i = 1, 2, ..., n-1$. The independence number of a graph is the maximum cardinality of a subset of pair-wise non-adjacent vertices. In this paper we study the independence number in k-step Hamiltonian graphs. We present sharp upper bounds as well as sharp lower bounds, and then present a construction that produces infinite families of k-step Hamiltonian graphs with arbitrarily large independence number.

Keywords: Independence number, Hamiltonian graph, k-Step Hamiltonian graph.

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1 Introduction

Let $G = (V, E)$ be a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. For a vertex $v \in V(G)$, let $N_G(v) = \{u | uv \in E(G)\}$ denote the *open neighborhood* of v and $N_G[v] = \{v\} \cup N_G(v)$ denote the *closed neighborhood* of v. The *degree* of a vertex v, $\deg_G(v)$, or just $\deg(v)$, in a graph G is the number of neighbors of v in G. We refer to $\Delta(G)$ and $\delta(G)$ as the maximum degree and the minimum degree of the vertices of G, respectively. We denote by C_n and P_n the cycle and the path on n vertices, respectively. The *distance* $d(u, v)$ between two vertices u and v in a graph G is the length of a shortest path from u

to v. A graph G is called *bipartite* if its vertex set can be partitioned into two sets X and Y such that any edge of G has one end-point in X and one end-point in Y. A bipartite graph is called *complete bipartite* if any vertex of each partite set is adjacent to all of the vertices of the other partite set. A complete bipartite graph whose partite sets have cardinality m and n, is denoted by $K_{m,n}$. The graph $K_{1,n-1}$ is called a *star*. A *double-star* is a tree with precisely two vertices that are not leaf (we refer to these two vertices as the centers of the double-star). We denote by $S(a, b)$ a double star in which one of the central vertices has degree a and the other central vertex has degree b. A graph is *triangle-free* if it does not contain a C_3 as an induced subgraph. For a subset S of vertices of G, we denote by $G[S]$ the subgraph of G induced by S. For other notations and terminologies not given here, we refer to [8].

A set S of vertices in a graph G is an *independent set* if no pair of vertices of S are adjacent. The *independence number* of G, denoted by $\alpha(G)$, is the maximum cardinality of an independent set in G. The concept of independent sets is an active area of research in graph theory and there are many papers dealed with independent sets which presented exact values or bounds for the independence number in graphs, see for example [1] - [3], [5] - [7] and [9].

A graph $G = (V, E)$ is *Hamiltonian* if there is a spanning cycle in G. There is no specific characterization to check the existence and non-existence of Hamiltonian cycle for a given graph G though the Hamiltonian problem has been widely studied in graph theory. All the works provide necessary conditions and sufficient conditions for a graph to be Hamiltonian. See [4] for a recent development and open problems related to Hamiltonicity of graphs. In [12], Lau et al. extended the concept of Hamiltonicity to k-step Hamiltonicity. They introduced the concept of $AL(k)-$traversal followed by k-step Hamiltonian graph as follows: For an integer $k \geq 1$, a graph G of order n is said to admit an $AL(k)-$traversal if we can arrange the vertices of G as the sequence of vertices $v_1, v_2, \ldots, v_n$ such that $d(v_i, v_{i+1}) = k$ for $i = 1, 2, \ldots, n-1$. A graph G is $k-$step Hamiltonian (or just $k-$SH) if it has an $AL(k)-$traversal and $d(v_1, v_n) = k$. Then, the sequence

$v_1, v_2, \ldots, v_n, v_1$ is called a $k-$step Hamiltonian walk of G. Clearly, k-SH graphs with $k = 1$ are the Hamiltonian graphs. The concept of k-SH graphs has been further studied in, for example [10], [11], [13] and [14].

In this paper we study the independence number in k-SH graphs. We present sharp upper bounds as well as sharp lower bounds, and then present a construction that produces infinite families of k-SH graphs with arbitrarily large independence number. We make use of the following.

Theorem 1 (Lau et al. [13]). *No bipartite graph is* 2*-SH.*

Theorem 2 (Lau et al. [13]). *The cycle C_n, $n \geq 3$ is k-SH for $k \geq 2$ if and only if $n \geq 2k+1$ and $gcd(n,k) = 1$.*

Proposition 1 (West [15]). *For a path P_n, $\alpha(P_n) = \lceil \frac{n}{2} \rceil$.*

2 Upper bounds

We first present an upper bound for the independence number in a k-SH graph with $k \geq 2$. Let $\mathcal{H}$ be the class of all graphs obtained from a double star $S(a,b)$ with $a \geq 3$ and $b \geq 3$ by adding at least two new vertices and joining each of them to both central vertices of $S(a,b)$.

Lemma 1. *A graph G of order n is a* 2*-SH graph with $\alpha(G) = n-2$ if and only if $G \in \mathcal{H}$.*

Proof. Let G be a 2-SH graph with $\alpha(G) = n-2$, D a maximum independent set in G, and $V(G) - D = \{x, y\}$. Since G is connected, $N(x) \cap N(y) \neq \emptyset$ and $D \subseteq N(x) \cup N(y)$. Since G is 2-SH, by Theorem 1, we obtain that x is adjacent to y. Let $A = N(x) - N(y)$, $B = N(y) - N(x)$ and $C = N(x) \cap N(y)$. Since G is $2-$SH, there exists a $2-$step Hamiltonian walk. Without loss of generality, we re-label the vertices of the $2-$step Hamiltonian walk such that $v_1 = x$. Then $v_2 \in B$. Let i be the minimum index such that $v_2, v_3, \ldots, v_i \in B$, $v_{i+1} \notin B$. Then, $v_{i+1} \in C$. There is an integer j such that $v_j = y$. Then, $v_{j-1} \in A$. Since $d(v_j, v_1) = 1$, we have $v_{j+1} \in A$. Thus, $|A| \geq 2$. Let s be the minimum integer such that $v_{j+1}, v_{j+2}, \ldots, v_s \in A$, $v_{s+1} \notin A$. Then, $v_{s+1} \in C$. Thus, $|C| \geq 2$. Also it is obvious that $v_n \in B$. Thus, $|B| \geq 2$. Consequently, $G \in \mathcal{H}$. Conversely, let $G \in \mathcal{H}$. Then, G is obtained from a double star $S(a, b)$ with $a \geq 3$ and $b \geq 3$ by adding at least two new vertices and joining each new vertex to both central vertices of $S(a, b)$. Let x and y be the central vertices of $S(a, b)$. Let $N(x) \cap N(y) = \{a_1, a_2, \ldots, a_t\}$, $N(x) - N(y) = \{b_1, b_2, \ldots, b_{t'}\}$ and $N(y) - N(x) = \{c_1, c_2, \ldots, c_{t''}\}$, where $t, t', t'' \geq 2$. Clearly $V(G) - \{x, y\}$ is an independent set in G and so $\alpha(G) \geq n - 2$. Since the vertices x, y and a_1 form a triangle in G, we obtain that $\alpha(G) \leq n - 2$. Thus $\alpha(G) = n - 2$. Now, $x, c_1, a_1, b_1, y, b_2, b_3, \ldots, b_{t'}, a_2, a_3, \ldots, a_t, c_2, c_3, \ldots, c_{t''}, x$ is a $2-$step Hamiltonian walk. Consequently, G is $2-$SH. $\square$

Theorem 3. *If G is a k-SH graph of order n for $k \geq 2$, then $\alpha(G) \leq n - \lceil \frac{k}{2} \rceil - 1$, with equality if and only if $k = 2$ and $G \in \mathcal{H}$.*

Proof. Let G be a $k-$SH graph of order n with $k \geq 2$. Let u and v be two consecutive vertices on a k-step Hamiltonian walk of G, and without loss of generality, assume that $v_1 v_2 ... v_k v_{k+1}$ be a shortest path in G from u to v, where $u = v_1$ and $v = v_{k+1}$. Let D be a maximum independent set in G. If $D \cap \{v_1, ..., v_{k+1}\} = \emptyset$, then $|D| \subseteq V(G) - \{v_1, ..., v_{k+1}\}$ and so $|D| \leq n - k - 1 \leq n - \lceil \frac{k}{2} \rceil - 1$. Thus assume that $D \cap \{v_1, ..., v_{k+1}\} \neq \emptyset$. Then $D \cap \{v_1, ..., v_{k+1}\}$ is an independent set in the graph $G[\{v_1, ..., v_{k+1}\}]$. Since $G[\{v_1, ..., v_{k+1}\}]$ is a path P_{k+1}, by Proposition 1, $|D \cap \{v_1, ..., v_{k+1}\}| \leq \lceil \frac{k+1}{2} \rceil$. Consequently, $|D| \leq \lceil \frac{k+1}{2} \rceil + n - (k + 1) = n - \lceil \frac{k}{2} \rceil$.

Suppose that $\alpha(G) = n - \lceil \frac{k}{2} \rceil$. Clearly D contains at most $\alpha(P_{k+1}) = \lceil \frac{k+1}{2} \rceil$ vertices of $G[\{v_1, ..., v_{k+1}\}]$, since $G[\{v_1, ..., v_{k+1}\}]$ is isomorphic to P_{k+1}. Since $|D| = n - \lceil \frac{k}{2} \rceil$, we obtain that $|D - \{v_1, ..., v_{k+1}\}| \geq n - k - 1$. Thus we find that $|D - \{v_1, ..., v_{k+1}\}| = n - k - 1$ and $|D \cap \{v_1, ..., v_{k+1}\}| = \alpha(P_{k+1}) = \lceil \frac{k+1}{2} \rceil$. Assume that k is even. Then $v_{2i+1} \in D$ for $i = 0, 1, ..., \frac{k}{2}$, $\deg(v_1) = \deg(v_{k+1}) = 1$ and $\deg(v_{2i+1}) = 2$ for each $i = 1, ..., \frac{k}{2} - 1$. Since $V(G) - \{v_1, ..., v_{k+1}\} \subseteq D$, $V(G) - \{v_1, ..., v_{k+1}\}$ is an independent set, and thus any vertex of $V(G) - \{v_1, ..., v_{k+1}\}$ is adjacent to some vertex in $\{v_{2i} : i = 1, 2, ..., \frac{k}{2}\}$.

If $k \geq 4$, then there is no vertex at distance k from v_3, a contradiction. Thus assume that $k = 2$. Since any vertex of $V(G) - \{v_1, v_2, v_3\}$ is adjacent to v_2, we obtain that G is a star. This is a contradiction by Theorem 1. Next assume that k is odd. Since $D \cap \{v_1, ..., v_{k+1}\}$ is an independent set of cardinality $\frac{k+1}{2}$, we have $D \cap \{v_1, v_{k+1}\} \neq \emptyset$. Without loss of generality, assume that $v_1 \in D$. Then $\deg(v_1) = 1$, and $\deg(x) = 2$ if $x \in \{v_2, ..., v_k\} \cap D$. As before, any vertex of $V(G) - \{v_1, ..., v_{k+1}\}$ is adjacent to some vertex in $(\{v_1, ..., v_{k+1}\} - D)$. Clearly $D \cap \{v_3, v_4\} \neq \emptyset$. If $k \geq 5$, then there is no vertex in G at distance k from the vertex in $D \cap \{v_3, v_4\}$, a contradiction. Thus assume that $k = 3$. If $v_3 \in D$, then there is no vertex at distance 3 from v_3, a contradiction. Thus assume that $v_4 \in D$. Then $\deg(v_4) = 1$, and so there is no vertex in G at distance three from v_2, a contradiction. We conclude that $\alpha(G) \leq n - \lceil \frac{k}{2} \rceil - 1$, as desired.

We next prove the equality part. Assume that $\alpha(G) = n - \lceil \frac{k}{2} \rceil - 1$. Let u,v,v_1,...,v_{k+1} and D be as described above. If $|D - \{v_1, ..., v_{k+1}\}| < n - k - 2$, then $n - \lceil \frac{k}{2} \rceil - 1 = |D| \leq |D \cap \{v_1, ..., v_{k+1}\}| + |D - \{v_1, ..., v_{k+1}\}| < \lceil \frac{k+1}{2} \rceil + n - k - 2$ which leads to a contradiction. Thus $|D - \{v_1, ..., v_{k+1}\}| \geq n - k - 2$. Let x and y be the vertices on the k-step Hamiltonian walk of G such that $d_G(x, v_{\lceil \frac{k+1}{2} \rceil}) = d_G(v_{\lceil \frac{k+1}{2} \rceil}, y) = k$. Clearly, $x, y \in V(G) - \{v_1, ..., v_{k+1}\}$. If $k \geq 6$, then the shortest path in G from $v_{\lceil \frac{k+1}{2} \rceil}$ to x contains at least three vertices in $V(G) - \{v_1, ..., v_{k+1}\}$, a contradiction, since $|D - \{v_1, ..., v_{k+1}\}| \geq n - k - 2$. If $k \in \{4, 5\}$, then the shortest path in G from $v_{\lceil \frac{k+1}{2} \rceil}$ to x contains at least two vertices in $V(G) - \{v_1, ..., v_{k+1}\}$, and the shortest path in G from $v_{\lceil \frac{k+1}{2} \rceil}$ to y contains at least two vertices in $V(G) - \{v_1, ..., v_{k+1}\}$, a contradiction, since $|D - \{v_1, ..., v_{k+1}\}| \geq n - k - 2$. Thus, $k \leq 3$. Suppose next that $k = 3$. Then $|D| = n - 3$. Suppose that $V(G) - \{v_1, ..., v_4\} \subseteq D$. Then $|D \cap \{v_1, ..., v_4\}| = 1$, and any vertex of $V(G) - \{v_1, ..., v_4\}$ is adjacent to some vertex in $\{v_1, ..., v_4\}$. If $v_4 \in D$, then v_4 has no neighbor in $V(G) - \{v_1, ..., v_4\}$, and so there is no vertex in G at distance three from v_2, a contradiction. Thus $v_4 \notin D$, and similarly, $v_1 \notin D$. If $v_3 \in D$, then v_3 has no neighbor in $V(G) - \{v_1, ..., v_4\}$, and so v_4 is the only vertex in G at distance three from v_1, a contradiction, since G is a 3-SH graph. Thus $v_3 \notin D$, and similarly, $v_2 \notin D$. This contradicts $|D \cap \{v_1, ..., v_4\}| = 1$. We deduce that $V(G) - \{v_1, ..., v_4\} \nsubseteq D$. Then $|D \cap \{v_1, ..., v_4\}| = 2$, and $|D - \{v_1, ..., v_4\}| = n - 5$. Clearly $D \cap \{v_1, ..., v_4\} = \{v_1, v_3\}$, $\{v_2, v_4\}$ or $\{v_1, v_4\}$. Assume that $D \cap \{v_1, ..., v_4\} = \{v_1, v_3\}$. If $\deg(v_1) = 1$, then $G[V(G) - \{v_1, ..., v_4\}]$ contains at least two edges, since there are two vertices in G at distance three from v_3. This is a contradiction. Thus $\deg(v_1) \geq 2$, and so v_1 has some neighbor in $V(G) - \{v_1, ..., v_4\}$. If any neighbor of v_1 in $V(G) - \{v_1, ..., v_4\}$ is a neighbor of v_3, then similarly we obtain a contradiction. Thus we assume that v_1 has a neighbor $v_1' \in V(G) - \{v_1, ..., v_4\} - N(v_3)$. Then v_3 has no neighbor in $V(G) - \{v_1, ..., v_4\}$, and so v_4 is the only vertex in G at distance three from v_1, a contradiction, since G is a 3-SH graph. Thus $D \cap \{v_1, ..., v_4\} \neq \{v_1, v_3\}$, and similarly, $D \cap \{v_1, ..., v_4\} \neq \{v_2, v_4\}$. Thus $D \cap \{v_1, ..., v_4\} = \{v_1, v_4\}$. Clearly, either v_1 has no neighbor in $V(G) - \{v_1, ..., v_4\}$ or v_4 has no neighbor in $V(G) - \{v_1, ..., v_4\}$. Without loss of generality, we may assume that v_4 has no neighbor in $V(G) - \{v_1, ..., v_4\}$. Since there are two vertices in G at distance three from v_2, $G[V(G) - \{v_1, ..., v_4\}]$ contains at least two edges, a contradiction. We conclude that $k = 2$. Now the result follows from Lemma 1. □

Corollary 1. *If G is a k-SH graph of order n for $k \geq 3$, then $\alpha(G) \leq n - \lceil \frac{k}{2} \rceil - 2$, and this bound is sharp for $k = 3$.*

Proof. The result follows from the Theorem 3. To see the sharpness for $k = 3$, consider the cycle C_7, and note that C_7 is $3-$SH by Theorem 2. □

We next give a new upper bound for the independence number of a k-SH graph for $k \geq 4$ which improves Corollary 1 for $k \geq 13$.

Theorem 4. *If G is a k-SH graph of order n for $k \geq 4$, then $\alpha(G) \leq n - k + \lceil \frac{k-1}{4} \rceil$ if k is odd, and $\alpha(G) \leq n - k + 1 + \lceil \frac{k-2}{4} \rceil$ if k is even.*

Proof. Let G be a $k-$SH graph of order n with $k \geq 4$. Let u and v be two consecutive vertices on a k-step Hamiltonian walk of G, and assume that $P : v_1v_2...v_kv_{k+1}$ is a shortest path in G from u to v, where $u = v_1$ and $v = v_{k+1}$. Let D be a maximum independent set in G. If $D \cap \{v_1, ..., v_{k+1}\} = \emptyset$, then $|D| \subseteq V(G) - \{v_1, ..., v_{k+1}\}$ and so $|D| \leq n - k - 1 \leq \min\{n - k + \lceil\frac{k-1}{4}\rceil, n - k + 1 + \lceil\frac{k-2}{4}\rceil\}$. Thus assume that $D \cap \{v_1, ..., v_{k+1}\} \neq \emptyset$. Then $D \cap \{v_1, ..., v_{k+1}\}$ is an independent set in the graph $G[\{v_1, ..., v_{k+1}\}]$. Since $G[\{v_1, ..., v_{k+1}\}]$ is a path P_{k+1}, by Proposition 1, $|D \cap \{v_1, ..., v_{k+1}\}| \leq \lceil\frac{k+1}{2}\rceil$. There is a vertex w in G at distance k from $v_{\lceil\frac{k+1}{2}\rceil}$, since G is a k-SH graph. Let $x_1x_2...x_{k+1}$ be a shortest path from $v_{\lceil\frac{k+1}{2}\rceil}$ to w, where $x_1 = v_{\lceil\frac{k+1}{2}\rceil}$ and $x_{k+1} = w$. Let i be the minimum integer such that $x_1, ..., x_i \in P$ and $x_{i+1} \notin P$. Then clearly, $x_{i+1}, ..., x_{k+1} \notin P$. Observe that $i \leq \frac{k+1}{2} + 1$ if k is odd, and $i \leq \frac{k}{2} + 1$ if k is even. Thus, $i \leq \lceil\frac{k+1}{2}\rceil + 1$. Let Q be the path $x_{i+1}, ..., x_{k+1}$. Then Q is isomorphic to P_{k+1-i}, and so D contains at most $\alpha(P_{k+1-i})$ vertices of Q. Thus

$$\begin{aligned}
|D| &\leq |D \cap \{v_1, ..., v_{k+1}\}| + |D \cap (V(G) - \{v_1, ..., v_{k+1}\})| \\
&\leq \lceil\frac{k+1}{2}\rceil + n - (k+1) - (k + 1 - i - \alpha(P_{k+1-i})).
\end{aligned}$$

Since $i \leq \lceil\frac{k+1}{2}\rceil + 1$, we obtain that

$$\begin{aligned}
|D| &\leq \lceil\frac{k+1}{2}\rceil + n - (k+1) - \\
&- (k + 1 - (\lceil\frac{k+1}{2}\rceil + 1) - \alpha(P_{k+1-(\lceil\frac{k+1}{2}\rceil+1)})) \\
&= \lceil\frac{k+1}{2}\rceil + n - (k+1) - (k - \lceil\frac{k+1}{2}\rceil - \alpha(P_{k-\lceil\frac{k+1}{2}\rceil})) \\
&= \lceil\frac{k+1}{2}\rceil + n - (k+1) - k + \lceil\frac{k+1}{2}\rceil + \lceil\frac{k - \lceil\frac{k+1}{2}\rceil}{2}\rceil.
\end{aligned}$$

Now a simple calculation shows that $|D| \leq n - k + \lceil\frac{k-1}{4}\rceil$ if k is odd, and $|D| \leq n - k + 1 + \lceil\frac{k-2}{4}\rceil$ if k is even. □

We note that a simple calculation shows that Theorem 4 improves Corollary 1 for $k \geq 13$. We purpose obtaining a sharp upper bound for the independence number of a k-SH graph for $k \geq 4$ as an open problem.

Problem 1. *What is the best upper bound for the independence number of a k-SH graph for $k \geq 4$?*

3 Lower bounds

We next present a lower bound for the independence number of a k-SH graph.

Theorem 5. *If G is a $k-SH$ graph for $k \geq 2$, then $\alpha(G) \geq \lceil \frac{k+1}{2} \rceil + \lfloor \frac{k-3}{4} \rfloor + 1$. This bound is sharp for $k = 2, 3, 4$.*

Proof. Let u and v be two consecutive vertices in a $k-$step Hamiltonian walk of G, and $P : v_1, v_2, \ldots, v_{k+1}$ be a shortest path in G from u to v, where $u = v_1$ and $v = v_{k+1}$. Clearly, $G[\{v_1, v_2, \ldots, v_{k+1}\}]$ is isomorphic to P_{k+1}. Thus $\alpha(G) \geq \alpha(G[\{v_1, v_2, \ldots, v_{k+1}\}]) = \alpha(P_{k+1}) \geq \lceil \frac{k+1}{2} \rceil$, by Proposition 1. The result is obvious if $k = 2$. Thus assume that $k \geq 3$. There is a vertex w in G at distance k from $v_{\lceil \frac{k+1}{2} \rceil}$, since G is a k-SH graph. Let $x_1x_2...x_{k+1}$ be a shortest path from $v_{\lceil \frac{k+1}{2} \rceil}$ to w, where $x_1 = v_{\lceil \frac{k+1}{2} \rceil}$ and $x_{k+1} = w$. Let i be the minimum integer such that $x_1, ..., x_i \in P$ and $x_{i+1} \notin P$. Then clearly, $x_{i+1}, ..., x_{k+1} \notin P$. Observe that $i \leq \frac{k+1}{2} + 1$ if k is odd, and $i \leq \frac{k}{2} + 1$ if k is even. Thus, $i \leq \lceil \frac{k+1}{2} \rceil + 1$. If there is an integer s with $\lceil \frac{k+1}{2} \rceil + 2 \leq s \leq k + 1$ such that x_s is adjacent to v_1, then $d(v_{\lceil \frac{k+1}{2} \rceil}, x_{k+1}) < k$, since the walk $v_{\lceil \frac{k+1}{2} \rceil}, v_{\lceil \frac{k+1}{2} \rceil - 1}, ..., v_1, x_s, x_{s+1}, ..., x_{k+1}$ is a path from $v_{\lceil \frac{k+1}{2} \rceil}$ to x_{k+1} of length less than k, a contradiction. Thus x_s is not adjacent to v_1 for each integer s with $\lceil \frac{k+1}{2} \rceil + 2 \leq s \leq k+1$. Similarly, x_s is not adjacent to v_k for each integer s with $\lceil \frac{k+1}{2} \rceil + 2 \leq s \leq k + 1$. For any vertex $a \in \{v_{2j+1} : 1 \leq j \leq \lceil \frac{k-1}{2} \rceil - 1\}$, since $d(v_{\lceil \frac{k+1}{2} \rceil}, a) < d(v_{\lceil \frac{k+1}{2} \rceil}, v_1)$ and $d(v_{\lceil \frac{k+1}{2} \rceil}, a) < d(v_{\lceil \frac{k+1}{2} \rceil}, v_k)$, we deduce that x_s is adjacent to no vertex in $\{v_{2j+1} : 0 \leq j \leq \lceil \frac{k-1}{2} \rceil\}$ for each integer s with $\lceil \frac{k+1}{2} \rceil + 2 \leq s \leq k+1$. Now,

$$\{v_{2j+1} : 0 \leq j \leq \lceil \frac{k-1}{2} \rceil\} \cup \{x_{\lceil \frac{k+1}{2} \rceil + 2 + 2t} : t = 0, 1, ..., \lfloor \frac{k-3}{4} \rfloor\}$$

is an independent set in G. Consequently, $\alpha(G) \geq \lceil \frac{k+1}{2} \rceil + \lfloor \frac{k-3}{4} \rfloor + 1$.

For the sharpness of the bound for $k = 2, 3, 4$, we prove a stronger result. We show that there is an infinite family of k-SH graphs G with $\alpha(G) = \lceil \frac{k+1}{2} \rceil + \lfloor \frac{k-3}{4} \rfloor + 1$ for each k. Let $2 \leq k \leq 4$. For a given integer $m \geq 1$, consider m copies of a cycle C_{2k+1}. Let $\{v_1^i, ..., v_{2k+1}^i\}$ be the vertex set of the i-th C_{2k+1}, where v_j^i is adjacent to v_{j+1}^i for $j = 1, 2, ..., 2k+1$ and the summation is in modulo $2k+1$. Let G be a graph obtained by joining v_j^i to the vertices v_j^r, v_{j-1}^r and v_{j+1}^r for each $r \in \{1, 2, ..., m\} - \{i\}$ and each $i = 1, 2, ..., m$ and $j = 1, 2, ..., 2k+1$. Then clearly $\alpha(G) = \alpha(C_{2k+1}) = k$ and it is easy to see that $k = \lceil \frac{k+1}{2} \rceil + \lfloor \frac{k-3}{4} \rfloor + 1$. If $k = 2$ then

$$v_1^1 v_3^1 v_5^1 v_2^1 v_4^1 v_1^2 v_3^2 v_5^2 v_2^2 v_4^2 ... v_1^m v_3^m v_5^m v_2^m v_4^m v_1^1$$

is a 2-step Hamiltonian walk for G, and thus G is 2-SH. If $k = 3$, then

$$v_1^1 v_4^1 v_7^1 v_3^1 v_6^1 v_2^1 v_5^1 v_1^2 v_4^2 v_7^2 v_3^2 v_6^2 v_2^2 v_5^2 ... v_1^m v_4^m v_7^m v_3^m v_6^m v_2^m v_5^m v_1^1$$

is a 3-step Hamiltonian walk for G, and thus G is 3-SH. If $k = 4$, then

$$v_1^1 v_5^1 v_9^1 v_4^1 v_8^1 v_3^1 v_7^1 v_2^1 v_6^1 v_1^2 v_5^2 v_9^2 v_4^2 v_8^2 v_3^2 v_7^2 v_2^2 v_6^2 ... v_1^m v_5^m v_9^m v_4^m v_8^m v_3^m v_7^m v_2^m v_6^m v_1^1$$

is a 4-step Hamiltonian walk for G, and thus G is 4-SH. □

We next characterize all triangle-free k-SH graphs G with $k \leq 4$ that achieve equality of the bound of Theorem 5.

Proposition 2. *For $2 \leq k \leq 4$, a triangle-free graph G is a $k-SH$ graph with $\alpha(G) = \lceil \frac{k+1}{2} \rceil + \lfloor \frac{k-3}{4} \rfloor + 1$ if and only if $G = C_{2k+1}$.*

Proof. Let G be a triangle-free k-SH graph with $\alpha(G) = \lceil \frac{k+1}{2} \rceil + \lfloor \frac{k-3}{4} \rfloor + 1$. First let $k = 2$. Then $\alpha(G) = 2$. Clearly, $N(x)$ is an independent set for any vertex x, since G is triangle-free. Consequently, $\Delta(G) = 2$. Since no path is 2-SH, we find that G is a cycle, and by Theorem 2 it can be seen that $G = C_5$.

Next assume that $k = 3$. Then $\alpha(G) = 3$. Suppose that $\Delta(G) > 2$. Let x be a vertex of maximum degree in G. If $\deg(x) > 3$, then $N(x)$ is an independent set of cardinality more that $\alpha(G)$, a contradiction. Thus $\deg(x) = 3$. There is a vertex y in G at distance three from

x, since G is 3-SH. Then $N(x) \cup \{y\}$ is an independent set in G, a contradiction. We deduce that $\Delta(G) = 2$. Since no path is 3-SH, we find that G is a cycle, and by Theorem 2, it can be seen that $G = C_7$.

Now assume that $k = 4$. Then $\alpha(G) = 4$. We show that $\Delta(G) = 2$. Suppose that $\Delta(G) \geq 3$. Let x be a vertex of maximum degree in G. If $\deg(x) > 4$, then $N(x)$ is an independent set of cardinality more that $\alpha(G)$, a contradiction. If $\deg(x) = 4$, then $N(x) \cup \{y\}$ is an independent set in G, where y is a vertex at distance four from x. This is a contradiction. Thus, $\deg(x) = 3$. There are two vertices z_1, z_2 at distance four from x, since G is 4-SH. If $z_1 \notin N(z_2)$, then $N(x) \cup \{z_1, z_2\}$ is an independent set in G, a contradiction. Thus assume that $z_1 \in N(z_2)$. Clearly, z_1 and z_2 have no common neighbor, since G is triangle-free. Let $y_1 \in N(z_1) - \{z_2\}$ and $y_2 \in N(z_2) - \{z_1\}$. Clearly, $y_1 \notin N(z_2)$. Now $N(x) \cup \{y_1, z_2\}$ is an independent set in G, a contradiction. We deduce that $\Delta(G) = 2$. Since no path is 4-SH, we find that G is a cycle, and by Theorem 2, it can be seen that $G = C_9$.

The converse is obvious. $\square$

We improve the bound of Theorem 5 for $k \geq 5$.

Theorem 6. *If G is a $k-SH$ graph for $k \geq 5$, then $\alpha(G) \geq \lceil \frac{k+1}{2} \rceil + \lfloor \frac{k-3}{4} \rfloor + 2$, and this bound is sharp for $k = 5, 6$.*

Proof. We follow the proof of Theorem 5. Let u, v, $P : v_1, v_2, \ldots, v_{k+1}$, w, and $x_1 x_2 ... x_{k+1}$ be as described in the proof of Theorem 5. As noted, the set

$$D = \{v_{2j+1} : 0 \leq j \leq \lceil \frac{k-1}{2} \rceil\} \cup \{x_{\lceil \frac{k+1}{2} \rceil + 2 + 2t} : t = 0, 1, ..., \lfloor \frac{k-3}{4} \rfloor\}$$

is an independent set in G of cardinality $\lceil \frac{k+1}{2} \rceil + \lfloor \frac{k-3}{4} \rfloor + 1$. Assume that $k \equiv 1$ or $2 \pmod 4$. Let z be a vertex of G at distance k from x_2. Note that z exists, since G is k-SH. If z is adjacent to a vertex $a \in D$, then $d(z, x_2) < k$, a contradiction. Thus z is adjacent to no vertex in D. Consequently, $D \cup \{z\}$ is an independent set in G, implying that $\alpha(G) \geq \lceil \frac{k+1}{2} \rceil + \lfloor \frac{k-3}{4} \rfloor + 2$. Next assume that $k \equiv 0$ or $3 \pmod 4$. Observe that $k \geq 7$. Let z' be a vertex of G at distance k from x_3. If

z' is adjacent to a vertex $a \in D$, then $d(z', x_3) < k$, a contradiction. Thus z' is adjacent to no vertex in D. Consequently, $D \cup \{z'\}$ is an independent set in G, implying that $\alpha(G) \geq \lceil \frac{k+1}{2} \rceil + \lfloor \frac{k-3}{4} \rfloor + 2$. To see the sharpness, consider the C_{2k+1} for $k = 5, 6$, and use Theorem 2. $\square$

We believe the Theorem 6 can be improved for $k \geq 7$.

Problem 2. *Find a sharp lower bound for the independence number of a k-SH graph for each $k \geq 7$.*

By Theorem 5, $\alpha(G) \geq \lceil \frac{k+1}{2} \rceil + \lfloor \frac{k-3}{4} \rfloor + 1$ for each $k \geq 2$ and any k-SH graph G. We next wish to prove that the difference $\alpha(G) - (\lceil \frac{k+1}{2} \rceil + \lfloor \frac{k-3}{4} \rfloor + 1)$ can be arbitrarily large. For this purpose, we prove a stronger result. We show that the quotient $\frac{\alpha(G)}{k}$ can be arbitrarily large in a k-SH graph G. We prove that for any positive integer $n \geq 1$, and $k \geq 2$, there is a graph G with $\frac{\alpha(G)}{k} = 2^n$. Note that for $k \geq 2$, and any integer $n \geq 1$, clearly, $2^n k > \lceil \frac{k+1}{2} \rceil + \lfloor \frac{k-3}{4} \rfloor + 1$. We define a construction on a graph as follows.

- ***A*-Construction**: For any graph G with vertex set $\{v_1, v_2, \ldots, v_n\}$, let $A(G)$ be a graph with vertex set $V(A(G)) = \{v_1, v_2, \ldots, v_n\} \cup \{u_i : i = 1, 2, ..., n\}$ and edge set $E(A(G)) = E(G) \cup \{u_i v_j : v_j \in N_G(v_i)\}$. Furthermore, we define recursively, $A^k(G)$ for any $k \geq 1$, by $A^1(G) = A(G)$, $A^2(G) = A(A(G))$, and $A^k(G) = A(A^{k-1}(G))$ if $k \geq 2$.

Proposition 3. *1. If G is a k-SH graph, then $A^m(G)$ is a $k-$SH graph for each $m \geq 1$. 2. $\alpha(A^m(G)) = 2^m \alpha(G)$ for each $m \geq 1$.*

Proof. We prove only for $m = 1$ and then the result follows by an induction on m.

1. Let $v_1, v_2, \ldots, v_n$ be a $k-$step Hamiltonian walk of G. From the construction of $A(G)$, it is clear that $d(u_i, u_{i+1}) = d(v_n, u_1) = d(u_n, v_1) = k$ for $i = 1, 2, \ldots, n-1$. Therefore, the sequence $v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n, v_1$ is a $k-$step Hamiltonian walk of $A(G)$, as desired.

2. Let D be a maximum independent set in G. Then $D \cup \{u_i : v_i \in D\}$ is an independent set for $A(G)$, and thus $\alpha(A(G)) \geq 2\alpha(G)$. Suppose that $\alpha(A(G)) > 2\alpha(G)$. Let I be a maximum independent set in $A(G)$. Then $|I \cap V(G)| < |I - V(G)|$. Then $\{v_i : u_i \in I - V(G)\}$ is an independent set for G of cardinality more than $\alpha(G)$, a contradiction. Consequently, $\alpha(A(G)) = 2\alpha(G)$. $\square$

Theorem 7. *For any positive integer $n \geq 1$ and $k \geq 2$, there is a graph H with $\frac{\alpha(H)}{k} = 2^n$.*

Proof. Given $k \geq 2$ and $n \geq 1$, let $G = C_{2k+1}$ and $H = A^n(G)$. Since G is a k-SH graph by Theorem 2 with $\alpha(G) = k$, by Proposition 3, H is a k-SH graph, with $\alpha(H) = 2^n k$. $\square$

Let $\beta(G)$ be the *vertex covering number* of a graph G. Note that $\beta(G)$ is the minimum cardinality of a set of vertices such that each edge of the graph is incident to at least one vertex of the set. It is known that $\alpha(G) + \beta(G) = n$. Thus from Theorems 3 and 5, we have the following.

Corollary 2. *If G is a k-SH graph of order n for $k \geq 2$, then $\lceil \frac{k}{2} \rceil + 1 \leq \beta(G) \leq n - \lceil \frac{k+1}{2} \rceil - \lfloor \frac{k-3}{4} \rfloor - 1$.*

4 Conclusion

The presented upper and lower bounds for the independence number in k-step Hamiltonian graphs given in this paper are sharp for small values of k. New sharp bounds for larger values of k are of sufficient interest.

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