Computer Science Journal of Moldova, vol.**25**, no.3(75), 2017

Stability Analysis of Efficient Portfolios in a Discrete Variant of Multicriteria Investment Problem with Savage's Risk Criteria

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Abstract

We consider a multicriteria discrete variant of investment portfolio optimization problem with Savage's risk criteria. Three combinations of norms in problem parameter spaces are considered. In each combination, one of the three spaces is endowed with Hölder's norm, and the other two spaces are endowed with Chebyshev's norm. The lower and upper attainable bounds on the stability radius of one Pareto optimal portfolio are obtained.

Keywords: Multicriteria problem, Pareto optimal portfolio, Savage's risk criteria, stability radius, Hölder's norms

1 Introduction

Modern financial environments require mitigation of the limitations of modern portfolio theory to make portfolio choice easier in the context of long-term and goal-based investing [1]. Investment managing problems are of the type with uncertainty of the initial data (see e.g. [2]). Usually, any separate investment asset has a higher level of risk and less return than the portfolio of those assets, and there is no reason to invest in one particular asset. Creating the portfolio by diversification and mixing a variety of investments an investor reduces the riskiness of the portfolio.

Following classical Markowitz's portfolio theory [3], [4], the investor plots on the graph an efficient frontier depending on various pairs of risk and expected return, and then he chooses portfolio drawing on individual risk-return preferences. It gives him a chance to construct a portfolio with the same expected return and less risk.

The model we consider is rather different from the classical models. The risk matrix is constructed for several market states related to each type of the risk. Unlike classic modern portfolio theories, where a portfolio consists of a percentage of each asset, in our model a Boolean decision vector is used to describe feasible portfolios. The problem consists in finding a set of Pareto optimal portfolios with Savage's risk criteria.

The model formulation requires statistical and expert evaluation of risks (e.g. financial or ecological) [5] to be specified as the initial data. To construct an efficient portfolio, the investor must be able to quantify risk and provide the necessary inputs. Usually, the collected data contain computational errors and inaccuracies. It leads to the situation when the initial data representing risk values are inaccurate and uncertain. One of the key points of portfolio choice analysis under uncertainty is estimation of perturbation ranges for initial data the optima. The quantitative measure of the data perturbation level that do not violate optimality is known as the stability radius. The concept is widely presented and analyzed in the recent literature focusing on finding analytical expressions and bounds (see e.g. [6]–[10]). Similar approaches were also developing in parallel in scheduling theory (see [11]–[13]).

Analytic formulas are pairwise comparisons of solutions depending on selected optimality principles. The structure of global perturbation of this problem and the structure of the solution set should be taken into account. The particular definition of the stability radius depends on choosing optimality principles (given the problem was multicriterial), an uncertain data and a type of distance metric used to measure the nearness in problem parameter spaces. Various types of metrics allow considering a specific of problem parameters perturbation. So in the case of the Chebyshev metric l_∞ the maximum changes in the initial data is taken into account only. Thus the perturbations are considered to be independent. In the case of the Manhattan metric l_1 every change of the initial data can be monitored in total. Hölder's metric l_p, $1 \leq p \leq$

∞, is the metric with the parameter and includes such extreme cases as the Chebyshev metric l_∞, Manhattan metric l_1 and also Euclidean metric l_2. Thus, l_p norm allows controlling the degree and type of admissible perturbations, and therefore gives the decision maker more flexibility. For more details on the issue of using Hölder's metric in portfolio optimization we refer the reader to [14].

Along with quantitative analysis, a qualitative approach is developed in parallel. This approach concentrates on specifying analytical conditions which will guarantee some certain pre-specified behavior of the set of optimal solutions. To highlight the ideas of this approach, it is worth mentioning papers [15],[16], where comparative analysis of five different types of stability is presented for multicriteria integer linear programming problem. Similar results were obtained for multicriteria combinatorial problems with bottleneck criteria [17] as well as with some other nonlinear criteria [18].

In the previous papers (see, e.g. [19]–[24]), some bounds on the stability radii were obtained in the cases where three-dimensional problem parameters space is equipped with different combinations of l_1 and l_∞ norms. In the present paper, we obtain the lower and upper bounds on the stability radius of one Pareto optimal portfolio for the multicriteria investment problem with Savage's risk criteria, where we assume that in one space an arbitrary l_p norm is defined with $1 \leq p \leq \infty$. At the same time, we measure distances with l_∞ norm in the remaining spaces. This allows the investor to make a more detailed control over changes in the initial data regarding to the three spaces. For example, the Euclidian metric is often used to deal with risks, and l_p norm can treat the case once the decision maker needs it.

2 Problem formulation and basic definitions

Consider a multicriteria discrete variant of the portfolio optimization problem. We assume the model can be described by the following primitives listed below. Let

$N_n = \{1, 2, \ldots, n\}$ be a variety of alternatives (investment assets, projects);

N_m be a set of possible financial market states (market situations, scenarios);

N_s be a set of possible risks;

r_{ijk} be a numerical measure of economic risk of type $k \in N_s$ if investor chooses project $j \in N_n$ given the market state $i \in N_m$;

$R = [r_{ijk}] \in \mathbf{R}^{m\times n\times s}$;

$x = (x_1, x_2, \ldots, x_n)^T \in \mathbf{E}^n$ be an investment portfolio, where $\mathbf{E} = \{0, 1\}$,

$$x_j = \begin{cases} 1 & \text{if investor chooses project } j, \\ 0 & \text{otherwise;} \end{cases}$$

$X \subset \mathbf{E}^n$ be a set of all admissible investment portfolios;

$\mathbf{R}^m$ be a financial market state space;

$\mathbf{R}^n$ be a portfolio space;

$\mathbf{R}^s$ be a risk space.

In our model, we assume that the risk measure is addictive, i.e. the total risk of one portfolio is a sum of risks of the projects included in the portfolio. The risk of each project can be measured, for instance, by means of the associated implementation cost.

The presence of a risk factor is integral feature of financial market functioning. One can find information about risk measurement methods and their classification in [25]. The last trend is to quantify risks using five R: robustness, redundancy, resourcefulness, response and recovery. The natural target of any investor is to minimize different types of risks. It creates a motivation for multicriteria analysis within risk modeling. It leads to the usage of multicriteria decision making tools [26].

Assume that the efficiency of a chosen portfolio (Boolean vector) $x \in X,\ |X| \geq 2$, is evaluated by a vector objective function

$$f(x, R) = (f_1(x, R_1), f_2(x, R_2), \ldots, f_s(x, R_s)),$$

each partial objective represents minimax Savage's risk criterion [27].

$$f_k(x, R_k) = \max_{i\in N_m} R_{ik}x = \max_{i\in N_m} \sum_{j\in N_n} r_{ijk}x_j \to \min_{x\in X}, \qquad k \in N_s,$$

where $R_k \in \mathbf{R}^{m\times n}$ – k-th cut $R = [r_{ijk}] \in \mathbf{R}^{m\times n\times s}$ with rows $R_{ik} = (r_{i1k}, r_{i2k}, \dots, r_{ink}) \in \mathbf{R}^n$, $i \in N_m$.

If investor chooses Savage's risk criterion [28], then (s)he minimizes the total risk of the selected portfolio in the worst (maximum risk state) case. This approach takes place when the decision maker has the most pessimistic expectations about the market.

The problem of finding Pareto optimal (efficient) portfolios is referred to as the multicriteria investment Boolean problem with Savage's risk criteria and denoted $Z^s(R)$, $s \in \mathbf{N}$. The set of Pareto optimal portfolios is defined as follows

$$P^s(R) = \{x \in X : \nexists x' \in X \ \ (g(x, x', R) \geq 0_{(s)} \ \ \& \ \ g(x, x', R) \neq 0_{(s)})\},$$

where

$$g(x, x', R) = (g_1(x, x', R_1), g_2(x, x', R_2), \ \dots, g_s(x, x', R_s)),$$

$$g_k(x, x', R_k) = f_k(x, R_k) - f_k(x', R_k) = \min_{i' \in N_m} \max_{i \in N_m} (R_{ik}x - R_{i'k}x'), k \in N_s,$$

$$0_{(s)} = (0, 0, \dots, 0) \in \mathbf{R}^s.$$

If $m = 1$, then the problem $Z^s(R)$ transforms into s-criteria linear Boolean programming problem:

$$Z^s_B(R): \quad Rx \to \min_{x \in X}, \tag{1}$$

where $X \subseteq \mathbf{E}^n$, $R = [r_{kj}] \in \mathbf{R}^{s\times n}$ is a matrix with rows $R_k = (r_{k1}, r_{k2}, \dots, r_{kn}) \in \mathbf{R}^n$, $k \in N_s$. The case $m = 1$ can be interpreted as a stable market with one state only.

While solving investment problems, it is necessary to take into account inaccuracy of initial information (statistical and expert risks evaluation errors) which are very common in real life. Under these conditions, it is highly recommended to get numerical bounds about possible changes in initial data preserving efficiency of the original Pareto optimal portfolio for any perturbation. Similarly to [20], [29], the number

$$\rho = \rho^s(x^0) = \begin{cases} \sup \ \Xi & \text{if } \ \Xi \neq \emptyset, \\ 0 & \text{if } \ \Xi = \emptyset, \end{cases}$$

is called a stability radius of a Pareto optimal solution $x^0 \in P^s(R)$, where

$$\Xi = \{\varepsilon > 0 : \ \forall R' \in \Omega(\varepsilon) \quad (x^0 \in P^s(R + R'))\},$$

$$\Omega(\varepsilon) = \{R' \in \mathbf{R}^{m \times n \times s} : \ \|R'\| < \varepsilon\}.$$

Here $\Omega(\varepsilon)$ is a set of feasible perturbation matrices, $P^s(R + R')$ is a Pareto set of perturbed problem $Z^s(R + R')$, $\|R'\|$ is the norm of the matrix $R' = [r'_{ijk}]$. This norm depends on norms specified in portfolio space $\mathbf{R}^n$, market state space $\mathbf{R}^m$ as well as risk space $\mathbf{R}^s$.

Further, we investigate the stability radius in three different cases depending on which of those three spaces $\mathbf{R}^n$, $\mathbf{R}^m$ or $\mathbf{R}^s$ is equipped with Hölder's l_p-norm, $1 \leq p \leq \infty$. For any dimension d and $1 \leq p \leq \infty$, the Hölder l_p norm of $a = (a_1, a_2, \dots, a_d) \in \mathbf{R}^d$ in $\mathbf{R}^d$ is defined by the following equation

$$\|a\|_p = \begin{cases} \left(\sum_{j \in N_d} |a_j|^p\right)^{1/p} & \text{if } \ 1 \leq p < \infty, \\ \\ \max\{|a_j| : \ j \in N_d\} & \text{if } \ p = \infty. \end{cases}$$

It is well-known that l_p norm, defined in $\mathbf{R}^d$, induces conjugated l_{p^*} norm in $(\mathbf{R}^d)^*$. For p and p^*, the following relations hold

$$\frac{1}{p} + \frac{1}{p^*} = 1, \qquad 1 < p < \infty.$$

Here as usual, we set $p^* = 1$ if $p = \infty$, and $p^* = \infty$ if $p = 1$. Thus, we assume that p and p^* vary within the range $[1, \infty]$. We also assume $1/p = 0$ if $p = \infty$.

It is easy to see that

$$\|z\|_p \|z\|_{p^*} = \|z\|_1 \ \text{ for } \ z \in \{-1, 0, 1\}^n, \ p \in [1, \infty]. \tag{2}$$

For any $\alpha > 0$ and $m \in \mathbf{N}$,

$$\|(\underbrace{\alpha, ..., \alpha}_{m})\|_p = m^{1/p} \alpha. \tag{3}$$

Further, we will use classical Hölder's inequality $ab \leq \|a\|_p \|b\|_{p^*}$, where $a = (a_1, a_2, \dots, a_n) \in \mathbf{R}^n$, $b = (b_1, b_2, \dots, b_n)^T \in \mathbf{R}^n$.

The following lemma can be easily proven by contradiction.

Lemma. Let $x^0 \in P^s(R)$, $\gamma > 0$. If for any portfolio $x \in X \setminus \{x^0\}$ and every perturbing matrix$R' \in \Omega(\gamma)$ there exists an index $l \in N_s$ such that $g_l(x, x^0, R_l + R'_l) > 0$, then x^0 is Pareto optimal in any perturbed problem $Z^s(R + R')$, i.e. $x^0 \in P^s(R + R')$ as $R' \in \Omega(\gamma)$.

3 Case A: portfolio space $\mathbf{R}^n$ is endowed with l_p

We endow portfolio space $\mathbf{R}^n$ with an arbitrary Hölder's l_p norm, $1 \leq p \leq \infty$, while in market state space $\mathbf{R}^m$ and risk space $\mathbf{R}^s$ we measure distances by means of l_∞. Thus, for any matrix $R = [r_{ijk}] \in \mathbf{R}^{m \times n \times s}$

$$\|R\|_{p\infty\infty} = \|(\|R_1\|_{p\infty}, \|R_2\|_{p\infty}, \dots, \|R_s\|_{p\infty})\|_\infty = \max_{k \in N_s} \|R_k\|_{p\infty},$$

where $\|R_k\|_{p\infty} = \|(\|R_{1k}\|_p,\ \|R_{2k}\|_p,\ \dots,\ \|R_{mk}\|_p)\|_\infty, \quad k \in N_s$. Obviously, $\|R_{ik}\|_p \leq \|R_k\|_{p\infty} \leq \|R\|_{p\infty\infty}, \quad i \in N_m,\ k \in N_s$. Additionally, due to Hölder's inequality, for any $x,\ x^0 \in X$ we get

$$\begin{aligned} R_{ik}x - R_{i'k}x^0 &\geq -(\|R_{ik}\|_p \|x\|_{p^*} + \|R_{i'k}\|_p \|x^0\|_{p^*}) \geq \\ &\geq -\|R_k\|_{p\infty}(\|x\|_{p^*} + \|x^0\|_{p^*}), \quad i, i' \in N_m,\ k \in N_s. \end{aligned} \tag{4}$$

In this context $\rho_1 = \rho_1^s(x^0, m, p, \infty, \infty)$ denotes the stability radius of x^0. For Pareto optimal portfolio x^0 in $Z^s(R)$, we will use the following notation

$$\varphi_1 = \varphi_1^s(x^0, m, p, \infty, \infty) = \min_{x \in X \setminus \{x^0\}} \frac{\|[g(x, x^0, R)]^+\|_\infty}{\|x\|_{p^*} + \|x^0\|_{p^*}},$$

$$\psi_1 = \psi_1^s(x^0, m, p, \infty, \infty) = \min_{x \in X \setminus \{x^0\}} \frac{\|[g(x, x^0, R)]^+\|_\infty}{\|x - x^0\|_{p^*}}.$$

Obviously, $\psi_1 \geq \varphi_1 \geq 0$. Here and henceforth we will use a vector $a = (a_1, a_2, \dots, a_s) \in \mathbf{R}^s$ projection operator to the nonnegative orthant:

$$[a]^+ = (a_1^+, a_2^+, \dots, a_s^+),$$

where sign " + " means the positive projection of the vector, i.e. $a_k^+ = \max\{0, a_k\}$, $k \in N_s$.

Theorem 1. *For any m, $s \in \mathbf{N}$ and $p \in [1, \infty]$, the stability radius $\rho_1^s(x^0, m, p, \infty, \infty)$ of Pareto optimal portfolio $x^0 \in P^s(R)$ in $Z^s(R)$ has the following upper and lower bounds*

$$\varphi_1^s(x^0, m, p, \infty, \infty) \leq \rho_1^s(x^0, m, p, \infty, \infty) \leq \psi_1^s(x^0, m, p, \infty, \infty). \quad (5)$$

Proof. Let $x^0 \in P^s(R)$. First we prove $\rho_1 \geq \varphi_1$. The claim is evident if $\varphi_1 = 0$. Assume $\varphi_1 > 0$. According to the definition of φ_1, for any portfolio $x \in X \setminus \{x^0\}$ the inequality

$$\|[g(x, x^0, R)]^+\|_\infty \geq \varphi_1(\|x\|_{p^*} + \|x^0\|_{p^*}) \quad (6)$$

holds. Further, we are going to prove by contradiction that

$$\forall R' \in \Omega(\varphi_1) \quad \exists l \in N_s \qquad (g_l(x, x^0, R_l + R'_l) > 0).$$

Suppose, there exists a perturbing matrix $R^0 \in \Omega(\varphi_1)$ with cuts R_k^0, $k \in N_s$ such that

$$g_k(x, x^0, R_k + R_k^0) \leq 0, \quad k \in N_s.$$

Then due to (4) for any $k \in N_s$, we obtain

$$0 \geq g_k(x, x^0, R_k + R_k^0) = \max_{i \in N_m}(R_{ik} + R_{ik}^0)x - \max_{i \in N_m}(R_{ik} + R_{ik}^0)x^0 =$$

$$= \min_{i' \in N_m} \max_{i \in N_m}(R_{ik}x - R_{i'k}x^0 + R_{ik}^0 x - R_{i'k}^0 x^0) \geq$$

$$\geq g_k(x, x^0, R_k) - \|R_k^0\|_{p\infty}(\|x\|_{p^*} + \|x^0\|_{p^*}) \geq$$

$$\geq g_k(x, x^0, R_k) - \|R^0\|_{p\infty\infty}(\|x\|_{p^*} + \|x^0\|_{p^*}) >$$

$$> g_k(x, x^0, R_k) - \varphi_1(\|x\|_{p^*} + \|x^0\|_{p^*}).$$

From the last, we deduce

$$\|[g(x, x^0, R)]^+\|_\infty < \varphi_1(\|x\|_{p^*} + \|x^0\|_{p^*}),$$

and it contradicts to (6). Finally, using Lemma, we get $x^0 \in P^s(R+R')$ for any $R' \in \Omega(\varphi_1)$. Hence, $\rho_1 \geq \varphi_1$.

Now we prove that $\rho_1 \leq \psi_1$. According to definition of $\psi_1 > 0$, there exists a portfolio $x^* \in X \setminus \{x^0\}$ such that

$$g_k(x^*, x^0, R_k) \leq [g_k(x^*, x^0, R_k)]^+ \leq$$

$$\leq \|[g(x^*, x^0, R)]^+\|_\infty = \psi_1 \|x^* - x^0\|_{p^*}, \qquad k \in N_s. \tag{7}$$

Assuming $\varepsilon > \psi_1$, consider a perturbing matrix $R^0 = [r^0_{ijk}] \in \mathbf{R}^{m \times n \times s}$ with elements

$$r^0_{ijk} = \delta \, \frac{x^0_j - x^*_j}{\|x^* - x^0\|_p}, \quad i \in N_m, \ \ j \in N_n, \ \ k \in N_s,$$

where $\psi_1 < \delta < \varepsilon$. Since in any cuts $R^0_k \in \mathbf{R}^{m \times n}$, $k \in N_s$, all the rows R^0_{ik}, $i \in N_m$, are the same (let A denotes such a row), we have

$$A = \delta \frac{(x^0 - x^*)^T}{\|x^* - x^0\|_p}. \tag{8}$$

Therefore, $\|R^0\|_{p\infty\infty} = \|R^0_k\|_{p\infty} = \|R^0_{ik}\|_p = \|A\|_p = \delta$, $i \in N_m$, $k \in N_s$, and, hence $R^0 \in \Omega(\varepsilon)$ for any $\varepsilon > \delta$. Further, due to (2) and (8), for any $p \in [1, \infty]$ the chain of equalities is true

$$A(x^* - x^0) = -\delta \frac{\|x^* - x^0\|_1}{\|x^* - x^0\|_p} = -\delta \|x^* - x^0\|_{p^*}.$$

Finally, using the above equalities along with (7), we conclude that for any $k \in N_s$ the following is true

$$g_k(x^*, x^0, R_k + R^0_k) = \max_{i \in N_m} (R_{ik} + A)x^* - \max_{i \in N_m} (R_{ik} + A)x^0 =$$

$$= g_k(x^*, x^0, R_k) + A(x^* - x^0) = g_k(x^*, x^0, R_k) - \delta \|x^* - x^0\|_{p^*} <$$

$$< g_k(x^*, x^0, R_k) - \psi_1 \|x^* - x^0\|_{p^*} \leq 0.$$

Therefore, $x^0 \notin P^s(R + R^0)$. And hence, $\rho_1 \leq \psi_1$. □

Attainability of the upper and lower bounds specified in (5) when $p = \infty$ follows also from the following evident statement, which is a direct consequence of Theorem 1.

Corollary 1 If for any investment portfolio $x \neq x^0$ the set $\{j \in N_n : x_j^0 = x_j = 1\}$ is empty, then for any number $m \in \mathbf{N}$ the formula

$$\rho_1^s(x^0, m, \infty, \infty, \infty) = \varphi_1^s(x^0, m, \infty, \infty, \infty) = \psi_1^s(x^0, m, \infty, \infty, \infty) =$$

$$= \min_{x \in X \setminus \{x^0\}} \frac{\|[g(x, x^0, R)]^+\|_\infty}{\|x + x^0\|_1}$$

holds.

From Theorem 1 it also follows the corollary below.

Corollary 2[[21]] For any $m \in \mathbf{N}$, the following bounds take place

$$\varphi_1^s(x^0, m, \infty, \infty, \infty) \leq \rho_1^s(x^0, m, \infty, \infty, \infty) \leq \psi_1^s(x^0, m, \infty, \infty, \infty).$$

The following theorem gives evidence about attainability of lower bound specified in Corollary 3, i.e. lower bound (5) while $p = \infty$.

Theorem 2. *There exists a class of problems $Z^s(R)$, such that for portfolio $x^0 \in P^s(R)$ the following relations are valid*

$$0 < \rho_1^s(x^0, m, \infty, \infty, \infty) = \varphi_1^s(x^0, m, \infty, \infty, \infty) < \psi_1(x^0, m, \infty, \infty, \infty). \tag{9}$$

Proof. Let $\varphi_1 > 0$. To fulfill the inequality $\varphi_1 < \psi_1$ it is sufficient that $\|x + x^0\|_1 > \|x - x^0\|_1$ holds for any $x \in X \setminus \{x^0\}$. To prove $\rho_1 = \varphi_1$, according to Theorem 1, it is sufficient to specify a class of problems with $\rho_1 \leq \varphi_1$. So, the rest of the proof is about this. From the definition of $\varphi_1 > 0$ there exists $x^* \in X \setminus \{x^0\}$ with

$$\varphi_1 \|x^* + x^0\|_1 \geq g_k(x^*, x^0, R_k), \quad k \in N_s. \tag{10}$$

Further conclusions are true for any $k \in N_s$. Denote

$$i(x^0) = \arg\max\{R_{ik}x^0 : \ i \in N_m\},$$

$$i(x^*) = \arg\max\{R_{ik}x^* : \ i \in N_m\},$$
$$\Delta = \|x^* + x^0\|_1 - \|x^* - x^0\|_1 > 0.$$

Further assume

$$(R_{i(x^*)k} - R_{i(x^0)k})x^* > \varphi_1\Delta, \tag{11}$$

which implies $i(x^0) \neq i(x^*)$ since $\varphi_1\Delta > 0$. For any $\varepsilon > \varphi_1$ the elements of the cut R_k^0 in the perturbing matrix R^0 we define as below

$$r_{ijk}^0 = \begin{cases} \delta & \text{if} \ \ i = i(x^0), \ \ x_j^0 = 1, \\ \\ -\delta & \text{otherwise}, \end{cases} \tag{12}$$

where

$$\min\left\{\varepsilon, \frac{1}{\Delta}(R_{i(x^*)k} - R_{i(x^0)k})x^*\right\} > \delta > \varphi_1. \tag{13}$$

Notice also that the last inequalities are valid due to (11). Due to the construction specific of R_k^0 we have

$$R_{ik}^0 x^* = -\delta\|x^*\|_1, \qquad i \in N_m \setminus \{i(x^0)\}, \tag{14}$$

$$R_{i(x^0)k}^0 x^0 = \delta\|x^0\|_1, \tag{15}$$

$$\|R_k^0\|_{p\infty} = \|R^0\|_{p\infty\infty} = \delta, \quad R^0 \in \Omega(\varepsilon).$$

Additionally,

$$R_{i(x^0)k}^0 x^* = \delta(\Delta - \|x^*\|_1). \tag{16}$$

Indeed, let

$$Q_1 = \{j \in N_n : \ \ x_j^* = x_j^0 = 1\},$$
$$Q_2 = \{j \in N_n : \ \ x_j^* = 1, \ x_j^0 = 0\}.$$

Then

$$|Q_1| = \Delta/2,$$
$$|Q_2| = \|x^*\|_1 - \Delta/2,$$
$$R_{i(x^0)k}^0 x^* = \delta(|Q_1| - |Q_2|),$$

and (16) follows.

Further we prove $g_k(x^*, x^0, R_k + R_k^0) < 0$. According to (15) we have

$$f_k(x^0, R_k + R_k^0) = \max_{i \in N_m}(R_{ik} + R_{ik}^0)x^0 = f_k(x^0, R_k) + \delta\|x^0\|_1. \quad (17)$$

Now we show

$$f_k(x^*, R_k + R_k^0) = f_k(x^*, R_k) - \delta\|x^*\|_1. \quad (18)$$

Using (14), we yield

$$f_k(x^*, R_k + R_k^0) = \max\Big\{(R_{i(x^*)k} + R_{i(x^*)k}^0)x^*, \max_{i \neq i(x^*)}(R_{ik} + R_{ik}^0)x^*\Big\} =$$

$$= \max\Big\{(f_k(x^*, R_k) - \delta\|x^*\|_1), \max_{i \neq i(x^*)}(R_{ik} + R_{ik}^0)x^*\Big\}.$$

Therefore, taking into account that

$$f_k(x^*, R_k) - \delta\|x^*\|_1 \geq (R_{ik} + R_{ik}^0)x^*, \quad i \in N_m \setminus \{i(x^0), i(x^*)\},$$

in order to prove (18) it is sufficient to check

$$f_k(x^*, R_k) - \delta\|x^*\|_1 \geq (R_{i(x^0)k} + R_{i(x^0)k}^0)x^*.$$

To do this, we use (13) and (16)

$$f_k(x^*, R_k) - \delta\|x^*\|_1 - (R_{i(x^0)k} + R_{i(x^0)k}^0)x^* =$$

$$= (R_{i(x^*)k} - R_{i(x^0)k})x^* - \delta\|x^*\|_1 - R_{i(x^0)k}^0 x^* >$$

$$> \delta(\Delta - \|x^*\|_1) - R_{i(x^0)k}^0 x^* = 0.$$

Finally, sequentially applying (17), (18), (10) and (13), for any index $k \in N_s$ we get

$$g_k(x^*, x^0, R_k + R_k^0) = g_k(x^*, x^0, R_k) - \delta\|x^* + x^0\|_1 \leq (\varphi_1 - \delta)\|x^* + x^0\|_1 < 0.$$

And hence, the formula below holds

$$\forall \varepsilon > \varphi_1 \quad \exists R^0 \in \Omega(\varepsilon) \qquad (x^0 \notin P^s(R + R^0)),$$

which due to $x^0 \in P^s(R)$ produces $\rho_1 \leq \varphi_1$. Summarizing, the correctness of (9) now becomes clear. $\square$

Now consider a numerical example illustrating Theorem 2.

Example 1. *Let* $m = 2,\ n = 3,\ s = 1,\ X = \{x^0, x^*\},\ x^0 = (1, 1, 0)^T$, $x^* = (0, 1, 1)^T$,

$$R = \begin{pmatrix} -5 & 2 & 2 \\ 1 & -1 & 0 \end{pmatrix}.$$

Then $f(x^0, R) = 0,\ f(x^*, R) = 4,\ x^0 \in P^1(R),\ \|x^* + x^0\|_1 = 4,\ \|x^* - x^0\|_1 = 2,\ i(x^0) = 2,\ i(x^*) = 1$. *Therefore* $\varphi_1 = 1,\ \psi_2 = 2,\ (R_{i(x^*)k} - R_{i(x^0)k})x^* = 5 > 2 = \varphi_1(\|x^* + x^0\|_1 - \|x^* - x^0\|_1)$.

If the perturbing matrix R^0 *is defined according to* (12)

$$R^0 = \begin{pmatrix} 0 & -\delta & -\delta \\ \delta & \delta & -\delta \end{pmatrix}, \quad 1 < \delta < 2.5,$$

it is easy to see that $\|R^0\| = \delta$ *and* $f(x^0, R + R^0) = 2\delta > 4 - 2\delta = f(x^*, R + R^0)$. *from the last and from the relations* $\|R^0\| > 1,\ x^0 \in P^1(R)$ *it follows that* $\rho_1 \leq 1$. *Therefore due to Theorem 1 we conclude* $\rho_1 = \varphi_1 = 1 < \psi_1 = 2$.

The following known result gives us the evidence about attainability of the upper bound on stability radius of $x^0 \in P^s(R)$ in $Z^s(R)$ for the case $m = 1$ (see (1)). In this context $\mathbf{R}^n$ is endowed with l_p, and $\mathbf{R}^s$ is endowed with l_∞.

Theorem 3 ([29])**.** *For any* $p \in [1, \infty]$ *and* $s \in \mathbf{N}$, *the stability radius of* $x^0 \in P^s(R)$ *in the linear Boolean programming problem* $Z^s_B(R)$, $R \in \mathbf{R}^{s \times n}$ *is expressed by the formula*

$$\rho^s_1(x^0) = \min_{x \in X \setminus \{x^0\}} \frac{\|[R(x - x^0)]^+\|_\infty}{\|x - x^0\|_{p^*}}.$$

4 Case B: market state space $\mathbf{R}^m$ is endowed with l_p

Now consider the case when portfolio space $\mathbf{R}^n$ and risk space $\mathbf{R}^s$ are endowed with l_∞, whereas market state space $\mathbf{R}^m$ is equipped with arbitrary Hölder's l_p norm, $1 \leq p \leq \infty$. Thus, the norm of matrix is defined by

$$\|R\|_{\infty p \infty} = \|(\|R_1\|_{\infty p}, \|R_2\|_{\infty p}, \ldots, \|R_s\|_{\infty p})\|_{\infty} = \max_{k \in N_s} \|R_k\|_{\infty p},$$

where

$$\|R_k\|_{\infty p} = \|(\|R_{1k}\|_{\infty}, \|R_{2k}\|_{\infty}, \ldots, \|R_{mk}\|_{\infty})\|_p, \quad k \in N_s.$$

Obviously,

$$\|R_{ik}\|_{\infty} \leq \|R_k\|_{\infty p} \leq \|R\|_{\infty p \infty}, \quad i \in N_m, \ k \in N_s.$$

Additionally, due to Hölder's inequality, for any x, $x^0 \in X$ we have

$$R_{ik}x - R_{i'k}x^0 \geq -(\|R_{ik}\|_{\infty}\|x\|_1 + \|R_{i'k}\|_{\infty}\|x^0\|_1) \geq$$

$$\geq -\|R_k\|_{\infty p}\|x + x^0\|_1, \quad i, i' \in N_m, \ k \in N_s. \tag{19}$$

In this context, $\rho_2^s(x^0) = \rho_2^s(x^0, m, \infty, p, \infty)$ is the stability radius of x^0. For Pareto optimal portfolio x^0 in $Z^s(R)$ we use the following notations

$$\varphi_2 = \varphi_2^s(x^0, m) = \min_{x \in X \setminus \{x^0\}} \frac{\|[g(x, x^0, R)]^+\|_{\infty}}{\|x + x^0\|_1},$$

$$\psi_2 = \psi_2^s(x^0, m) = \min_{x \in X \setminus \{x^0\}} \frac{\|[g(x, x^0, R)]^+\|_{\infty}}{\|x - x^0\|_1}.$$

Evidently, $\psi_2 \geq \varphi_2 \geq 0$.

Theorem 4. *For any m, $s \in \mathbf{N}$ and $p \in [1, \infty]$ the stability radius $\rho_2^s(x^0, m)$ of Pareto optimal portfolio $x^0 \in P^s(R)$ in $Z^s(R)$ the following bounds are valid*

$$\varphi_2^s(x^0, m) \leq \rho_2^s(x^0) \leq m^{1/p}\psi_2^s(x^0, m).$$

Proof. Let $x^0 \in P^s(R)$. First we prove $\rho_2 \geq \varphi_2$. It is evident if $\varphi_2 = 0$. Let $\varphi_2 > 0$. According to the definition of φ_2, for any portfolio $x \in X \setminus \{x^0\}$ the inequality

$$\|[g(x, x^0, R)]^+\|_\infty \geq \varphi_2 \|x + x^0\|_1 \tag{20}$$

is true. Further, by contradiction, we show the correctness of the formula given below

$$\forall R' \in \Omega(\varphi_2) \quad \exists l \in N_s \qquad (g_l(x, x^0, R_l + R'_l) > 0).$$

From contrary, let it be so that there exists a perturbing matrix $R^0 \in \Omega(\varphi_2)$ with cuts R^0_k, $k \in N_s$ such that

$$g_k(x, x^0, R_k + R^0_k) \leq 0, \quad k \in N_s.$$

Then according to (19) for any index $k \in N_s$ we obtain

$$0 \geq g_k(x, x^0, R_k + R^0_k) = \max_{i \in N_m}(R_{ik} + R^0_{ik})x - \max_{i \in N_m}(R_{ik} + R^0_{ik})x^0 =$$

$$= \min_{i' \in N_m} \max_{i \in N_m}(R_{ik}x - R_{i'k}x^0 + R^0_{ik}x - R^0_{i'k}x^0) \geq$$

$$\geq g_k(x, x^0, R_k) - \|R^0_k\|_{\infty p} \|x + x^0\|_1 \geq$$

$$\geq g_k(x, x^0, R_k) - \|R^0\|_{\infty p \infty} \|x + x^0\|_1 > g_k(x, x^0, R_k) - \varphi_2 \|x + x^0\|_1.$$

From the last we deduce inequality

$$\|[g(x, x^0, R)]^+\|_\infty < \varphi_2 \|x + x^0\|_1,$$

which contradicts to (20). Finally, applying Lemma, we have $x^0 \in P^s(R + R')$ for every perturbing matrix $R' \in \Omega(\varphi_2)$. Hence, $\rho_2 \geq \varphi_2$.

Now we prove $\rho_2 \leq m^{1/p}\psi_2$. According to the definition $\psi_2 > 0$, there exists a portfolio $x^* \in X \setminus \{x^0\}$ such that

$$g_k(x^*, x^0, R_k) \leq [g_k(x^*, x^0, R_k)]^+ \leq$$

$$\leq \|[g(x^*, x^0, R)]^+\|_\infty = \psi_2 \|x^* - x^0\|_1, \qquad k \in N_s. \tag{21}$$

Assuming $\varepsilon > m^{1/p}\psi_2$, consider a perturbing matrix $R^0 = [r^0_{ijk}] \in \mathbf{R}^{m\times n\times s}$ whose elements are defined as follows

$$r^0_{ijk} = \delta(x^0_j - x^*_j), \quad i \in N_m, \ \ j \in N_n, \ \ k \in N_s,$$

where $\varepsilon/m^{1/p} > \delta > \psi_2$. Since all the rows R^0_{ik}, $i \in N_m$ in the cut $R^0_k \in \mathbf{R}^{m\times n}$, $k \in N_s$ are the same in the matrix $\mathbf{R}^0$, then we have (let $A \in \mathbf{R}^m$ denote such a row)

$$A = \delta(x^0 - x^*)^T. \tag{22}$$

$$\|R^0_{ik}\|_\infty = \|A\|_\infty = \delta, \quad i \in N_m, \ k \in N_s.$$

From the last and (3), we get

$$\|R^0_k\|_{\infty p} = m^{1/p}\delta, \quad k \in N_s,$$

$$\|R^0\|_{\infty p\infty} = m^{1/p}\delta \geq m^{1/p}\psi_2.$$

Thus $R^0 \in \Omega(\varepsilon)$ for any $\varepsilon > m^{1/p}\psi_2$. Further due to (22), we have

$$A(x^* - x^0) = -\delta\|x^* - x^0\|_1.$$

Finally, combining the equality above and (21), we conclude that for any $k \in N_s$ the following relations are true

$$g_k(x^*, x^0, R_k + R^0_k) = \max_{i\in N_m}(R_{ik} + A)x^* - \max_{i\in N_m}(R_{ik} + A)x^0 =$$

$$= g_k(x^*, x^0, R_k) + A(x^* - x^0) = g_k(x^*, x^0, R_k) - \delta\|x^* - x^0\|_1 <$$

$$< g_k(x^*, x^0, R_k) - \psi_2\|x^* - x^0\|_1 \leq 0.$$

Thus $x^0 \notin P^s(R + R^0)$. Hence, $\rho_2 \leq m^{1/p}\psi_2$. □

The following known result confirms attainability on the upper bound of the stability radius of $x^0 \in P^s(R)$ in $Z^s(R)$ for the case $m = 1$ (see (1)). In this context, both $\mathbf{R}^n$ and $\mathbf{R}^s$ are equipped with l_∞.

Theorem 5 ([30]). *For the stability radius of $x^0 \in P^s(R)$ in the Boolean linear programming problem $Z^s_B(R)$, $R \in \mathbf{R}^{s\times n}$, and $s \in \mathbf{N}$ the following analytical expression holds*

$$\rho^s_2(x^0) = \min_{x\in X\setminus\{x^0\}} \frac{\|[R(x - x^0)]^+\|_\infty}{\|x - x^0\|_1}.$$

5 Case C: risk space $\mathbf{R}^s$ is endowed with l_p

Now assume we measure distances by means of l_∞ in portfolio space $\mathbf{R}^n$ and market state space $\mathbf{R}^m$ while in risk space $\mathbf{R}^s$ we use l_p, $1 \le p \le \infty$. In this case under the norm of the matrix R we understand the number

$$\|R\|_{\infty\infty p} = \|(\|R_1\|_{\infty\infty}, \|R_2\|_{\infty\infty}, \dots, \|R_s\|_{\infty\infty})\|_p,$$

where

$$\|R_k\|_{\infty\infty} = \|(\|R_{1k}\|_\infty, \|R_{2k}\|_\infty, \dots, \|R_{mk}\|_\infty)\|_\infty, \quad k \in N_s.$$

Obviously,

$$\|R_{ik}\|_\infty \le \|R_k\|_{\infty\infty} \le \|R\|_{\infty\infty p}, \quad i \in N_m, \ k \in N_s.$$

It is easy to check that for any portfolios x and x' the inequalities hold

$$R_{ik}x - R_{i'k}x' \ge -\|R_k\|_{\infty\infty}\|x + x'\|_1, \quad i, \ i' \in N_m, \quad k \in N_s. \tag{23}$$

In this context, $\rho_3 = \rho_3^s(x^0, m, \infty, \infty, p)$ denotes the stability radius of x^0. For Pareto optimal portfolio x^0 in $Z^s(R)$, we introduce the notation

$$\varphi_3 = \varphi_3^s(x^0, m, \infty, \infty, p) = \min_{x \in X \setminus \{x^0\}} \frac{\|[g(x, x^0, R)]^+\|_p}{\|x + x^0\|_1},$$

$$\psi_3 = \psi_3^s(x^0, m, \infty, \infty, p) = \min_{x \in X \setminus \{x^0\}} \frac{\|[g(x, x^0, R)]^+\|_p}{\|x - x^0\|_1}.$$

Evidently, $\psi_3 \ge \varphi_3 \ge 0$.

Theorem 6. *For any $m, \ s \in \mathbf{N}$ and $p \in [1, \infty]$, the stability radius $\rho_3^s(x^0, m, \infty, \infty, p)$ of portfolio $x^0 \in P^s(R)$ in $Z^s(R)$ has the following lower and upper bounds*

$$\varphi_3^s(x^0, m, \infty, \infty, p) \le \rho_3^s(x^0, m, \infty, \infty, p) \le \psi_3^s(x^0, m, \infty, \infty, p).$$

Proof. Let $x^0 \in P^s(R)$. First we prove $\rho_3 \geq \varphi_3$. Without loss of generality, assume $\varphi_3 > 0$ (otherwise inequality $\rho_3 \geq \varphi_3$ is obvious). According to the definition of φ_3, for any $x \neq x^0$ the following is true

$$\|[g(x, x^0, R)]^+\|_p \geq \varphi_3 \|x + x^0\|_1. \tag{24}$$

To prove the lower bound, it is necessary to show that the formula below is true

$$\forall R' \in \Omega(\varphi_3) \quad \exists l \in N_s \qquad (g_l(x, x^0, R_l + R'_l) > 0). \tag{25}$$

From contrary, assume there exists a perturbing matrix $R^0 \in \Omega(\varphi_3)$ such that

$$g_k(x, x^0, R_k + R^0_k) \leq 0, \quad k \in N_s.$$

Then using (23), we easily deduce

$$0 \geq g_k(x, x^0, R_k + R^0_k) = \min_{i' \in N_m} \max_{i \in N_m} (R_{ik}x - R_{i'k}x^0 + R^0_{ik}x - R^0_{i'k}x^0) \geq$$

$$\geq g_k(x, x^0, R_k) - \|R^0_k\|_{\infty\infty} \|x + x^0\|_1,$$

i.e.

$$[g_k(x, x^0, R_k)]^+ \leq \|R^0_k\|_{\infty\infty} \|x + x^0\|_1, \quad k \in N_s.$$

Thus, due to $R^0 \in \Omega(\varphi_3)$ while $p \in [1, \infty]$ we have

$$\|[g(x, x^0, R)]^+\|_p \leq \|R^0\|_{\infty\infty p} \|x + x^0\|_1 < \varphi_3 \|x + x^0\|_1.$$

This contradicts to (24), and hence (25) is true. From here, according to the Lemma, $x^0 \in P^s(R + R')$ for any $R' \in \Omega(\varphi_3)$. Hence, $\rho^s_3(x^0, m, \infty, \infty, p) \geq \varphi^s_3(x^0, m, \infty, \infty, p)$.

Further, we prove that $\rho_3 \leq \psi_3$ holds for any $p \in [1, \infty]$. Let $\varepsilon > \psi_3 > 0$, and portfolio $x^* \neq x^0$ is such that

$$\|[g(x^*, x^0, R)]^+\|_p = \psi_3 \|x - x^0\|_1.$$

Then, taking into account that the norm l_p depends on vector continuously, we take $\delta \in \mathbf{R}^s$ with positive components such that

$$\delta_k \|x^* - x^0\|_1 > [g_k(x^*, x^0, R_k)]^+, \qquad k \in N_s, \tag{26}$$

and $\varepsilon > \|\delta\|_p > \psi_3$. Then we construct a perturbing matrix $R^0 \in \Omega(\varepsilon)$, where $\varepsilon > \|\delta\|_p$, with cuts R_k^0, $k \in N_s$ such that for every $k \in N_s$ the inequality

$$g_k(x^*, x^0, R_k + R_k^0) < 0 \tag{27}$$

holds. Using components of vector δ, we define the elements of any k-th cut $R_k^0 = [r_{ijk}^0] \in \mathbf{R}^{m\times n}$ of the perturbing matrix $R^0 = [r_{ijk}^0] \in \mathbf{R}^{m\times n\times s}$ using the formula

$$r_{ijk}^0 = \begin{cases} \delta_k & \text{if } \ i \in N_m, \quad x_j^0 \geq x_j^*, \\ -\delta_k & \text{if } \ i \in N_m, \quad x_j^0 < x_j^*. \end{cases}$$

Then, we have

$$\|R_k^0\|_{\infty\infty} = \delta_k, \quad k \in N_s.$$

Therefore, it is easy to see that $\|R^0\|_{\infty\infty p} = \|\delta\|_p < \varepsilon$. Additionally, all the rows R_{ik}^0 $(i \in N_m)$ in the cut R_k^0, $k \in N_s$ are the same and contain components δ_k and $-\delta_k$ only. Denoting such a row A_k, we obtain

$$A_k(x^* - x^0) = -\delta_k\|x^* - x^0\|_1, \quad k \in N_s.$$

From this for any $k \in N_s$ due to (26) we get (27):

$$g_k(x^*, x^0, R_k + R_k^0) = g_k(x^*, x^0, R_k) + A(x^* - x^0) =$$

$$= g_k(x^*, x^0, R_k) - \delta_k\|x^* - x^0\|_1 \leq$$

$$\leq [g_k(x^*, x^0, R_k)]^+ - \delta_k\|x^* - x^0\|_1 < 0, \quad k \in N_s.$$

Thus, while $\varepsilon > \psi_3$ there exists a perturbing matrix $R^0 \in \Omega(\varepsilon)$ such that $x^0 \in P^s(R)$ is not Pareto optimal in the perturbed problem $Z^s(R + R^0)$. This implies that for any $\varepsilon > \psi_3$ we have $\rho_3 < \varepsilon$. Hence, $\rho_3 \leq \psi_3^s$, and then $p \in [1, \infty]$. □

The following statement gives the evidence about attainability of the lower and upper bounds specified in Theorem 6.

Corollary 3 If for any $x \neq x^0$ the set $\{j \in N_n : \quad x_j^0 = x_j = \ 1\}$ is empty, then for any $m \in \mathbf{N}$ any $p \in [1, \infty]$ the following holds

$$\rho_3^s(x^0, m, \infty, \infty, p) = \varphi_3^s(x^0, m, \infty, \infty, p) =$$

$$= \psi_3^s(x^0, m, \infty, \infty, p) = \min_{x \in X \setminus \{x^0\}} \frac{\|[g(x, x^0, R)]^+\|_p}{\|x + x^0\|_1}.$$

If $m = 1$, as it was pointed out before, $Z^s(R)$ transforms into s-criteria Boolean linear programming problem $Z_B^s(R)$, $R \in \mathbf{R}^{s \times n}$ (see (1)). In this context, $\mathbf{R}^n$ is equipped with l_∞, and $\mathbf{R}^s$ is equipped with l_p, $1 \le p \le \infty$. The following known result illustrates the fact that the upper bound specified in Theorem 6 is right.

Theorem 7 ([29]). *For any $p \in [1, \infty]$ and $s \in \mathbf{N}$, the stability radius of $x^0 \in P^s(R)$ in $Z_B^s(R)$, $R \in \mathbf{R}^{s \times n}$ is expressed by the formula*

$$\rho_3^s(x^0) = \min_{x \in X \setminus \{x^0\}} \frac{\|[R(x - x^0)]^+\|_p}{\|x - x^0\|_1}.$$

6 Conclusion

While composing a portfolio, the investor's intention to minimize different types of risks motivates the use of multicriteria environment within the corresponding mathematical and economical models. This approach allows using a variety of multicriteria decision making methods [26], [31]. In this paper, to maintain the different types of risks, we used the bottleneck partial objectives, forcing the investor to choose a portfolio with the minimal total aggregated risk in the worst case scenario, i.e. in the situation where the values of risks are at their maximum.

Another type of uncertainty is related to inaccuracy of statistical observations and expert evaluations while measuring different risks. In this context there is a necessity to conduct post-optimal analysis in order to quantify the extreme level of initial data changes not violating the portfolio optimality. In this work different cases are analyzed depending on a type of the metric used in problem parameter spaces. In all the cases analyzed, the lower and upper bounds on stability radius of an efficient portfolio are presented.

The results give an investor information on reliability of chosen optimal portfolio and prevent the situation, when his/her portfolio will lose its property with unexpected changes in the initial data. Using Hölder's norm the control of changes in the initial data can vary depending on the space of the problem parameters. The results could be potentially interesting for construction the investment models in which the investor wants to merge different types of risks at the presence of unreliable information and forecasted financial market states.

The straightforward application of the results to practical calculation is limited due to enumerating structure of analytical expressions which may need a number of comparisons growing exponentially with n and s. In the case when direct calculation is time consuming (it may happen if $n \geq 40$ and $s \geq 3$), getting the values should be calculated heuristically, for example some multicriteria genetic algorithms can be used.

It is also important to note that sometimes the stability radius does not give us complete information about the quality of a given solution in the case when problem data are located outside of the stability region. Some attempts to study a quality of the problem solution in this case are connected with concepts of stability and accuracy functions. These functions were first introduced in [32],[33] for the scalar combinatorial optimization problem. In [34], the results were later extended to the vector linear discrete optimization problem with Pareto and lexicographic optimality principles. Similar results were obtained for Boolean linear programming [35], game theory problem formulations [36] and some scheduling models [37]. Moreover, as it was shown recently (see, e.g. [38], [39]), calculating stability and accuracy functions is closely related to analyzing problem robustness. Robust optimization in that context is understood as a process aiming to produce solutions that optimize an additionally constructed objective. The objective must assure that the optimal solution will remain feasible under worst case realization of uncertain problem input parameters. Robust optimization is also known as worst-case or minmax regret optimization, and optimal solutions of worst case optimization are often referred to as robust solutions (see, e.g. [40]). Conducting similar research for investment models could be an interesting direction for further investigations.

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Received December 1, 2017

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