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On the locating matrix of a graph and its spectral analysis

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Abstract

We introduce a new matrix representation for a graph by defining the locating matrix $\mathbf{Lo}(G)$ of G. We define the locating eigenvalues, the locating spectrum, and locating energy of the graph and we calculate them for some standard graphs. We also obtain bounds for the locating energy for regular and strongly regular graphs.

Keywords: Locating eigenvalues (of graph), Locating Spectrum (of graph), Locating energy (of graph).

1 Introduction

A graph is completely determined by either its adjacencies or its incidences. This information can be conveniently stated in matrix form. It is often possible to make use of these matrices in order to identify certain properties of a graph. The **adjacency matrix** $A(G) = A = [a_{ij}]$ of a labeled graph G with p points is the $p \times p$ matrix in which $a_{ij} = 1$ if v_i is adjacent with v_j and $a_{ij} = 0$ otherwise. Thus, there is a one-to-one correspondence between labeled graphs with p vertices and $p \times p$ symmetric binary matrices with zero diagonal. The eigenvalues are the roots of the characteristic polynomial

$$\phi(G;\lambda) = det(\lambda I - A)$$
$$= \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_{n-1}\lambda + c_n,$$

where I is the $n \times n$ identity matrix. The eigenvalues of $A(G)$ are eigenvalues of G. Since A is a real symmetric matrix with zero trace, these eigenvalues are all real with sum equal to zero. The Spectrum of a graph is the list of distinct eigenvalues $\lambda_1 > \lambda_2 > \ldots > \lambda_r$ of G, with multiplicities $m_1, m_2, \ldots, m_r$, represented by

$$\mathrm{Spec(G)} = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_r \\ m_1 & m_2 & \cdots & m_r \end{pmatrix}.$$

The energy of the graph G is defined in [8] as the sum of the absolute values of its eigenvalues:

$$E(G) = \sum_{i=1}^{n} |\gamma_i| \ .$$

Details on the theory of graph energy can be found in the book [20], whereas details on its chemical applications in the book [14] and in the review [11].

Lemma 1. *[6] For the standard graphs K_p, $K_{m,n}$ and C_n, we have*

- $Spec(K_p) = \begin{pmatrix} n-1 & -1 \\ 1 & n-1 \end{pmatrix}$.
- $Spec(K_{m,n}) = \begin{pmatrix} \sqrt{mn} & -\sqrt{mn} & 0 \\ 1 & 1 & m+n-2 \end{pmatrix}$.
- $Spec(C_n) = \begin{cases} \begin{pmatrix} 2 & 2\cos\frac{2\pi}{n} & \cdots & 2\cos\frac{2(n-1)\pi}{n} \\ 1 & 2 & \cdots & 2 \end{pmatrix}, & \textit{if } n \textit{ is odd;} \\ \begin{pmatrix} 2 & 2\cos\frac{2\pi}{n} & \cdots & 2\cos\frac{2(n-2)\pi}{n} & -2 \\ 1 & 2 & \cdots & 2 & 1 \end{pmatrix}, & \textit{if } n \textit{ is even.} \end{cases}$

2 The Locating Spectrum of a Graph

Definition 1. *Let $G = (V, E)$ be a connected graph with vertex set $V = \{v_1, v_2, ..., v_n\}$. A locating function of G denoted by $\mathbf{L}(G)$ is a function $\mathbf{L}(G) : V(G) \to \mathbb{R}^n$ such that $\mathbf{L}(v_i) = \overrightarrow{v_i} = (d(v_1, v_i), d(v_2, v_i), ..., d(v_n, v_i))$, where $d(v_i, v_j)$ is the distance between the vertices v_i and v_j in G. The vector $\overrightarrow{v_i}$ is called the locating vector*

corresponding to the vertex v_i.
The locating product of two locating vectors $\overrightarrow{v_i}$ *and* $\overrightarrow{v_j}$ *in a graph* G *is denoted by* $L(\overrightarrow{v_i}.\overrightarrow{v_j})$ *and defined as:*

$$L(\overrightarrow{v_i}.\overrightarrow{v_j}) = \begin{cases} \overrightarrow{v_i}.\overrightarrow{v_j}, & \text{if } i \neq j \text{ and } v_i \text{ adjacent to } v_j; \\ 0 & \text{otherwise,} \end{cases}$$

where $\overrightarrow{v_i}.\overrightarrow{v_j}$ *is the dot product of the vectors* $\overrightarrow{v_i}$ *and* $\overrightarrow{v_j}$ *in the Euclidean space* $\mathbb{R}^n$.

The locating matrix of G *is then* $\mathbf{Lo} = \mathbf{Lo}(G) = [l_{ij}]$, *where*

$$l_{ij} = L(\overrightarrow{v_i}.\overrightarrow{v_j}).$$

The characteristic polynomial $det(\gamma\, \boldsymbol{I} - \mathbf{Lo}(G))$ *of* $\mathbf{Lo}(G)$ *is called the* ***Lo-characteristic polynomial*** *of* G *and is denoted by* $P_{Lo}(G) = \sum_{i=0}^{n} a_i \gamma^{n-i}$. *The eigenvalues of the matrix* $\mathbf{Lo}(G)$, *which are the zeros of* $|\gamma \boldsymbol{I} - \mathbf{Lo}(G)|$ *are called the Lo-eigenvalues of* G *and form its Spectrum denoted by* $Spec_{Lo}(G)$. *If the distinct Lo-eigenvalues of* G *are* $\gamma_1, \gamma_2 ..., \gamma_m$ *with multiplicities* $t_1, t_2, ..., t_m$ *respectively, then,* $Spec_{Lo}(G)$ *is written as:* $\begin{pmatrix} \gamma_1 & \gamma_2 & ... & \gamma_m \\ t_1 & t_2 & ... & t_m \end{pmatrix}$.

By the above definition, the locating matrix is a real symmetric $n \times n$ matrix. Therefore its eigenvalues $\gamma_1, \gamma_2, \ldots, \gamma_m$ are real numbers. Since the trace of $\mathbf{Lo}(G)$ is zero, the sum of its eigenvalues is also equal to zero.
In this paper, by graph, we mean a simple, finite, undirected, connected graph and for short by $\overrightarrow{v_i}.\overrightarrow{v_j}$, we mean the locating product of the two locating vectors $\overrightarrow{v_i}$ and $\overrightarrow{v_j}$ in G. For graph theoretic terminology we refer to Charatrand and Lesniak [5].

Lemma 2. *Let* G *be a connected graph with* n *vertices and let* $\gamma_1, \gamma_2, ..., \gamma_n$ *be its Lo-eigenvalues. Then*

1. $\sum_{i=1}^{n} \gamma_i = 0$
2. $\sum_{i=1}^{n} \gamma_i^2 = 2 \sum_{1 \leq i < j \leq n} (\overrightarrow{v_i}.\overrightarrow{v_j})^2$

Proof. (1) $\sum_{i=1}^{n} \gamma_i = trace(Lo(G)) = \sum_{i=1}^{n} a_{ii} = 0$.
(2) For $i = 1, 2, ..., n$, the (i,i) entry of $(Lo(G))^2$ is equal to the $trace(Lo(G))^2$

$$trace[Lo(G)]^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}(\overrightarrow{v_i}.\overrightarrow{v_j})^2$$
$$= 2\sum_{1\le i<j\le n}(\overrightarrow{v_i}.\overrightarrow{v_j})^2.$$

□

Definition 2. *The* ***locating energy*** *of the graph G is*

$$E_{Lo} = E_{Lo}(G) = \sum_{i=1}^{n} |\gamma_i| \ .$$

Theorem 1. *For the complete graph K_n of order $n \ge 2$,*

$$Spec_{Lo}(K_n) = \begin{pmatrix} (n-1)(n-2) & -(n-2) \\ 1 & n-1 \end{pmatrix},$$

and $E_{Lo}(K_n) = 2(n-1)(n-2)$.

Proof. Let $G = K_n$ with vertices $v_1, v_2, ..., v_n$ and let $\overrightarrow{v_i}$ be the locating vector corresponding to the vertex v_i. Then $\overrightarrow{v_i} = (a_1, ..., a_n)$, where $a_i = 0$ and $a_j = 1$. Thus for any two vectors $\overrightarrow{v_i}, \overrightarrow{v_j}$, where $i \ne j$, we have

$$\overrightarrow{v_i}.\overrightarrow{v_j} = n-2.$$

Therefore, $\mathbf{Lo}(K_n) = (n-2)\,\mathbf{A}(K_n)$, were $A(K_n)$ is the adjacency matrix of K_n, and by Lemma 1 it is easy to see that $Spec_{Lo}(K_n) = \begin{pmatrix} (n-1)(n-2) & -(n-2) \\ 1 & n-1 \end{pmatrix}$. Hence $E_{Lo}(K_n) = 2(n-1)(n-2)$. □

We now determine the Lo-spectrum and Lo-energy of any cycle C_n.

Theorem 2. *Let $n \ge 2$ be an even integer. Then for the cycle C_n, we have*

$$Spec_{Lo}(C_n) =$$

$$= \begin{pmatrix} \frac{n(n-2)^2}{6} & \frac{n(n-2)^2}{6}\cos\frac{2\pi}{n} & \cdots & \frac{n(n-2)^2}{6}\cos\frac{2(n-2)\pi}{n} & -\frac{n(n-2)^2}{6} \\ 1 & 2 & \cdots & 2 & 1 \end{pmatrix}.$$

Further $E_{Lo}(C_n) = \frac{n(n-2)^2}{12} E(C_n)$, *where* $E(C_n)$ *is the energy of* C_n.

Proof. By labelling the vertices of the cycle C_n in the anticlockwise direction as $\{v_1, v_2, ..., v_n\}$, we observe that,

$$\overrightarrow{v_1} = \Big(0, 1, 2, 3, ..., \frac{n}{2}, \frac{n}{2} - 1, \frac{n}{2} - 2, ..., 1\Big)$$
$$\overrightarrow{v_2} = \Big(1, 0, 1, 2, ..., \frac{n}{2} - 1, \frac{n}{2}, \frac{n}{2} - 1, ..., 2\Big)$$
$$\overrightarrow{v_3} = \Big(2, 1, 0, 1, ..., \frac{n}{2} - 2, \frac{n}{2} - 1, \frac{n}{2}, ..., 3\Big)$$
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$$\overrightarrow{v_n} = \Big(1, 2, 3, ..., \frac{n}{2}, \frac{n}{2} - 1, \frac{n}{2} - 2, \frac{n}{2} - 3, ..., 0\Big).$$

Then, by symmetry,

$$\begin{aligned} \overrightarrow{v_i}.\overrightarrow{v_{i+1}} &= 2\Big((2)(1) + (3)(2) + (4)(3) + ... + \frac{n}{2}(\frac{n}{2} - 1)\Big) \\ &= 2\sum_{i=2}^{\frac{n}{2}} i(i-1) \\ &= 2\sum_{i=2}^{\frac{n}{2}} i^2 - 2\sum_{i=2}^{\frac{n}{2}} i \\ &= 2\bigg(\frac{\frac{n}{2}(\frac{n}{2}+1)(n+1)}{6} - 1\bigg) - 2\bigg(\frac{\frac{n}{2}(\frac{n}{2}+1)}{2} - 1\bigg) \\ &= \frac{n(n-2)^2}{12}. \end{aligned}$$

Therefore $\mathbf{Lo}(C_n) = \frac{n(n-2)^2}{12}\mathbf{A}(C_n)$.

Hence by Lemma 1, we get,

$$Spec_{Lo}(C_n) =$$

$$= \begin{pmatrix} \frac{n(n-2)^2}{6} & \frac{n(n-2)^2}{6}\cos\frac{2\pi}{n} & \cdots & \frac{n(n-2)^2}{6}\cos\frac{2(n-2)\pi}{n} & -\frac{n(n-2)^2}{6} \\ 1 & 2 & \cdots & 2 & 1 \end{pmatrix}.$$

Also clearly

$$E_{Lo}(C_n) = \frac{n(n-2)^2}{12} E(C_n),$$

where $E(C_n)$ is the energy of C_n. □

Theorem 3. *Let $n \geq 3$ be an odd integer. Then for the cycle C_n, we have,*

$$Spec_{Lo}(C_n) =$$

$$= \begin{pmatrix} \frac{(n-1)(n-2)(n+3)}{6} & \frac{(n-1)(n-2)(n+3)}{6}\cos\frac{2\pi}{n} & \cdots & \frac{(n-1)(n-2)(n+3)}{6}\cos\frac{2(n-1)\pi}{n} \\ 1 & 2 & \cdots & 2 \end{pmatrix}.$$

Further $E_{Lo}(C_n) = \frac{(n-1)(n-2)(n+3)}{6} E(C_n)$, where $E(C_n)$ is the energy of $E(C_n)$.

Proof. Let G be a cycle C_n with odd number n of vertices. By labeling the vertices of G with anticlockwise direction as $\{v_1, v_2, ..., v_n\}$, we observe that,

$$\overrightarrow{v_1} = \left(0, 1, 2, 3, ..., \frac{n-1}{2}, \frac{n-1}{2}, \frac{n-1}{2} - 1, \frac{n-1}{2} - 2, ..., 1\right)$$

$$\overrightarrow{v_2} = \left(1, 0, 1, 2, ..., \frac{n-1}{2} - 1, \frac{n-1}{2}, \frac{n-1}{2}, \frac{n-1}{2} - 1, ..., 2\right)$$

$$\overrightarrow{v_3} = \left(2, 1, 0, 1, ..., \frac{n-1}{2} - 2, \frac{n-1}{2} - 1, \frac{n-1}{2}, \frac{n-1}{2}, ..., 3\right)$$

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$$\overrightarrow{v_n} = \left(1, 2, 3, ..., \frac{n-1}{2}, \frac{n-1}{2}, \frac{n-1}{2} - 1, \frac{n-1}{2} - 2, ..., 0\right).$$

Then, by symmetry,

$$\begin{aligned}\overrightarrow{v_i}.\overrightarrow{v_{i+1}} &= 2\left[(2)(1)+(3)(2)+(4)(3)+...+\frac{n-1}{2}\left(\frac{n-1}{2}-1\right)\right]+\\ &+\frac{n-1}{2}\left(\frac{n-1}{2}\right)\\ &= 2\sum_{i=2}^{\frac{n-1}{2}} i(i-1)+\frac{(n-1)^2}{4}\\ &= 2\sum_{i=2}^{\frac{n-1}{2}} i^2 - 2\sum_{i=2}^{\frac{n-1}{2}} i+\frac{(n-1)^2}{4}\\ &= 2\left[\frac{\frac{n-1}{2}(\frac{n-1}{2}+1)(2\frac{n-1}{2}+1)}{6}-1\right]-\\ &-2\left[\frac{\frac{n-1}{2}(\frac{n-1}{2}+1)}{2}-1\right]+\frac{(n-1)^2}{4}\\ &= \frac{(n-1)(n-2)(n+3)}{12}.\end{aligned}$$

Therefore,

$$Spec_{Lo}(C_n) =$$

$$\begin{pmatrix} \frac{(n-1)(n-2)(n+3)}{6} & \frac{(n-1)(n-2)(n+3)}{6}\cos\frac{2\pi}{n} & \cdots & \frac{(n-1)(n-2)(n+3)}{6}\cos\frac{2(n-1)\pi}{n} \\ 1 & 2 & \cdots & 2 \end{pmatrix}.$$

Further $E_{Lo}(C_n) = \frac{(n-1)(n-2)(n+3)}{12}E(C_n)$. □

Theorem 4. *Let G be a complete bipartite graph $K_{a,b}$, where $1 \le a \le b$. Then*

$$Spec_{Lo}(K_{a,b}) = \begin{pmatrix} (2a+2b-4)\sqrt{ab} & -(2a+2b-4)\sqrt{ab} & 0 \\ 1 & 1 & a+b-2 \end{pmatrix}.$$

Further $E_{Lo}(K_{a,b}) = 4(a+b-2)\sqrt{ab}$.

Proof. Let the vertices of $K_{a,b}$ be labelled such that v_i are adjacent to v_{a+j} for all $1 \le i \le a$ and $1 \le j \le b$.

Now, it is obvious that the locating vectors $\overrightarrow{v_i}$ of v_i are given by:

$$\overrightarrow{v}_1 = \left(0, \underbrace{2,...,2}_{a-1}, \underbrace{1,1,...,1}_{b}\right),\ \overrightarrow{v_2} = \left(2, 0, \underbrace{2,...,2}_{a-2}, \underbrace{1,1,...,1}_{b}\right)$$

$$\overrightarrow{v}_3 = \left(2, 2, 0, \underbrace{2,...,2}_{a-3}, \underbrace{1,1,...,1}_{b}\right),...,$$

$$\overrightarrow{v}_a = \left(\underbrace{2,...,2}_{a-1}, 0, \underbrace{1,1,...,1}_{b}\right),\ \overrightarrow{v}_{a+1} = \left(\underbrace{1,...,1}_{a}, 0, \underbrace{2,...,2}_{b-1}\right),$$

$$\overrightarrow{v}_{a+2} = \left(\underbrace{1,...,1}_{a}, 2, 0, \underbrace{2,...,2}_{b-2}\right),...,\ \overrightarrow{v_{a+b}} = \left(\underbrace{1,...,1}_{a}, \underbrace{2,...,2}_{b-1}, 0\right).$$

Then it is easy to see that for any two locating vertices v_i, v_j in $K_{a,b}$, $\overrightarrow{v_i}.\overrightarrow{v_j} = 2(a+b-2)$. Therefore,

$$\mathbf{Lo}(K_{a,b}) = 2(a+b-2)\,\mathbf{A}(K_{a,b}).$$

Also by using Lemma 1, we get

$$Spec_{Lo}(K_{a,b}) = \begin{pmatrix} (2a+2b-4)\sqrt{ab} & -(2a+2b-4)\sqrt{ab} & 0 \\ 1 & 1 & a+b-2 \end{pmatrix},$$

and hence

$E_{Lo}(K_{a,b}) = 4(a+b-2)\sqrt{ab}.$ □

The following results are obtained straightforward from Theorem 4.

1. Let G be a complete bipartite graph $K_{n,n}$, where $n \geq 1$. Then $E_{Lo}(K_{n,n}) = 8n(n-1)$.

2. Let G be any star graph $K_{1,n}$. Then

$$E_{Lo}(K_{1,n}) = 4\sqrt{n}(n-1)$$

3 Locating Spectrum And Energy of Regular and Strongly Regular Graphs

One of the most important family of regular graphs is the strongly regular graphs (abbreviated SRG), which has so many beautiful properties. There are many SRGs arising from combinatorial concepts such as orthogonal arrays, latin squares, conference matrices, designs and geometric graphs.

A strongly regular graph (SRG) with parameters (n, k, λ, μ) is a graph on n vertices which is regular with valency k and has the following properties:

- any two adjacent vertices have exactly λ common neighbours;
- any two nonadjacent vertices have exactly μ common neighbours.

Theorem 5. *[7] Let G be a strongly regular graph with parameters (n, k, λ, μ). Then the eigenvalues of G satisfy the following properties:*

1. G has exactly three distinct eigenvalues which are k, θ and τ where
$$\theta = \frac{1}{2}(\lambda - \mu + \sqrt{(\lambda-\mu)^2 + 4(k-\mu)}),$$
and
$$\tau = \frac{1}{2}(\lambda - \mu - \sqrt{(\lambda-\mu)^2 + 4(k-\mu)}).$$
2. The multiplicity of the eigenvalue k is 1 and the multiplicities of θ and τ are f and g respectively, where
$$f = n - 1 + \frac{(n-1)(\mu-\lambda) - 2k}{\sqrt{(\lambda-\mu)^2 + 4(k-\mu)}},$$
and
$$g = n - 1 - \frac{(n-1)(\mu-\lambda) - 2k}{\sqrt{(\lambda-\mu)^2 + 4(k-\mu)}}.$$

3. If $(n-1)(\mu-\lambda)-2k \neq 0$, then the eigenvalues θ and τ are integers. On the other hand if $(n-1)(\mu-\lambda)-2k = 0$, then $f = g$ and θ and τ need not be integers. The strongly regular graph is called a conference graph in this case.

Theorem 6. *Let G be a strongly regular graph with parameters (n,k,λ,μ) and $Spec(G) = \begin{pmatrix} k & \theta & \tau \\ 1 & f & g \end{pmatrix}$. Then*

$$Spec_{Lo}(G) = \begin{pmatrix} k\delta & \theta\delta & \tau\delta) \\ 1 & f & g \end{pmatrix},$$

further, $E_{Lo}(G) = \delta E(G)$, where $\delta = \lambda + 4(n-k-1)$ and $E(G)$ is the energy of G.

Proof. Let G be a strongly regular graph with the parameters (n,k,λ,μ). Let u and v be any two adjacent vertices and suppose that $P^1_{11}(u,v)$ is the number of vertices which are adjacent to both of the vertices u and v, $P^1_{12}(u,v)$ is the number of vertices which are adjacent to u but not adjacent to v, $P^1_{21}(u,v)$ the number of vertices which are adjacent to v but not adjacent to u and $P^1_{22}(u,v)$ is the number of vertices which are not adjacent to both of the vertices u and v. As in [4], we have

$$P^1_{12}(u,v) = P^1_{21}(u,v) = n_1 - P^1_{11}(u,v) - 1$$

and

$$P^1_{22}(u,v) = n_2 - n_1 + P^1_{11}(u,v) + 1,$$

where $n_1 = k$ and $n_2 = n-k-1$.
Note that the diameter of G is atmost two. Thus for any vertex v in G there are k vertices that have distance one from v and $n-k-1$ vertices that have distance two from v. Suppose that $\overrightarrow{v}$, $\overrightarrow{u}$ are the locating vectors corresponding to the adjacent vertices u and v respectively. Then

$$P^1_{12}(u,v) = P^1_{21}(u,v) = k - \lambda - 1$$

and

$$P^1_{22}(u,v) = n - 2k + \lambda.$$

Therefore

$$\overrightarrow{v}.\overrightarrow{u} = P_{11}^1(u,v) + 2P_{12}^1(u,v) + 2P_{21}^1(u,v) + 4P_{22}^1(u,v).$$

Hence

$$\overrightarrow{v}.\overrightarrow{u} = \lambda + 4(n-k-1).$$

Then

$$Lo(G) = (\lambda + 4(n-k-1))A(G).$$

If we put $\delta = \lambda + 4(n-k-1)$, then,

$$Spec_{Lo}(G) = \begin{pmatrix} k\delta & \theta\delta & \tau\delta \\ 1 & f & g \end{pmatrix},$$

and $E_{Lo}(G) = \delta E(G)$ □

Notation:

If $\begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_r \\ m_1 & m_2 & \cdots & m_r \end{pmatrix}$ is the spectrum of a graph G, then we write

$$\delta\ spec(G) = \delta \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_r \\ m_1 & m_2 & \cdots & m_r \end{pmatrix} = \begin{pmatrix} \delta\lambda_1 & \delta\lambda_2 & \cdots & \delta\lambda_r \\ m_1 & m_2 & \cdots & m_r \end{pmatrix},$$

for any real number δ.

We can generalize the Theorem 6 as the following:

Theorem 7. *Let $G = (V,E)$ be a k-regular graph of diameter two in which any two adjacent vertices have t common neighbours. If $Spec(G) = \begin{pmatrix} k & \lambda_2 & ... & \lambda_m \\ 1 & t_2 & ... & t_m \end{pmatrix}$, then*

$$Spec_{Lo}(G) = \delta Spec(G).$$

Proof. Let u and v be any two adjacent vertices in G. We can partition the remaining vertices of G into four sets given by:

1. $A = \{w \in V(G) | d(u,w) = 1, d(v,w) = 1\}$.
2. $B = \{w \in V(G) | d(u,w) = 1, d(v,w) = 2\}$.
3. $C = \{w \in V(G) | d(u,w) = 2, d(v,w) = 1\}$.
4. $D = \{w \in V(G) | d(u,w) = 2, d(v,w) = 2\}$.

Since G is k-regular graph with diameter two, we have, $|A| = P^1_{11}(u,v)$, $|B| = P^1_{12}(u,v)$, $|C| = P^1_{21}(u,v)$ and $|D| = P^1_{22}(u,v)$. Since any two adjacent vertices u, v have t common neighbours, then $P^1_{11}(u,v) = t$, $P^1_{12}(u,v) = P^1_{21}(u,v) = k-1-t$ and as the sets A, B, C and D partition the set $V(G) - \{u,v\}$, we have $n-2 = 2(k-t-1)+t+|D|$. Hence $P^1_{22}(u,v) = n-2k+t$.

Thus,

$\overrightarrow{u}.\overrightarrow{v} = P^1_{11}(u,v) + 2P^1_{12}(u,v) + 2P^1_{21}(u,v) + 4P^1_{22}(u,v) = t + 4(n-k-1)$.

Putting $t + 4(n-k-1) = \delta$, we obtain $\mathbf{Lo}(G) = \delta,\ \mathbf{A}(G)$. Hence

$$Spec_{Lo}(G) = \begin{pmatrix} k\delta & \delta\lambda_2 & ... & \delta\lambda_m \\ 1 & t_2 & ... & t_m \end{pmatrix}.$$

□

Theorem 8. *Let $G = (V,E)$ be a regular graph of diameter two and without triangles. Then*

$$Spec_{Lo}(G) =$$
$$= \begin{pmatrix} 4k(n-k-1)) & 4(n-k-1)\lambda_2 & ... & 4(n-k-1)\lambda_m \\ 1 & t_2 & ... & t_m \end{pmatrix}.$$

Proof. When G has no triangles in Theorem 8, that means $t = 0$. Hence we obtain the following:

If $Spec(G) = \begin{pmatrix} k & \lambda_2 & ... & \lambda_m \\ 1 & t_2 & ... & t_m \end{pmatrix}$, then

$$Spec_{Lo}(G) =$$
$$= \begin{pmatrix} 4k(n-k-1)) & 4(n-k-1)\lambda_2 & ... & 4(n-k-1)\lambda_m \\ 1 & t_2 & ... & t_m \end{pmatrix}.$$

□

4 Bounds for the Locating Energy $E_{Lo}(G)$

Theorem 9. *If G is a graph with locating vectors $\overrightarrow{v_1}, \overrightarrow{v_2}, ..., \overrightarrow{v_n}$, then*

$$\sqrt{\alpha} \leq E_{Lo}(G) \leq \sqrt{n\alpha}\ ,$$

where $\alpha = 2\sum_{1\leq i<j\leq n}(\overrightarrow{v_i}.\overrightarrow{v_j})^2$.

Proof Let $\gamma_1, \gamma_2, \ldots, \gamma_n$ be the Lo–eigenvalues of G. The Cauchy–Schwarz inequality states that if $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ are n-vectors, where $a_1, a_2, ..., a_n, b_1, b_2, ..., b_n \in \mathbb{R}$, then,

$$\left(\sum_{i=1}^{n} a_i\, b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right).$$

Now, by setting $a_i = 1$ and $b_i = |\gamma_i|$, $i = 1, 2, \ldots, n$, in the above inequality, we obtain

$$\left(\sum_{i=1}^{n} |\gamma_i|\right)^2 \leq \left(\sum_{i=1}^{n} 1^2\right)\left(\sum_{i=1}^{n} |\gamma_i|^2\right).$$

Hence by using Lemma 2 we get

$$\sum_{i=1}^{n} |\gamma_i| \leq \sqrt{n\sum_{i=1}^{n} |\gamma_i|^2} = \sqrt{2n \sum_{1\leq i<j\leq n} (\overrightarrow{v_i}.\overrightarrow{v_j})^2}$$

and so

$$E_{Lo}(G) \leq \sqrt{n\alpha}.$$

For the lower bound, we have

$$(E_{Lo}(G))^2 = \left(\sum_{i=1}^{n} |\gamma_i|\right)^2 \geq \sum_{i=1}^{n} (|\gamma_i|)^2 = 2\sum_{1\leq i<j\leq n} (\overrightarrow{v_i}.\overrightarrow{v_j})^2.$$

Thus

$$E_{Lo}(G) \geq \sqrt{\alpha}.$$

□

Theorem 10. *For any connected graph of diameter d with n vertices, $n \geq 2$,*

$$(n-2)\sqrt{2(n-1)} \leq E_{Lo}(G) \leq n(n-2)\ d^2\sqrt{n-1}\ .$$

Proof. Let G be a connected graph with vertices $v_1, v_2, ..., v_n$, with diameter d, then for any locating vectors $\overrightarrow{v_i}$, $\overrightarrow{v_j}$ in G the locating product $\overrightarrow{v_i}.\overrightarrow{v_j}$ is at least equal to $n-2$, that means,

$$\overrightarrow{v_i}.\overrightarrow{v_j} \geq n-2,$$

and hence

$$(\overrightarrow{v_i}.\overrightarrow{v_j})^2 \geq (n-2)^2.$$

The number of $(\overrightarrow{v_i}.\overrightarrow{v_j})^2$ for which $1 \leq i < j \leq n$ and the locating product $\overrightarrow{v_i}.\overrightarrow{v_j}$ does not equal to zero is at least $n-1$ because G is connected.

Therefore

$$\sum_{1 \leq i<j \leq n}^{n} (\overrightarrow{v_i}.\overrightarrow{v_j})^2 \geq (n-1)(n-2)^2.$$

By using the lower bound found in Theorem 9 we have,

$$E_{Lo}(G) \geq \sqrt{2\sum_{1 \leq i<j \leq n} (\overrightarrow{v_i}.\overrightarrow{v_j})^2} \geq (n-2)\sqrt{2(n-1)}.$$

Similarly, to obtain the upper bound, we have

$$\overrightarrow{v_i}.\overrightarrow{v_j} \leq (n-2)d^2,$$

and we have,

$$(\overrightarrow{v_i}.\overrightarrow{v_j})^2 \leq (n-2)^2\ d^4.$$

Obviously, for any connected graph G there exist at most $n(n-1)/2$ of the locating product $\overrightarrow{v_i}.\overrightarrow{v_j}$ which are not zero. Therefore

$$\sum_{1 \leq i<j \leq n}^{n} (\overrightarrow{v_i}.\overrightarrow{v_j})^2 \leq \frac{1}{2}(n(n-1)(n-2)^2\ d^4).$$

By using the upper bound which is obtained in Theorem 9, we have,

$$E_{Lo}(G) \leq \sqrt{2n \sum_{1\leq i<j\leq n} (\overrightarrow{v_i}.\overrightarrow{v_j})^2} \leq n(n-2)\ d^2\sqrt{n-1}.$$

□

Theorem 11. *Let G be k-regular graph with n vertices and diameter two. Then*

$$E_{Lo}G \leq n(4n-3k-5)\sqrt{n-1}.$$

Proof. Let G be a connected k-regular graph with diameter two and its vertices are $v_1, v_2, ..., v_n$. For any two locating vectors $\overrightarrow{v_i}$ and $\overrightarrow{v_j}$ the locating product $\overrightarrow{v_i}.\overrightarrow{v_j}$ in G has maximum value if the vertices which have distance two from v_i also have distance two from v_j. Since G is regular, there are $n-k-1$ vertices that have distance two from v_i and k vertices that have distance one. Hence for any adjacent vertices v_i and v_j, we have $\overrightarrow{v_i}.\overrightarrow{v_j} \leq 4n-3k-5$, that implies to

$$(\overrightarrow{v_i}.\overrightarrow{v_j})^2 \leq (4n-3k-5)^2.$$

Also since $Lo(G)$ contains at most $n(n-1)$ terms that are not zero, therefore, as earlier,

$$\sum_{1\leq i<j\leq n} (\overrightarrow{v_i}.\overrightarrow{v_j})^2 \leq \frac{1}{2}(n(n-1)(4n-3k-5)^2).$$

Hence by Theorem 9

$$E_{Lo}G \leq \sqrt{n\alpha} = \sqrt{2n}\sqrt{\sum_{1\leq i<j\leq n} (\overrightarrow{v_i}.\overrightarrow{v_j})^2}.$$

Hence,

$$E_{Lo}G \leq n(4n-3k-5)\sqrt{n-1}.$$

□

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