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A remark on the weak Turán's Theorem

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Abstract

A subset S of vertices of a graph G is an *independent set* if no pair of vertices of S are adjacent. The *independence number*, $\alpha(G)$ of G, is the maximum cardinality of an independent set of G. In this note, we present an improvement of the weak Turán's theorem.

Keywords: Independent set, Turán's Theorem, Probabilistic methods.

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1 Introduction

For graph theory notation and terminology not given here we refer to [4], and for the probabilistic methods notation and terminology we refer to [1], [5]. We consider finite, undirected and simple graphs G with vertex set $V = V(G)$ and edge set $E(G)$. The number of vertices of G is called the *order* of G and is denoted by $n = n(G)$, and the number of edges of G is called the *size* of G. The *open neighborhood* of a vertex $v \in V$ is $N(v) = N_G(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood* of v is $N[v] = N_G[v] = N(v) \cup \{v\}$. The *degree* of a vertex v, denoted by $\deg(v)$ (or $\deg_G(v)$ to refer to G), is the cardinality of its open neighborhood. We denote by $\delta(G)$ and $\Delta(G)$, the minimum and maximum degrees among all vertices of G, respectively. For a subset S of vertices of G, we denote by $G[S]$ the subgraph of G induced by S. A subset S of vertices of G is an *independent set* if $G[S]$ has no edge. The *independence number*, $\alpha(G)$ of G, is the maximum cardinality of an independent set.

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Turán [6] proved his best-known result, namely *Turán's Graph Theorem* or just *Turán's Theorem*, by determining those graphs of order n, not containing the complete graph K_k of order k, and extremal with respect to size (that is, with as many edges as possible). Much have been written about Turán's Theorem, see for example [1], [2] and [5]. The Turán's Theorem states that if G is a graph with n vertices such that G is K_{r+1}-free, then the number of edges in G is at most $(1 - 1/r)\frac{n^2}{2}$. There is an equivalent theorem referred as the dual version of the Turán's Theorem, (or sometimes the Turán's Theorem, too) that states that any graph G of order n and size m contains an independent set of size at least $\frac{n}{d+1}$, where $d = \frac{2m}{n}$ is the average degree of G. A weak version of Turán's Theorem has been proved by several authors by probabilistic methods.

Theorem 1 (A weak Turán's theorem, [1], [5]) *If G is a graph of order n, and size m, and $d = \frac{2m}{n} \geq 1$ is the average degree, then $\alpha(G) \geq \frac{n}{2d}$.*

In this note, we present an improvement of the weak Turán's Theorem by the same probabilistic methods. We use the following.

Theorem 2 (Caro [3] and Wei [7]) *For any graph G,*

$$\alpha(G) \geq \sum_{v \in V(G)} \frac{1}{1 + \deg(v)}.$$

2 Main result

Theorem 3 *If G is a graph of order n, size m, maximum degree Δ, minimum degree δ, and $d = \frac{2m}{n} \geq 1$ is the average degree, then $\alpha(G) \geq \frac{n}{2d} + \frac{n}{\Delta+1}\left[\frac{1}{d} + \left(1 - \frac{2}{d}\right)\left(1 - \frac{1}{d}\right)^{\delta}\right]$.*

Proof. Select a random subset of vertices $S \subseteq V(G)$ in such a way that we insert every vertex into S independently with probability $p = \frac{1}{d}$. Let A be the set of all non-isolated vertices of $G[S]$, $G' = G[A]$, and I_S be a maximum independent set in G'. Let $X = \{v \in V(G) - S : N_G[v] \cap S = \emptyset\}$, and J_S be a maximum independent set in $G[X]$. From Theorem 2, we find that

$$|I_S| \geq \frac{|A|}{\Delta(G') + 1} \geq \frac{|A|}{\Delta(G) + 1} = \frac{|A|}{\Delta + 1},$$

and

$$|J_S| \geq \frac{|X|}{\Delta(G[X])) + 1} \geq \frac{|X|}{\Delta(G) + 1} = \frac{|X|}{\Delta + 1}.$$

Let $X = |S|$, Y denotes the number of edges of $G[S - I_S]$, and $Z = |J_S|$. We compute the expectation of $X - Y + Z$. Clearly $E(X) = np$. Any vertex of I_S is incident with at least one edge in $G[S]$. Thus, the number of edges of $G[S - I_S]$ is bounded above by the number of edges of $G[S]$ minus the number of vertices of I_S. Let Y_1 denotes the number of edges of $G[S]$, and Y_2 denotes the number of vertices of $|I_S|$. Then $Y \leq Y_1 - Y_2$. Observe that $E(Y_1) = mp^2$, and $E(Y_2) = E(|I_S|) \geq E(\frac{|A|}{\Delta+1}) = \frac{1}{\Delta+1} E(|A|)$. For a vertex v, $Pr(v \in A) = p(1 - (1-p)^{\deg(v)})$, and $Pr(v \in X) = (1-p)^{1+\deg(v)}$. Thus,

$$\begin{aligned} E(Y) &\leq E(Y_1) - E(Y_2) \\ &\leq mp^2 - \frac{1}{\Delta + 1} E(|A|) \\ &\leq mp^2 - \frac{1}{\Delta + 1} np\left(1 - (1-p)^{\delta}\right). \end{aligned}$$

Moreover,

$$E(|J_S|) \geq E(\frac{|X|}{\Delta + 1}) = \frac{1}{\Delta + 1} E(|X|) \geq \frac{n}{\Delta + 1} (1-p)^{1+\delta}.$$

Now,

$$\begin{aligned} & E(X - Y + Z) \geq \\ & np - mp^2 + \frac{1}{\Delta + 1} np\left(1 - (1-p)^{\delta}\right) + \frac{n}{\Delta + 1}(1-p)^{1+\delta} = \\ & = \frac{n}{2d} + \frac{n}{\Delta + 1}\left[\frac{1}{d} + \left(1 - \frac{2}{d}\right)\left(1 - \frac{1}{d}\right)^{\delta}\right]. \end{aligned}$$

Thus there exists a specific set S for which the number of vertices of S minus the number of edges in $G[S-I_S]$ plus the number of vertices of J_S is at least $\frac{n}{2d}+\frac{n}{\Delta+1}\left[\frac{1}{d}+\left(1-\frac{2}{d}\right)\left(1-\frac{1}{d}\right)^{\delta}\right]$. Select one vertex from each edge of $G[S-I_S]$ and delete it. This leaves a set S^* with at least $\frac{n}{2d}+\frac{n}{\Delta+1}\left[\frac{1}{d}+\left(1-\frac{2}{d}\right)\left(1-\frac{1}{d}\right)^{\delta}\right]$ vertices which is an independent set. ■

Remark: Although our main result is weaker than Theorem 2, it would be interesting for researchers interested to weak Turán's Theorem.

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