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Choice Numbers of Multi-Bridge Graphs

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Abstract

Suppose $ch(G)$ and $\chi(G)$ denote, respectively, the choice number and the chromatic number of a graph $G = (V, E)$. If $ch(G) = \chi(G)$, then G is said to be chromatic-choosable. Here, we find the choice numbers of all multi-bridge or l-bridge graphs and classify those that are chromatic-choosable for all $l \geq 2$.

Keywords: List coloring, chromatic-choosable, l-bridge

1 Preliminaries

In this paper, $G = (V, E)$ denotes a simple connected *graph*, where $V = V(G)$ and $E = E(G)$ denote, respectively, the set of *vertices* and the set of *edges* of G. An edge $e \in E$ with endpoints $u, v \in V$ is denoted by uv. Also, we denote by $N(u) = N_G(u) = \{x \in V \mid ux \in E\}$ the *(open) neighbor* set in G of $u \in V$. $\Delta = \Delta(G)$, K_n and C_n denote, respectively, the maximum degree of G, a complete graph and a cycle on n vertices. The *join* of two graphs G_1 and G_2, denoted by $G_1 \vee G_2$, is the graph G whose vertex set is $V(G) = V(G_1) \cup V(G_2)$, a disjoint union, and whose edge set is $E(G) = E(G_1) \cup E(G_2) \cup \{u_1u_2 \mid u_1 \in V(G_1),\ u_2 \in V(G_2)\}$. For other basic notions of graphs, see [15].

A *list assignment* to the graph $G = (V, E)$ is a function L which assigns a finite set (list) $L(v)$ to each vertex $v \in V$. A *proper L-coloring* of G is a function $\phi : V \to \cup_{v \in V} L(v)$ satisfying, for every u, $v \in V$, (i) $\phi(v) \in L(v)$ and (ii) $uv \in E \to \phi(v) \neq \phi(u)$.

The *choice number* of G, denoted by $ch(G)$, is the smallest integer k such that there is always a proper L-coloring of G if L satisfies $|L(v)| \geq k$ for every $v \in V$. We define G to be *k-choosable* if it admits a proper

L-coloring whenever $|L(v)| \geq k$ for all $v \in V$; so $ch(G)$ is the smallest integer k such that G is k-choosable. The following theorem is useful in the estimation of choice number.

Theorem A. (Erdős, Rubin and Taylor [3]) *If G is a connected graph that is neither a complete graph nor an odd cycle, then $ch(G) \leq \Delta(G)$.*

Corollary A. *For any graph G, $ch(G) \leq \Delta(G) + 1$.*

The proof of Corollary A follows from a "greedy coloring" argument.

Clearly, $\chi(G) \leq ch(G)$ since the chromatic number $\chi(G)$ is similarly defined with the restriction that the list assignment is to be constant and there are many graphs whose choice number exceeds (sometimes greatly) their chromatic number. The two planar graphs in Figure 1 are some examples, where it is not too hard to see that, given the list assignment for each graph G, $ch(G) = 3 > 2 = \chi(G)$.

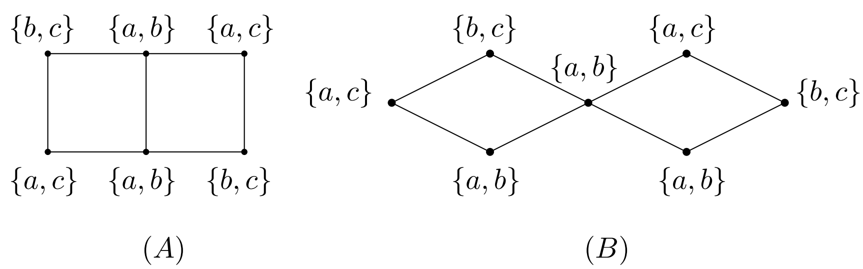

Figure 1: Two graphs with two list assignments.

Any graph G for which the extremal case $\chi(G) = ch(G)$ holds is said to be *chromatic-choosable*. Cycles, cliques and trees are some examples of chromatic-choosable graphs.

Historically, the topic of list colorings is believed to be first introduced by Vizing [9] and independently by Erdős, Rubin and Taylor [3]. Ever since, many researchers (see for e.g., [1], [4]–[7]) have sought to classify chromatic-choosable graphs. It is worth noting that the problem of finding chromatic-choosable graphs contains the famous list coloring conjecture [9]: *the line graph of any graph is chromatic-choosable.* In fact, this conjecture has been partially proved by Galvin [4] in

Theorem B.(Galvin [4]) *The line graph of any bipartite multigraph is chromatic-choosable.*

Recently, Reed et al. [6] settled the well-known Ohba's conjecture [7]. We state their result (or Ohba's conjecture) without proof, in the next theorem.

Theorem C.(Noel, Reed and Wu [6]) *If* $|V(G)| \leq 2\chi(G) + 1$*, then* G *is chromatic-choosable.*

Because the proposed bound is obviously weak in characterizing chromatic-choosable graphs with low chromatic numbers, we classify a class of acyclic graphs with low chromatic number ($\chi \leq 3$) and arbitrarily large Δ.

2 Choice number of some l-bridge graphs

The length of a path is the number of its edges and two paths are said to be *internally disjoint* if they have no common internal vertex.

An *l-bridge* (or *multi-bridge*) graph $\Theta(a_1, \ldots, a_l)$ is the graph obtained by connecting two distinct vertices u and v with l internally disjoint paths P_{a_i} of lengths $a_i \geq 1$. It is customary to assume $l \geq 3$ since when $l = 2$, $\Theta(a_1, a_2)$ is a cycle on $a_1 + a_2$ vertices; the trivial case when $l = 1$, $\Theta(a_1) \cong uv$, an edge. L-bridge graphs are planar and when $l = 3$, figure 1(A) depicts an example of $\Theta(1, 3, 3)$. For the rest of this article, it causes no confusion to denote $P_{a_i} := uv$ if some $a_i = 1$, and $P_{a_i} := ux_{i_1}x_{i_2}\ldots x_{i_{a_i}-1}v$, a sequence of edges for all $a_i \geq 2$.

Recall, the *core* of a connected graph is the graph obtained by deleting all vertices of degree 1, and then all vertices of degree 1 in what remains, and so forth, until there are no vertices of degree 1 remaining; except that, in case of K_2, delete only one vertex. Erdős, Rubin and Taylor have described the structure of 2-choosable graphs (which are necessarily bipartite) in the next theorem.

Theorem D.(Erdős, Rubin and Taylor [3]). *A connected graph G is 2-choosable if and only if the core of G is K_1, an even cycle, or of the form $\Theta(2,2,2t)$, where t is a positive integer.*

Not surprisingly, there has been no characterization of k-choosable graphs, $k \geq 3$. Alon and Tarsi [2] showed that every bipartite planar graph is 3-choosable and there has been several attempts at characterizing triangle free planar graphs in order to strengthen Alon and Tarsi's result. See for instance, [8], [10]–[14]. Clearly, since each graph G in Figure 1 is bipartite and planar, it follows from Alon and Tarsi's result that $ch(G) = 3$, given the list assignment. It is important to point out that l-bridge graphs are not necessarily bipartite as they may contain odd cycles. Here, we show that they are 3-choosable and later, we classify them based on their choice number.

Proposition 1. *If $G = \Theta(a_1, \ldots, a_l)$, then G is 3-choosable.*

Suppose L is a list assignment to G satisfying $|L(w)| \geq 3$ for each $w \in V(G)$. Because every path is 2-choosable, color properly the vertices (including u,v) of some path P_{a_i}. Suppose, in coloring P_{a_i}, $\phi(u) = c_1$ and $\phi(v) = c_2$, where c_1 and c_2 are not necessarily distinct colors. For each vertex $y \in V(G \backslash P_{a_i})$, define $L'(y) = L(y) - \{c_1, c_2\}$. If $|L'(y)| \geq 2$ for each $y \in V(G \backslash P_{a_i})$, color properly the vertices on each independent path $P_{a_j} - uv$, $j \neq i$. Or else, there exists a vertex $z \in V(G \backslash P_{a_i})$ such that, for some $k \neq i$, $|L'(z)| \geq 1$. This implies that $N(u) = z = N(v)$, i.e., $P_{a_k} := uzv$. In this case, color z with the color left in its palette, giving a proper L-coloring of G.

□

Theorem 1. *Suppose $G = \Theta(a_1, \ldots, a_l)$ is any l-bridge graph with $l \geq 3$. $ch(G) = 3$ if and only if G is not $\Theta(2,2,2t)$, for all $t \geq 1$.*

Proof. Clearly if $G = \Theta(2,2,2t)$, then it follows from Theorem D that $ch(G) = 2$. Now, if G is an l-bridge that contains an odd cycle, then the result follows from Proposition 1. Thus, to complete the proof, we can assume that G is neither $\Theta(2,2,2t)$ nor contains an odd cycle and show that $ch(G) > 2$. In each upcoming claim we present a list assignment which is left up to the reader to verify in order to establish the result.

Claim A. If $G_1 = \Theta(a_1, \ldots, a_l)$ and each a_i is odd, then G_1 is not 2-choosable, for all $l \geq 3$.

Let $H = \Theta(a_1, a_2, a_3)$ such that each a_i is odd, for $i = 1, 2, 3$. Clearly H contains no odd cycle. Define a list assignment L_1 satisfying, for each $w \in V(H)$

(i) $L_1(u) = L_1(v) = L_1(x_{1_j}) = \{a, b\}$ for $1 \leq j \leq a_1 - 1$

(ii) $L_1(x_{2_1}) = \ldots = L_1(x_{2_{a_2-2}}) = L_1(x_{3_{a_3-1}}) = \{a, c\}$

(iii) $L_1(x_{3_1}) = \ldots = L_1(x_{3_{a_3-2}}) = L_1(x_{2_{a_2-1}}) = \{b, c\}$.

It is easy to see that every proper L_1-coloring of P_{a_1} will require distinct colors a, b for the vertices u, v, forcing $L_1(x_{i_j}) = \emptyset$, for some $i \neq 1$ and $1 \leq j \leq a_i - 1$. Hence, $ch(H) > 2$. Because $H \subseteq G_1$, for all $l \geq 3$, G_1 is not 2-choosable.

□

Claim B. If $G_2 = \Theta(2r, 2s, 2t)$, then G_2 is not 2-choosable for all $r \geq 1$, and $s, t \geq 2$.

Denote x_1, x_2 and y_1, y_2 the vertices on the paths P_{2s} and P_{2t}, respectively, such that $x_1 = N(u)$, $x_2 = N(x_1)$, $y_1 = N(u)$, and $y_2 = N(y_1)$. Then for each $w \in V(G_3)$, define the list assigment L_2 such that

(i) $L_2(u) = L_2(v) = L_2(z) = \{a, b\}$ for $z \notin \{x_1, x_2, y_1, y_2\}$

(ii) $L_2(x_1) = L_2(y_2) = \{b, c\}$, $L_2(y_1) = L_2(x_2) = \{a, c\}$.

It is easy to see that G_2 does not admit a proper L_2-coloring. □

Observe that the previous claim completely resolves the case of l-bridge graphs (with even paths) that are not of the form $\Theta(2,2,2t)$ for $l = 3$.

Claim C. If $G_3 = \Theta(a_1, \ldots, a_l)$ and each a_i is even, then G_3 is not 2-choosable, for all $l \geq 4$.

For $l \geq 4$, we present a list assignment L_3 to G_3 when $a_i = a_j$, for each $i \neq j$. A similar list assignment can easily be derived when some $a_i \neq a_k$ by letting $L_3(x_{k_{a_k/2}})$ be a specific 2-subset of $\{a, b, c, d\}$.

Let $H = \Theta(a_1, a_2, a_3, a_4)$ such that $a_1 = a_2 = a_3 = a_4$. Now define L_3 to be a list assignment satisfying, for each $w \in V(H)$:

(i) $L_3(u) = \{a, b\}$ and $L_3(v) = \{c, d\}$

(ii) $L_3(x_{1_{a_1/2}}) = \{a, c\}$, $L_3(x_{2_{a_2/2}}) = \{a, d\}$, $L_3(x_{3_{a_3/2}}) = \{b, c\}$, $L_3(x_{4_{a_4/2}}) = \{b, d\}$

(iii) $L_3(x_{i_j}) = \{a, b\}$ for $1 \leq j < a_i/2$ and $L_3(x_{i_j}) = \{c, d\}$ for $a_i/2 < j \leq a_i - 1$

It is easy to verify that $H \subseteq G_3$ admits no proper L_3-coloring. □

Thus, if G contains only even cycles and G is not $\Theta(2, 2, 2t)$, G must satisfy one of the previous claims. The result follows for all l-bridge graphs, with $l \geq 3$. □

Corollary 1. *Suppose $G = \Theta(a_1, \ldots, a_l)$, $l \geq 3$. G is chromatic-choosable if and only if G contains an odd cycle or G is of the form $\Theta(2, 2, 2t)$, where t is a positive integer.*

Proof. Suppose $G = \Theta(a_1, \ldots, a_l)$ is chromatic-choosable. It follows from Proposition 1 that, either (i) $\chi(G) = 2 = ch(G)$ or (ii) $\chi(G) = 3 = ch(G)$. Case (i) follows from Theorem D. In which case $G \cong \Theta(2, 2, 2t)$ while in case (ii) it is clear that G must contain an odd cycle. Conversely, if G contains an odd cycle, then $\chi(G) = 3$. It follows from Theorem 1 that G is chromatic-choosable. Moreover, if $G \cong \Theta(2, 2, 2t)$, then G contains no odd cycle and $\chi(G) = 2$. It follows from Theorem D that G is chromatic-choosable. □

We end this article with the next lemma which gives an estimate on the choice number of any connected graph. A graph G of order greater than r is said to be *r-connected* if G remains connected whenever fewer than any r number of vertices of G are removed.

Lemma 1. *Suppose G is an r-connected graph with components $G_1, \ldots, G_m$. If $k = \max_{1 \le i \le m} \{ch(G_i)\}$, then G is $(k+r)$-choosable for all $k, r \ge 1$ and $m \ge 2$.*

Proof. Suppose L is a list assignment to $v \in V(G)$ satisfying $|L(v)| \ge k+r$ with $k, r \ge 1$. Denote $S \subset V(G)$ a set of r vertices whose deletion produces the non-empty components $G_1, \ldots, G_m$, $m \ge 2$. Color each element of S using distinct r colors, and remove those colors from the palette of each vertex $u \in V(G) \backslash S$. Let L' be the resulting list assigment for each vertex u. It follows that $|L'(u)| \ge k$ for each $u \in V(G_i)$, $1 \le i \le m$. By the hypothesis, each G_i is k-choosable so we color each vertex $u \in V(G_i)$. Because G is r-connected, together with the r-colorings of S, we have a proper L-coloring of G. □

Notice that this bound is sharp for some 1-connected cyclic graphs. See for instance, Figure 1(B). From this proposition follows

Corollary 2. *Suppose S is a clique on r vertices and for some graphs H_i, $k = \max_{1 \le i \le m} \{ch(H_i)\}$. If $G = S \vee \{H_i\}_{i=1}^{m}$, then $ch(G) = k + r$.*

Proof. Because every proper coloring of $S \subset G$ uses exactly r colors, the result follows from similar steps as in Lemma 1. □

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