Comparison of the maximal inaccuracies for two experiments

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Abstract

In this paper we refine and generalize some previous our results on the inaccuracy (error) theory. We define conditions, which characterize different types of functions. Via these functions an indirectly measurable variable Y can be analytically represented. We also present criteria for comparison of the maximal absolute and relative inaccuracies of the indirectly measurable variable Y in the first and in the second order for two experiments. We correct some of our previous conclusions regarding the application of the dimensionless scale for evaluation of the quality of an experiment. Furthermore we give two numerical contra examples.

Keywords: indirectly measurable variable; maximal absolute inaccuracy; maximal relative inaccuracy; dimensionless scale.

1 Introduction

Let an indirectly measurable variable Y be represented as a function of a finite number of directly measurable variables $X_1, X_2, ..., X_n$, i.e. $Y = f(X_1, X_2, ..., X_n)$ and let f be a differentiable function of each of its real variables. If in an experiment we have k number of observations $x_{i1}, x_{i2}, ..., x_{ik}$ of the directly measurable varialbe $X_i \quad (i = 1, 2, ..., n)$, then it is assumed that the arithmetic mean $\bar{x}_i = \frac{1}{k}\sum\limits_{m=1}^{k} x_{im}$ is the most probable (the most reliable) value of X_i. We denote $|\Delta x_{im}| = |x_{im} - \bar{x}_i|, \; i = 1, 2, ..., n, \; m = 1, 2, ..., k$.

The value of the maximal absolute inaccuracy $\Delta^1 Y$ of an indirectly measurable variable Y according to the classical method is

$$\Delta^1 Y = \frac{1}{k} \sum_{m=1}^{k} \sum_{i=1}^{n} \left| \frac{\partial f}{\partial X_i} (x_{1m}, ..., x_{nm}) \right| |\Delta x_{im}|, \qquad (1)$$

and the value of the maximal relative inaccuracy of Y is $\frac{\Delta^1 Y}{Y}$, where $\Delta^1 Y$ is defined by (1) and

$$Y = \frac{1}{k} \sum_{m=1}^{k} |f (x_{1m}, ..., x_{nm})|, \qquad (2)$$

[6, 7].

The value of the maximal absolute inaccuracy $\Delta^1 Y$ according to our method [1] is

$$\Delta^1 Y = \sum_{i=1}^{n} A_i |\Delta X_i|, \qquad (3)$$

where

$$A_i = \frac{1}{k} \sum_{m=1}^{k} \left| \frac{\partial f}{\partial X_i} (x_{1m}, ..., x_{nm}) \right|, \; i = 1, ..., n \qquad (4)$$

and

$$|\Delta X_i| = \frac{1}{k} \sum_{j=1}^{k} |\Delta x_{ij}| \;, \; i = 1, 2, ..., n. \qquad (5)$$

The value of the maximal relative inaccuracy $\frac{\Delta^1 Y}{Y}$ according to our method [2, 3] is

$$\frac{\Delta^1 Y}{Y} = \sum_{i=1}^{n} B_i \left| \frac{\Delta X_i}{X_i} \right|, \qquad (6)$$

where

$$B_i = \frac{1}{k} \sum_{m=1}^{k} \left| \frac{x_{im}}{f (x_{1m}, ..., x_{nm})} \frac{\partial f}{\partial X_i} (x_{1m}, ..., x_{nm}) \right|, \; i = 1, ..., n \qquad (7)$$

and

$$\left|\frac{\Delta X_i}{X_i}\right| = \frac{1}{k}\sum_{j=1}^{k}\left|\frac{\Delta x_{ij}}{x_{ij}}\right|, \ i = 1, 2, ..., n. \tag{8}$$

We note, that in (4) and in (7)

$$\frac{\partial f}{\partial X_i}(x_{1m}, ..., x_{nm}) \text{ and } \frac{x_{im}}{f(x_{1m}, ..., x_{nm})}\frac{\partial f}{\partial X_i}(x_{1m}, ..., x_{nm})$$

are respectively the values of $\frac{\partial f}{\partial X_i}$ and $\frac{X_i}{f}\frac{\partial f}{\partial X_i}$, calculated on the m-th observation. A_i and B_i are the arithmetic means of these values for $m = 1, 2, ..., k$.

In [4, 5] we denote the values of the maximal absolute inaccuracy $\Delta^2 Y$ and of the maximal relative inaccuracy $\frac{\Delta^2 Y}{Y}$ of second order of $Y = f(X_1, X_2, ..., X_n)$ respectively by

$$\Delta^2 Y = \sum_{i,j=1}^{n} A_{ij}\left|\Delta X_i\right|\left|\Delta X_j\right| \text{ and } \frac{\Delta^2 Y}{Y} = \sum_{i,j=1}^{n} B_{ij}\left|\frac{\Delta X_i}{X_i}\right|\left|\frac{\Delta X_j}{X_j}\right| \tag{9}$$

where A_{ij} and B_{ij} for $\Delta^2 Y$ and $\frac{\Delta^2 Y}{Y}$ are defined as follows:

$$A_{ij} = \frac{1}{k}\sum_{m=1}^{k}\left|\frac{\partial^2 f}{\partial X_i \partial X_j}(x_{1m}, ..., x_{nm})\right|, \ i, j = 1, 2, ..., n \tag{10}$$

and

$$B_{ij} = \frac{1}{k}\sum_{m=1}^{k}\left|\frac{x_{im}x_{jm}}{f(x_{1m}, ..., x_{nm})}\frac{\partial^2 f}{\partial X_i \partial X_j}(x_{1m}, ..., x_{nm})\right|, \quad i, j = 1, 2, ..., n. \tag{11}$$

We note, that in (10) and in (11)

$$\frac{\partial^2 f}{\partial X_i \partial X_j}(x_{1m}, ..., x_{nm}) \text{ and } \frac{x_{im}x_{jm}}{f(x_{1m}, ..., x_{nm})}\frac{\partial^2 f}{\partial X_i \partial X_j}(x_{1m}, ..., x_{nm})$$

are respectively the values of $\frac{\partial^2 f}{\partial X_i \partial X_j}$ and $\frac{X_i X_j}{f}\frac{\partial^2 f}{\partial X_i \partial X_j}$, calculated at the m-th observation. A_{ij} and B_{ij} are the arithmetic means of these values for $m = 1, 2, ..., k$.

The maximum absolute inaccuracy ΔY of an indirectly measurable variable Y in the second degree of approximation, according to [4, 5], is

$$\Delta Y = \Delta^1 Y + \frac{1}{2}\Delta^2 Y, \tag{12}$$

and the maximum relative inaccuracy $\frac{\Delta Y}{|Y|}$ of Y in the second degree of approximation is

$$\frac{\Delta Y}{|Y|} = \frac{\Delta^1 Y}{|Y|} + \frac{1}{2}\frac{\Delta^2 Y}{|Y|}. \tag{13}$$

In this paper we give some conditions that characterize some type of functions. An indirectly measurable variable can be analytically represented via these functions. Thus we obtain some necessary and sufficient conditions for comparison of the values of the maximal inaccuracies for two experiments. We correct some of our previous conclusions regarding the dimensionless scale application for evaluation of the quality of an experiment. We show two numerical counterexamples.

2 Conditions that characterize different types of functions by which an indirectly measurable variable can be represented analytically

Theorem 1. If $f(x_1, ..., x_n)$ is a function with domain R^n and there exist the first partial derivatives of f in respect to all its variables, then the following holds:

$$\frac{\partial f}{\partial x_i} = a_i,\ a_i \in \mathrm{R},\ i = 1, ..., n \tag{14}$$

if and only if

$$f\left(x_1,...,x_n\right)=a_1x_1+...+\ a_nx_n+c,\quad c,\,a_i\in \mathrm{R},\ i=1,...,n. \qquad (15)$$

Proof. If (15) is true, then obviously (14) holds true.

Contrariwise, let (14) is true. Then from $\frac{\partial f}{\partial x_i}=a_i$ it follows $\partial f=a_i\partial x_i$, $a_i\in \mathrm{R}$. Therefore

$$f=a_ix_i+c_i\left(x_1,...,x_{i-1},x_{i+1},...,x_n\right), \qquad (16)$$

where $c_i\left(x_1,...,x_{i-1},x_{i+1},...,x_n\right)$ is a real function of $x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n$.

We will prove that

$$f\left(x_1,...,x_n\right)=a_1x_1+...+\ a_ix_i+c_i\left(x_{i+1},...,x_n\right), \qquad (17)$$

by induction on i, $1\le i\le n$.

Indeed, for $i=1$ the equality (17) follows from (16). Assume the equality (17) is true for $i-1\ge 1$, i.e.

$$f\left(x_1,...,x_n\right)=a_1x_1+...+a_{i-1}x_{i-1}+c_{i-1}\left(x_i,...,x_n\right). \qquad (18)$$

Since from (14) and (18) it follows $a_i=\frac{\partial f}{\partial x_i}=\frac{\partial c_{i-1}}{\partial x_i}$, then $\partial c_{i-1}=a_i\partial x_i$. Therefore

$$c_{i-1}\left(x_i,...,x_n\right)=a_ix_i+c_i\left(x_{i+1},...,x_n\right). \qquad (19)$$

As we substitute $c_{i-1}\left(x_i,...,x_n\right)$ from (19) in (18), then we obtain the equality (17). Therefore formula (17) is proved by induction on i, $1\le i\le n$.

Let $i=n$. Then from (17) we have

$$f=a_1x_1+...+a_nx_n+c,$$

where $c=c_n\in \mathrm{R}$.

The theorem is prooved. □

Theorem 2. If $f(x_1,...,x_n)$ is a function with domain R^n and there exist the first partial derivatives of f in respect to all its variables, then the following holds

$$\frac{x_i}{f}\frac{\partial f}{\partial x_i}=k_i,\ k_i\in\mathrm{R},\ i=1,...,n \tag{20}$$

if and only if

$$f=cx_1^{k_1}...x_n^{k_n},\quad c,k_i\in\mathrm{R}^+,\ i=1,...,n. \tag{21}$$

Proof. Let (21) holds true. Then

$$\frac{x_i}{f}\frac{\partial f}{\partial x_i}=\frac{x_i\,k_i c x_1^{k_1}...x_{i-1}^{k_{i-1}}x_i^{k_i-1}x_{i+1}^{k_{i+1}}...x_n^{k_n}}{cx_1^{k_1}...x_n^{k_n}}=k_i,$$

i.e. (20) holds true.

Contrariwise, let (20) holds true. Let us denote $y=f(x_1,...,x_n)$. Then from (20) it follows that $\frac{dy}{y}=\frac{k_i}{x_i}\partial x_i$. We obtain $\ln|y|=k_i\ln|x_i|+\ln|c_i(x_1,...,x_{i-1},x_{i+1},...,x_n)|$, where $c_i(x_1,...,x_{i-1},x_{i+1},...,x_n)$. Then

$$y=\pm x_i^{k_i}c_i(x_1,...,x_{i-1},x_{i+1},...,x_n)\,. \tag{22}$$

We will prove by induction for i, $1\le i\le n$, that

$$y=\pm x_1^{k_1}...x_i^{k_i}c_i(x_{i+1},...,x_n)\,. \tag{23}$$

Indeed for $i=1$ the equality (23) is the proved formula (22). Let us assume, that (23) holds true for $i-1\ge 1$, i.e.

$$y=\pm x_1^{k_1}...x_{i-1}^{k_{i-1}}c_{i-1}(x_i,...,x_n)\,. \tag{24}$$

From formulas (20) and (24) we have

$$k_i=\frac{x_i}{y}\frac{\partial y}{\partial x_i}=\frac{x_i\,x_1^{k_1}...x_{i-1}^{k_{i-1}}\frac{\partial c_{i-1}}{\partial x_i}}{x_1^{k_1}...x_{i-1}^{k_{i-1}}c_{i-1}(x_i,...,x_n)}.$$

Therefore we obtain the following formulas

$$\frac{\partial c_{i-1}}{c_{i-1}} = \frac{k_i}{x_i}\partial x_i,\ \ln|c_{i-1}| = k_i \ln|x_i| + \ln|c_i(x_{i+1},...,x_n)|,$$

$$c_{i-1} = \pm x_i^{k_i} c_i(x_{i+1},...,x_n).$$

We substitute the last formula in (23) and we get

$$y = \pm x_1^{k_1}...x_{i-1}^{k_i} c_i(x_{i+1},...,x_n).$$

Thus formula (23) is proved by induction on i, $1 \leq i \leq n$.

Let $i = n$. From (23) we have

$$y = \pm x_1^{k_1}...x_n^{k_n} c,$$

where $c = c_n \in \mathrm{R}$.

The theorem is prooved. □

Theorem 3. If $f = f(x_1,...,x_n)$ is a second degree polynomial with unknown quantities $x_1,...,x_n$, represented in the form

$$f(x_1,...,x_n) = \sum_{i,j=1}^{n} a_{ij}x_ix_j + \sum_{i=1}^{n} a_ix_i + a, a_{ji} = a_{ij}, \quad a_j, a_i, a \in \mathrm{R}, \tag{25}$$

then for each $i, j = 1,...,n$ the equality $\frac{\partial^2 f}{\partial x_i \partial x_j} = 2a_{ij}$ holds.

Proof. Let us denote f in the form

$$f(x_1,...,x_n) = \sum_{i=1}^{n} a_{ii}^2 x_i^2 + 2\sum_{i=1}^{n-1}\sum_{j=2}^{n} a_{ij}x_ix_j + \sum_{i=1}^{n} a_ix_i + a.$$

Then

$$\frac{\partial f}{\partial x_i} = 2a_{ii}x_i + 2\sum_{j>i} a_{ij}x_j + a_i.$$

For $j \neq i$ we have $\frac{\partial^2 f}{\partial x_i \partial x_j} = 2a_{ij}$, and for $j = i$ it follows $\frac{\partial^2 f}{\partial x_i \partial x_i} = 2a_{ii}$.

The theorem is prooved. □

Theorem 4. If the function $f = f(x_1, ..., x_n)$ has the form

$$f = cx_1^{k_1}...x_i^{k_i}...x_j^{k_j}...x_n^{k_n}, \quad c, k_i \in \mathrm{R}, \tag{26}$$

then for each i, j the following holds true:

$$\frac{x_i x_j}{f} \frac{\partial^2 f}{\partial x_i \partial x_j} = \begin{cases} k_i k_j, \text{ if } (i)\, j \neq i, \\ k_i (k_i - 1), \text{ if } (ii)\ j = i \text{ and } k_i \neq 1, \\ 0, \text{ if } j = i \text{ and } k_i = 1. \end{cases} . \tag{27}$$

Proof. If $j \neq i$, then the following equalities are true

$$\frac{x_i x_j}{f} \frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{x_i x_j}{cx_1^{k_1}...x_n^{k_n}} ck_i k_j x_1^{k_1}...x_i^{k_i-1}...x_j^{k_j-1}...x_n^{k_n} = k_i k_j.$$

If $j = i$ and $k_i \neq 1$, then

$$\frac{x_i^2}{f} \frac{\partial^2 f}{\partial x_i \partial x_i} = \frac{x_i^2}{cx_1^{k_1}...x_n^{k_n}} ck_i (k_i - 1) x_1^{k_1}...x_i^{k_i-2}...x_n^{k_n} = k_i (k_i - 1).$$

If $j = i$ and $k_i = 1$, then obviously the third part of (27) holds true. □

3 Some necessary and sufficient conditions for comparison of the values of the maximal inaccuracies for two experiments

1) Let $\Delta^1 Y$ and $\Delta^1 \tilde{Y}$ be the maximal absolute inaccuracies of the first order for two experiments, i.e.

$$\Delta^1 Y = \sum_{i=1}^{n} A_i |\Delta X_i|, \quad \Delta^1 \tilde{Y} = \sum_{i=1}^{n} \tilde{A}_i \left|\Delta \tilde{X}_i\right|, \tag{28}$$

where $|\Delta X_i|$ and $\left|\Delta \tilde{X}_i\right|$ are defined from formula (5).

1.1) If in (28) $A_i = \frac{\partial f}{\partial x_i}$ are constant values, $i = 1, ..., n$, then according to formula (3)

$$\Delta^1 Y = \sum_{i=1}^{n} A_i \left|\Delta X_i\right|, \;\; \Delta^1 \tilde{Y} = \sum_{i=1}^{n} A_i \left|\Delta \tilde{X}_i\right|. \qquad (29)$$

Thus obviously the following statement is true.

Criterion 1. If $A_i = \frac{\partial f}{\partial x_i} = const$, $i = 1, 2, ..., n$, then the first experiment of the maximal absolute inaccuracy of Y is more accurate than the second one if and only if

$$\sum_{i=1}^{n} A_i \left(\left|\Delta \tilde{X}_i\right| - \left|\Delta X_i\right|\right) \geq 0. \qquad (30)$$

Both experiments have equal accuracy if and only if

$$\sum_{i=1}^{n} A_i \left(\left|\Delta \tilde{X}_i\right| - \left|\Delta X_i\right|\right) = 0.$$

In this case for the inaccuracy of the experiments, calculated by the classical way from (1) and (28) we have

$$\Delta^1 Y = \frac{1}{k} \sum_{i=1}^{n} \sum_{m=1}^{k} A_i \left|\Delta x_{im}\right| = \sum_{i=1}^{n} A_i \left|\Delta \bar{X}_i\right| = \sum_{i=1}^{n} A_i \left|\Delta X_i\right|.$$

Therefore this result match with our result from (29).

In particular, by $n = 1$ the first experiment is more accurate than the second one if and only if $\left|\Delta X_1\right| \leq \left|\Delta \tilde{X}_1\right|$.

Both experiments have equal accuracy if and only if $\left|\Delta X_1\right| = \left|\Delta \tilde{X}_1\right|$.

As an example for this case we can consider the function from Theorem 1

$$f\left(x_1, ..., x_n\right) = \sum_{i=1}^{n} a_i X_i + c\,, \quad a_i, c \in \mathrm{R}.$$

1.2) Let in (28) $\Delta X_1, ..., \Delta X_n$ are constant values.

Criterion 2. If $\Delta X_i = const \quad (i = 1, 2, ..., n)$, then the first experiment of the maximal absolute inaccuracy of Y is more accurate than the second one if and only if

$$\sum_{i=1}^{n} \left(\tilde{A}_i - A_i\right) |\Delta X_i| \geq 0. \tag{31}$$

Both experiments have equal accuracy if and only if

$$\sum_{i=1}^{n} \left(\tilde{A}_i - A_i\right) |\Delta X_i| = 0.$$

2) Let $\frac{\Delta^1 Y}{Y}$ and $\frac{\Delta^1 \tilde{Y}}{\tilde{Y}}$ be the maximal relative inaccuracies of the first order for two experiments, i.e.

If $\frac{X_i}{f}\frac{\partial f}{\partial X_i} = B_i \quad (i = 1, 2, ..., n)$ are constant values, then according to formula (6)

$$\frac{\Delta^1 Y}{Y} = \sum_{i=1}^{n} |B_i| \left|\frac{\Delta X_i}{X_i}\right|, \quad \frac{\Delta^1 \tilde{Y}}{\tilde{Y}} = \sum_{i=1}^{n} \left|\tilde{B}_i\right| \left|\frac{\Delta \tilde{X}_i}{\tilde{X}_i}\right|, \tag{32}$$

where $\left|\frac{\Delta X_i}{X_i}\right|$ and $\left|\frac{\Delta \tilde{X}_i}{\tilde{X}_i}\right|$ are defined from (8).

Criterion 3. If $\frac{x_i}{f}\frac{\partial f}{\partial x_i} = const \quad (i = 1, 2, ..., n)$, then the first experiment of the maximal relative inaccuracy of Y is more accurate than the second one if and only if

$$\sum_{i=1}^{n} |B_i| \left(\left|\frac{\Delta \tilde{X}_i}{\tilde{X}_i}\right| - \left|\frac{\Delta X_i}{X_i}\right|\right) \geq 0. \tag{33}$$

Both experiments have equal accuracy if and only if

$$\sum_{i=1}^{n} |B_i| \left(\left|\frac{\Delta \tilde{X}_i}{\tilde{X}_i}\right| - \left|\frac{\Delta X_i}{X_i}\right|\right) = 0.$$

In particular, for $n = 1$ the first experiment is more accurate than the second one if and only if $\left|\frac{\Delta X_1}{X_1}\right| \leq \left|\frac{\Delta \tilde{X}_1}{\tilde{X}_1}\right|$. Both experiments have equal accuracy if and only if $\left|\frac{\Delta X_1}{X_1}\right| = \left|\frac{\Delta \tilde{X}_1}{\tilde{X}_1}\right|$.

As an example for this case we can consider the function from Theorem 2

$$f\left(X_1,...,X_n\right) = cX_1^{k_1}...X_n^{k_n}\,,\quad c,k_i \in \mathrm{R}^+\,,\; i=1,...,n, \quad k_1 \neq 0.$$

3) Let $\Delta^2 Y$ and $\Delta^2 \tilde{Y}$ are the maximal absolute inaccuracies of the second order of two experiments.

3.1) If $\frac{\partial^2 f}{\partial X_i \partial X_j} = A_{ij}$ are constants, then according to formula (9)

$$\Delta^2 Y = \sum_{i,j=1}^{n} A_{ij} \left|\Delta X_i\right| \left|\Delta X_j\right|, \;\; \Delta^2 \tilde{Y} = \sum_{i,j=1}^{n} A_{i,j} \left|\Delta \tilde{X}_i\right| \left|\Delta \tilde{X}_j\right|. \qquad (34)$$

Criterion 4. If $\frac{\partial^2 f}{\partial x_i \partial x_j} = A_{ij} = const \quad (i,j=1,2,...,n)$, then the first experiment of the maximal absolute inaccuracy of the second order of Y is more accurate than the second one if and only if

$$\sum_{i,j=1}^{n} A_{ij}\left(\left|\Delta \tilde{X}_i\right| \left|\Delta \tilde{X}_j\right| - \left|\Delta X_i\right| \left|\Delta X_j\right|\right) \geq 0. \qquad (35)$$

Both experiments have equal accuracy if and only if

$$\sum_{i,j=1}^{n} A_{ij}\left(\left|\Delta \tilde{X}_i\right| \left|\Delta \tilde{X}_j\right| - \left|\Delta X_i\right| \left|\Delta X_j\right|\right) = 0.$$

As an example for this case we can consider the function from Theorem 3

$$f\left(X_1,...,X_n\right) = \sum_{i,j=1}^{n} a_{ij} X_i X_j + \sum_{i=1}^{n} a_i X_i + a\,, \quad a_{ij}, a_i, a \in \mathrm{R}.$$

3.2) Let $\Delta X_1,...,\Delta X_n$ are constant values and

$$\Delta^2 Y = \sum_{i,j=1}^{n} A_{ij} \left|X_i\right| \left|X_j\right|, \;\; \Delta^2 \tilde{Y} = \sum_{i,j=1}^{n} \tilde{A}_{ij} \left|X_i\right| \left|X_j\right|.$$

Criterion 5. If $\Delta X_i = const$, $(i = 1, 2, ..., n)$, then the first experiment of the maximal absolute inaccuracy of the second order of Y is more accurate than the second one if and only if

$$\sum_{i,j=1}^{n} \left(\tilde{A}_{ij} - A_{ij}\right) |\Delta X_i|\, |\Delta X_j| \geq 0. \tag{36}$$

Both experiments have equal accuracy if and only if

$$\sum_{i,j=1}^{n} \left(\tilde{A}_{ij} - A_{ij}\right) |\Delta X_i|\, |\Delta X_j| = 0.$$

4) Let $\frac{\Delta^2 Y}{Y}$ and $\frac{\Delta^2 \tilde{Y}}{\tilde{Y}}$ are the maximal relative inaccuracies of the second order of two experiments. If $\frac{X_i X_j}{f} \frac{\partial^2 f}{\partial X_i \partial X_j} = B_{ij}$ are constant values, then from (9) we have

$$\frac{\Delta^2 Y}{Y} = \sum_{i,j=1}^{n} B_{ij} \left|\frac{\Delta X_i}{X_i}\right| \left|\frac{\Delta X_j}{X_j}\right|, \quad \frac{\Delta^2 \tilde{Y}}{\tilde{Y}} = \sum_{i,j=1}^{n} B_{ij} \left|\frac{\Delta \tilde{X}_i}{\tilde{X}_i}\right| \left|\frac{\Delta \tilde{X}_j}{\tilde{X}_j}\right|. \tag{37}$$

Criterion 6. If $\frac{x_i x_j}{f} \frac{\partial^2 f}{\partial x_i \partial x_j} = const \quad (i = 1, 2, ..., n)$, then the first experiment of the maximal relative inaccuracy of the second order of Y is more accurate than the second one if and only if

$$\sum_{i,j=1}^{n} B_{ij} \left(\left|\frac{\Delta \tilde{X}_i}{\tilde{X}_i}\right| \left|\frac{\Delta \tilde{X}_j}{\tilde{X}_j}\right| - \left|\frac{\Delta X_i}{X_i}\right| \left|\frac{\Delta X_j}{X_j}\right| \right) \geq 0. \tag{38}$$

Both experiments have equal accuracy if and only if

$$\sum_{i,j=1}^{n} B_{ij} \left(\left|\frac{\Delta \tilde{X}_i}{\tilde{X}_i}\right| \left|\frac{\Delta \tilde{X}_j}{\tilde{X}_j}\right| - \left|\frac{\Delta X_i}{X_i}\right| \left|\frac{\Delta X_j}{X_j}\right| \right) = 0.$$

4 Counterexamples to the dimensionless scale and improvement of its application

In [1] we considered $\Delta X_1, \Delta X_2, ..., \Delta X_n, \pm Y$ as a system of generalized orthogonal coordinates. Then for $n \geq 2$ we get an $(n+1)$-dimensional Euclidean space, where (3) is an equation of a plane that passes through the origin of the coordinate system.

Thus we take ε for *sample plane in the space of the absolute inaccuracy* which represents an imaginary ideal perfectly accurate experiment.

If $\alpha : \Delta Y = A_1\Delta X_1 + A_2\Delta X_2 + ... + A_n\Delta X_n$, then ε is determined by $A_1 = A_2 = ... = A_n = 0$, i.e.

$$\varepsilon : \Delta Y = 0.$$

In [2, 3] we considered the angle between the normal vectors $\overrightarrow{n_\alpha}\,(A_1, A_2, ..., A_n, -1)$ of the plane α of the real experiment and $\overrightarrow{n_\varepsilon}\,(0, 0, ..., 0, -1)$ of the palne ε. Then the value of the cosine

$$k_\alpha = \cos\angle\,(\overrightarrow{n_\alpha}, \overrightarrow{n_\varepsilon}) = \frac{1}{\sqrt{A_1^2 + A_2^2 + ... + A_n^2 + 1}} \tag{39}$$

of this angle can be chosen for a coefficient of accuracy in a dimensionless scale, i.e. for a numerical characteristic of the quality of the experiment.

Since $k_\alpha = \cos\angle\,(\overrightarrow{n_\alpha}, \overrightarrow{n_\varepsilon})$, then the scale for evaluating the quality of the experiment is the interval $[0, 1]$. The value $k_\alpha = 1$ represents the ideal perfectly accurate experiment and the value $k_\alpha = 0$ represents the ideal absolutely inaccurate experiment. The conclusions we have made in [1, 2, 3] regarding the application of the scale are not absolutely correct. We will prove this with the following numerical examples, applying the criteria from section 3.

Example 1) Let $S = f\,(t) = gt$ be the distance that the uniformly moving object passes with constant velocity v during time t. Thus $f\,(t)$ has the form from Theorem 1.

For the first experiment we choose $t_{11} = 4,\ t_{12} = 2$. Then $\bar{t}_1 = 3,\ |\Delta t_{11}| = 1,\ |\Delta t_{12}| = 1,\ |\Delta t_1| = 1$. Since $\frac{df}{dt} = v$, then according to formula (4)

$$A_1 = \frac{1}{2} \sum_{m=1}^{2} \left| \frac{df}{dt} (t_{1m}) \right| = \frac{1}{2} \sum_{m=1}^{2} |v| = v.$$

From (3) we find the value of the maximal absolute inaccuracy for the first experiment

$$\Delta^1 Y = \Delta^1 f = A_1 |\Delta t_1| = v.1 = v.$$

For the second experiment we choose $\tilde{t}_{11} = 3,6$, $\tilde{t}_{12} = 2,2$. Then $\bar{t}_1 = 2,9$, $\left|\Delta\tilde{t}_{11}\right| = 0,7$, $\left|\Delta\tilde{t}_{12}\right| = 0,7$, $\left|\Delta\tilde{t}_1\right| = 0,7$. From (4), since $\frac{df}{dt} = v$, we calculate

$$A_2 = \frac{1}{2} \sum_{m=1}^{2} \left| \frac{df}{dt} (t_{1m}) \right| = \frac{1}{2} \sum_{m=1}^{2} |v| = v.$$

From (3) we find the value of the maximal absolute inaccuracy for the second experiment

$$\Delta^1 \tilde{Y} = \Delta^1 \tilde{f} = A_2 \left|\Delta\tilde{t}_1\right| = 0,7v.$$

Since $A_1 = A_2$, then from formula (38) we have the following relationship between the coefficients of accuracy:

$$k_1 = \frac{1}{\sqrt{A_1^2 + 1}} = \frac{1}{\sqrt{A_2^2 + 1}} = k_2,$$

i.e. regarding [1, 2, 3] we can conclude that both experiments have the same accuracy. But

$$\Delta^1 Y = \Delta^1 f = g > 0,7g = \Delta^1 \tilde{Y} = \Delta^1 \tilde{f}.$$

Therefore the second experiment is more accurate than the first one. This counterexample contradicts the conclusions in [1, 2, 3] for the dimensionless scale.

From the necessary and sufficient conditions we have presented in section 4, for $A_1 = A_2$, according to Criterion 1, it follows that the second experiment is more accurate than the first one, because $\left|\Delta\tilde{t}_1\right| < |\Delta t_1|$. Therefore Criterion 1 gives us more precise conclusion.

Example 2) Let $S = f(t) = \frac{gt^2}{2}$ be the distance that free falling object passes during time t (in vacuum) and $g = 9,8\ m/s^2$ is the earth gravitational acceleration. Thus $f(t)$ has the form from Theorem 2.

For the first experiment we choose $t_{11} = 2$, $t_{12} = 1,6$. Then $\bar{t}_1 = 1,8$, $|\Delta t_{11}| = 0,2$, $|\Delta t_{12}| = 0,2$ and $|\Delta t_1| = 0,2$. Since $\frac{df}{dt} = gt$, then from formula (4) we find $A_1 = \frac{1}{2}\sum\limits_{m=1}^{2}\left|\frac{df}{dt}(t_{1m})\right| = \frac{1}{2}\sum\limits_{m=1}^{2}|gt| = \frac{1}{2}g\sum\limits_{m=1}^{2}|t| = \frac{1}{2}g(2+1,6) = 1,8g$. From formula (3) we calculate the value of the maximal absolute inaccuracy for the first experiment

$$\Delta^1 Y = \Delta^1 f = A_1|\Delta t_1| = 1,8g \times 0,2 = 0,36g.$$

For the second experiment we choose $\tilde{t}_{11} = 1,8$, $\tilde{t}_{12} = 1,9$. Then $\bar{t}_1 = 1,85$, $\left|\Delta\tilde{t}_{11}\right| = 0,05$, $\left|\Delta\tilde{t}_{12}\right| = 0,05$, $\left|\Delta\tilde{t}_1\right| = 0,05$. From formula (4) we find

$$A_2 = \frac{1}{2}\sum_{m=1}^{2}\left|\frac{df}{dt}(t_{1m})\right| = \frac{1}{2}\sum_{m=1}^{2}|gt| = \frac{1}{2}g\sum_{m=1}^{2}|t| =$$

$$= \frac{1}{2}g(1,8+1,9) = 1,85g.$$

From formula (3) we find the value of the maximal absolute inaccuracy for the second experiment

$$\Delta^1\tilde{Y} = \Delta^1\tilde{f} = A_2\left|\Delta\tilde{t}_1\right| = 1,85g \times 0,05 = 0,0925g.$$

Since $A_1 < A_2$, then from formula (39) we have the following relationship between the coefficients of accuracy:

$$k_1 = \frac{1}{\sqrt{A_1^2+1}} > \frac{1}{\sqrt{A_2^2+1}} = k_2,$$

i.e. according to [2, 3] the value of the maximal absolute inaccuracy $\Delta^1 Y$ for the first experiment is more accurate than the value $\Delta^1\tilde{Y}$ of the second one. But

$$\Delta^1\tilde{Y} = \Delta^1\tilde{f} = 0,0925g < 0,36g = \Delta^1 Y = \Delta^1 f.$$

Therefore we can conclude that the second experiment is more accurate than the first one. This counterexample contradicts the conclusions in [1, 2, 3] for the dimensionless scale.

Both examples show that the conclusions we have made in [1, 2, 3] regarding the dimensionless scale and the sample plane in the spaces of the absolute and relative inaccuracies, have to be improved.

For correct application of the dimensionless scale in [1, 3], we give the following supplements.

Definition 5. We will say that the vector $A = (A_1, A_2, ..., A_n)$ is less than or equal to the vector $B = (B_1, B_2, ..., B_n)$ (coordinate by coordinate) and we will denote with $\bar{A} \le \bar{B}$, if $A_i \le B_i$ for each $i = 1, 2, ..., n$.

Let for fixed values of $\Delta X_1, \Delta X_2, ..., \Delta X_n$ for an experiment we have two different forms for representation of the maximal absolute inaccuracy ΔY, i.e.:

$\Delta^1 Y = A_1\Delta X_1 + A_2\Delta X_2 + ... + A_n\Delta X_n$ and $\Delta^1\tilde{Y} = B_1\Delta X_1 + B_2\Delta X_2 + ... + B_n\Delta X_n$.

Then obviously the following conclusion is true:

Theorem 6. For fixed values of $\Delta X_1, \Delta X_2, ..., \Delta X_n$ between two experiments with planes $\alpha:\ \Delta^1 Y = A_1\Delta X_1 + A_2\Delta X_2 + ... + A_n\Delta X_n$ and $\beta:\ \Delta^1\tilde{Y} = B_1\Delta X_1 + B_2\Delta X_2 + ... + B_n\Delta X_n$ the more accurate is that one, the normal vector of which is less than or equal to the other.

If there are two vectors $A = (A_1, A_2, ..., A_n)$, $B = (B_1, B_2, ..., B_n)$ and $\bar{A} \le \bar{B}$, then $k_\alpha \ge k_\beta$ and for the fixed $\Delta X_1, \Delta X_2, ..., \Delta X_n$ it follows that $\Delta^1 Y \le \Delta^1\tilde{Y}$. However it is not true the statement that we formulated in [1, 3], that from $k_\alpha \ge k_\beta$ it follows $\Delta^1 Y \le \Delta^1\tilde{Y}$.

Let us consider that the maximal absolute inaccuracy ΔY has the same representation $\Delta Y = A_1\Delta X_1 + A_2\Delta X_2 + ... + A_n\Delta X_n$ for two provided experiments, i.e. the values of the coefficients $A_1, A_2, ..., A_n$ are fixed. Then obviously for different experiments with measured values $x_{11}, x_{12}, ..., x_{1n}$ and $x_{21}, x_{22}, ..., x_{2n}$ of $\Delta X_1, \Delta X_2, ..., \Delta X_n$ the following conclusion is true:

Theorem 7. For the fixed values $A_1, A_2, ..., A_n$ between experiments with measured values $x_{11}, x_{12}, ..., x_{1n}$ and $x_{21}, x_{22}, ..., x_{2n}$ of $\Delta X_1, \Delta X_2, ..., \Delta X_n$, the more accurate experiment is that one, the vector of which is less than or equal (coordinate by coordinate) to the other.

Thus, if $x_1 = (x_{11}, x_{12}, ..., x_{1n})$, $x_2 = (x_{21}, x_{22}, ..., x_{2n})$ and $\bar{x}_1 \leq \bar{x}_2$, then $\Delta^1 Y \leq \Delta^1 \tilde{Y}$. In this case the reverse statement is not true.

The most accurate experiment will be that one, where the values of the variables and the normal vector (coordinate by coordinate) are the least possible.

Analogical conclusions as Theorem 6 and Theorem 7 can be formulated also for the maximal relative inaccuracy $\frac{\Delta Y}{Y}$ of an indirectly measurable variable Y.

5 Discussion

The suggested by us method for determining the numerical values of the maximal and relative inaccuracy of an indirectly measurable variable is of great importance for every experimental science, in which the studied processes can be modelled via functions. The values of the maximal inaccuracies can be compared very easily when we have two experiments.

6 Conclusion

In this paper we give necessary and sufficient conditions for comparison of the values of the maximal inaccuracies for two experiments. We consider some of the most common in the practice classes of functions. We give numerical counterexamples regarding the introduced by us dimensionless scale in [1, 2, 3] for evaluation of two experiments. We also give some conditions for the correct application of the scale. Thus we improve the conclusions we have made in [1, 2, 3].

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