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Nash equilibrium set function in dyadic mixed-strategy games

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Abstract

Dyadic two-person mixed strategy games form the case for which Nash equilibrium sets can be determined simply. In this paper the set of Nash equilibria in a particular game is determined as an intersection of graphs of optimal reaction mappings of the first and the second players. In contrast to other games, it is obtained not only an algorithm, but a multi-valued Nash equilibrium set function that gives directly as its values the Nash equilibrium sets corresponding to the values of payoff matrix instances. To give an expedient form to such a function definition, it is used a code written in the Wolfram language. Additionally, it is also applied a Wolfram language code to prove the main theoretic result.

Keywords: Dyadic mixed-strategy game, Nash equilibrium, Nash equilibrium set function, NES function, graph of best response mapping.

MSC 2010: 91A05, 91A10, 91A43, 91A44.

1 Introduction

The problem of Nash equilibrium [2] computing in noncooperative games [1] remains important in game theory at least for last decades. Moreover, the problem of the entire Nash equilibrium set computing becomes important too because of non equivalence of Nash equilibrium instances.

We studied the problem of Nash equilibrium sets computing in a series of works [5]–[8], [10], [11], [15], [16]. The main idea of those works is based on the concept of a graph of best response mapping and the representation of a Nash equilibrium set as the intersection of best-response mapping graphs. The idea was extended to Pareto-Nash-Stackelberg games too [9], [12]–[14], [17]–[21].

In this paper, we present a formula of the Nash equilibrium set function $NES(A, B)$ (NES function). It is a multi-valued function that has as its domain the Cartesian product $\mathbb{R}^{2\times 2} \times \mathbb{R}^{2\times 2}$ of two real spaces of two 2×2 matrices and as a NES function image all possible sets of Nash equilibria in dyadic bimatrix mixed-strategy games. These types of games where studied earlier in a series of works, e.g. [3]–[5].

2 Game statement and its simplification

The dyadic two-player mixed strategy game is defined by a tuple

$$\Gamma' = \langle N, X, Y, f_1(\mathbf{x}, \mathbf{y}), f_2(\mathbf{x}, \mathbf{y}) \rangle,$$

where

$N = \{1, 2\}$ is a set of players,

$X = \{\mathbf{x} \in \mathbb{R}^2 : x_1 + x_2 = 1, x_1 \geq 0, x_2 \geq 0\}$ is a set of strategies of the first player,

$Y = \{\mathbf{y} \in \mathbb{R}^2 : y_1 + y_2 = 1, y_1 \geq 0, y_2 \geq 0\}$ is a set of strategies of the second player,

$f_i : X \times Y \to R$ is a player's $i \in N$ payoff function,
$f_1(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T A \mathbf{y}, \; A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \; A \in \mathbb{R}^{2\times 2},$

$f_2(\mathbf{x}, \mathbf{y}) = \mathbf{x}^{\mathbf{T}} B \mathbf{y}, \; B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \; B \in \mathbb{R}^{2\times 2}.$

We suppose that every player maximizes the value of his payoff function. It is a requirement that is dictated by a need to be consistent with the Wolfram language code.

Remark, the strategy set is a segment for each player, i.e. it is a hypotenuse of the correspondent unit simplex (right triangle) in $\mathbb{R}^2$.

We reduce the game Γ' to a simpler game Γ by substitutions:

$$x_1 = x,\ x_2 = 1 - x,\ 0 \le x \le 1,$$

$$y_1 = y,\ y_2 = 1 - y,\ 0 \le y \le 1.$$

Thereby, in the game Γ both the players have as their strategy sets the segment $[0, 1]$ and as their payoff functions:

$$f_1(x, y) = (\alpha y - \alpha_0)\, x + (a_{21} - a_{22})\, y + a_{22},$$

$$f_2(x, y) = (\beta x - \beta_0)\, y + (b_{21} - b_{22})\, x + b_{22},$$

where

$$\alpha = a_{11} - a_{21} + a_{22} - a_{12},$$

$$\alpha_0 = a_{22} - a_{12},$$

$$\beta = b_{11} - b_{12} + b_{22} - b_{21},$$

$$\beta_0 = b_{22} - b_{21}.$$

Remark 2.1. *To compute their optimal strategies, the players can omit the last two members of their payoff functions because the first player chooses a value for the variable x and the second chooses a value for the variable y. Thus, they solve the simplified game:*

$$\Gamma = \langle [0, 1]\, , [0, 1]\, ;\ \tilde{f}_1(x, y) = (\alpha y - \alpha_0)\, x,\ \tilde{f}_2(x, y) = (\beta x - \beta_0)\, y \rangle.$$

Proposition 2.1. *From the strategy point of view, the games Γ' and Γ are equivalent.*

Proof. The truth of the statement follows from the reduction of the game Γ' to the game Γ provided above. □

Further, we will use both the game Γ' and the game Γ to construct the sets of Nash equilibria.

3 Optimal Value Functions and Best Response Mappings

The game Γ represents a game on a unit square. The payoff functions are bilinear, i.e. for a fixed value of one variable, the functions become linear functions in relation to the other variable. To choose his best strategy, every player must solve a parametric linear programming problem on a unit segment.

For any player, we can define the optimal value function and the best response mapping:

$\varphi_1(y) = \max\limits_{x\in[0,1]} f_1(x,y)$ — the optimal value function of the first player,

$\gamma_1(y) = \operatorname{Arg}\max\limits_{x\in[0,1]} f_1(x,y) = \operatorname{Arg}\max\limits_{x\in[0,1]} \tilde{f}_1(x,y)$ — the best response mapping of the first player,

$\varphi_2(x) = \max\limits_{y\in[0,1]} f_1(x,y)$ — the optimal value function of the second player,

$\gamma_2(x) = \operatorname{Arg}\max\limits_{y\in[0,1]} f_2(x,y) = \operatorname{Arg}\max\limits_{y\in[0,1]} \tilde{f}_2(x,y)$ — the best response mapping of the second player.

To determine Nash equilibrium sets for particular instances of matrices A and B, we need the graphs of best response mappings:

$$Gr_1(A,B) = \left\{ (x,y) \in [0,1]\times[0,1] : \begin{array}{l} y \in [0,1] \\ x \in \operatorname{Arg}\max_{x\in[0,1]} \tilde{f}_1(x,y) \end{array} \right\},$$

$$Gr_2(A,B) = \left\{ (x,y) \in [0,1]\times[0,1] : \begin{array}{l} x \in [0,1] \\ y \in \operatorname{Arg}\max_{y\in[0,1]} \tilde{f}_2(x,y) \end{array} \right\}.$$

As these graphs are functions of payoff matrices A and B, it is evidently that the Nash equilibrium set is a function of the payoff matrices A and B too:

$$NES(A,B) = Gr_1(A,B) \bigcap Gr_2(A,B).$$

Remark 3.1. *It is important to observe once again that the graphs of best responses $Gr_1(A,B)$ and $Gr_2(A,B)$ are multi-valued (set-valued) functions of the matrices A and B. As the Nash equilibrium set in a particular game is determined as the intersection of these graphs, the Nash equilibrium set function (NES function) is a function of the matrices A and B. Such a function must have eight arguments corresponding to eight elements of the matrices A and B:*

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad A \in \mathbb{R}^{2\times 2},$$

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \quad B \in \mathbb{R}^{2\times 2}.$$

Because of the initial game Γ' simplification to the game Γ, we define both the graphs of best response mappings and the NES function as the functions of the four arguments: α, α_0, and β, β_0, the formulas for which are defined above as the functions of the eight matrix elements.

Now, according to the above remark, let us consider the problem of a Nash equilibrium set computing as a problem of defining a Nash equilibrium set function with the payoff matrices A and B (or α, α_0, and β, β_0) as their arguments, i.e. let us consider analytically the problem of a NES function definition and its computing. In the same context, let us consider the graphs of best response mappings as the functions of α, α_0, and β, β_0.

$$\mathrm{g1}(\alpha,\alpha_0) = \left\{ (x,y) \in [0,1]\times[0,1] : \begin{array}{l} y \in [0,1] \\ x \in \operatorname{Arg}\max\limits_{x\in[0,1]} (\alpha y - \alpha_0)\, x \end{array} \right\},$$

$$\mathrm{g2}(\beta,\beta_0) = \left\{ (x,y) \in [0,1]\times[0,1] : \begin{array}{l} x \in [0,1] \\ y \in \operatorname{Arg}\max\limits_{y\in[0,1]} (\beta x - \beta_0)\, y \end{array} \right\}.$$

Optimal solutions of the optimization problems in the expressions for $\mathrm{g1}(\alpha,\alpha_0)$ and $\mathrm{g2}(\beta,\beta_0)$ are attained either on one of the extremities, or on the whole segment $[0,1]$. As the results depend of both the values of $\mathrm{g1}(\alpha,\alpha_0)$ and $\mathrm{g2}(\beta,\beta_0)$, we can establish exactly expressions for these graphs as piecewise functions.

The graph of best response mapping of the first player is

$$\mathrm{g1}(\alpha,\alpha_0)=\begin{cases}
[0,1]\times[0,1] & \text{if } \alpha=0\&\alpha_0=0,\\
[[1,0],[1,1]] & \text{if } (\alpha\geq 0\&\alpha0>0)\|\\
 & \quad(\alpha<0\&\alpha+\alpha0>0),\\
[[0,0],[0,1]] & \text{if } (\alpha\leq 0\&\alpha0<0)\|\\
 & \quad(\alpha>0\&\alpha+\alpha0<0),\\
[[0,0],[1,0]]\bigcup[[1,0],[1,1]] & \text{if } \alpha>0\&\alpha0=0,\\
\left[[0,0],\left[0,-\frac{\alpha0}{\alpha}\right]\right]\bigcup & \text{if } \alpha>0\&\alpha0<0\&\\
\left[\left[0,-\frac{\alpha0}{\alpha}\right],\left[1,-\frac{\alpha0}{\alpha}\right]\right]\bigcup & \quad\alpha+\alpha0>0,\\
\left[\left[1,-\frac{\alpha0}{\alpha}\right],[1,1]\right] & \\
[[0,0],[0,1]]\bigcup[[0,1],[1,1]] & \text{if } \alpha>0\&\alpha+\alpha0=0,\\
[[1,0],[0,0]]\bigcup[[0,0],[0,1]] & \text{if } \alpha<0\&\alpha0=0,\\
\left[[1,0],\left[1,-\frac{\alpha0}{\alpha}\right]\right]\bigcup & \text{if } \alpha<0\&\alpha0>0\&\\
\left[\left[1,-\frac{\alpha0}{\alpha}\right],\left[0,-\frac{\alpha0}{\alpha}\right]\right]\bigcup & \quad\alpha+\alpha0<0,\\
\left[\left[0,-\frac{\alpha0}{\alpha}\right],[0,1]\right] & \\
[[1,0],[1,1]]\bigcup[[1,1],[0,1]] & \text{if } \alpha<0\&\alpha+\alpha0=0.
\end{cases}$$

Remark 3.2. *Above, we denoted the unit square by* $[0,1]\times[0,1]$ *and the segment which connects two points, e.g.* $[0,0]$ *and* $[1,0]$*, by an expression of the type* $[[0,0],[1,0]]$.

Remark 3.3. *From the expression for the graph* $\mathrm{g1}(\alpha,\alpha_0)$*, we can conclude that the set of its values or the image of the function* $\mathrm{g1}(\alpha,\alpha_0)$ *is formed by a union of the following four alternatives:*

> *a unit square,*
>
> *a unit segment,*
>
> *a union of two connected segments on the boundary of the unit square (one horizontal and one vertical),*

a union of three connected segments (two vertical, on the opposite vertical sides of the unit square, connected by the third interior horizontal segment, from one lateral side to the other).

It is important to observe that the condition specified by the concrete values of α and $\alpha 0$ corresponds to an entire class of matrices $A \in \mathbb{R}^{2\times 2}$. More the more, accordingly to the formulas that define α and α_0, the function $\mathrm{g1}(\alpha, \alpha_0)$ is defined on entire space $\mathbb{R}^{2\times 2}$, i.e. it is defined for any numeric dyadic matrix A.

The graph of best response mapping of the second player is

$$\mathrm{g2}(\beta,\beta_0) = \begin{cases} [0,1]\times[0,1] & \text{if } \beta = 0 \& \beta_0 = 0, \\ [[0,1],[1,1]] & \text{if } (\beta \geq 0 \& \beta 0 > 0) \| \\ & \quad (\beta < 0 \& \beta + \beta 0 > 0), \\ [[0,0],[1,0]] & \text{if } (\beta \leq 0 \& \beta 0 < 0) \| \\ & \quad (\beta > 0 \& \beta + \beta 0 < 0), \\ [[0,0],[0,1]] \bigcup [[0,1],[1,1]] & \text{if } \beta > 0 \& \beta 0 = 0, \\ \left[[0,0], \left[-\frac{\beta 0}{\beta}, 0\right]\right] \bigcup & \text{if } \beta > 0 \& \beta 0 < 0 \& \\ \left[\left[-\frac{\beta 0}{\beta}, 0\right], \left[-\frac{\beta 0}{\beta}, 1\right]\right] \bigcup & \quad \beta + \beta 0 > 0, \\ \left[\left[-\frac{\beta 0}{\beta}, 1\right], [1,1]\right] & \\ [[0,0],[1,0]] \bigcup [[1,0],[1,1]] & \text{if } \beta > 0 \& \beta + \beta 0 = 0, \\ [[0,1],[0,0]] \bigcup [[0,0],[1,0]] & \text{if } \beta < 0 \& \beta 0 = 0, \\ \left[[0,1], \left[-\frac{\beta 0}{\beta}, 1\right]\right] \bigcup & \text{if } \beta < 0 \& \beta 0 > 0 \& \\ \left[\left[-\frac{\beta 0}{\beta}, 1\right], \left[-\frac{\beta 0}{\beta}, 0\right]\right] \bigcup & \quad \beta + \beta 0 < 0, \\ \left[\left[-\frac{\beta 0}{\beta}, 0\right], [1,0]\right] & \\ [[0,1],[1,1]] \bigcup [[1,1],[1,0]] & \text{if } \beta < 0 \& \beta + \beta 0 = 0. \end{cases}$$

For the graph $\mathrm{g2}(\beta, \beta_0)$ analogical conclusions are valid as for the graph $\mathrm{g1}(\alpha, \alpha_0)$.

4 Nash equilibria and NES function

As every graph of best response mappings has 9 different possible forms for particular instances of the payoff matrices A and B, their intersection abstractly may generate 81 possible instances/cases of Nash equilibrium sets. Some of them may coincide.

Next theorem has to highlight distinct cases in the form of the NES function definition on the matrices A and B. Remark once again that such a function is a set-valued function.

Theorem 4.1. *In the dyadic mixed strategy game, the Nash equilibrium set function* $nes[\alpha, \alpha 0, \beta, \beta 0]$ *may be defined as a piecewise function formed by 36 distinct pieces.*

Proof. The proof is constructive. It enumerates distinct possible cases in the form of the Wolfram language code.

First, the best response mapping graphs are defined as the Wolfram language functions.

Second, the NES function is defined as the Wolfram language function too.

In the end, we present a Wolfram language code to manipulate all the elements together and to highlight results for different initial data. The Wolfram language primitives, such as Rectangle, Line, Point, have clear meanings and are used as a simple language means to expose both the proof, and a Mathematica 10.4.1 program.

g1[$\alpha\_, \alpha 0\_$]:=Piecewise[{
{Rectangle[{{0,0}, {1,1}}], $\alpha == 0 \&\& \alpha 0 == 0$},
{Line[{{1,0},{1,1}}], $(\alpha >= 0 \&\& \alpha 0 > 0) \| (\alpha < 0 \&\& \alpha + \alpha 0 > 0)$},
{Line[{{0,0},{0,1}}], $(\alpha <= 0 \&\& \alpha 0 < 0) \| (\alpha > 0 \&\& \alpha + \alpha 0 < 0)$},
{Line[{{0,0},{1,0},{1,1}}], $\alpha > 0 \&\& \alpha 0 == 0$},
{Line[{{0,0}, {0,-$\frac{\alpha 0}{\alpha}$}, {1,-$\frac{\alpha 0}{\alpha}$}, {1,1}}], $\alpha > 0 \&\& \alpha 0 < 0 \&\& \alpha + \alpha 0 > 0$},
{Line[{{0,0}, {0,1}, {1,1}}], $\alpha > 0 \&\& \alpha + \alpha 0 == 0$},
{Line[{{1,0}, {0,0}, {0,1}}], $\alpha < 0 \&\& \alpha 0 == 0$},
{Line[{{1,0},{1,-$\frac{\alpha 0}{\alpha}$},{0, $-\frac{\alpha 0}{\alpha}$}, {0, 1}}], $\alpha < 0 \&\& \alpha 0 > 0 \&\& \alpha + \alpha 0 < 0$},
{Line[{{1,0},{1,1},{0,1}}], $\alpha < 0 \&\& \alpha + \alpha 0 == 0$}
}]

g2[$\beta\_, \beta 0\_$]:= Piecewise[{
{Rectangle[{{0,0},{1,1}}], $\beta == 0 \&\& \beta 0 == 0$},
{Line[{{0,1},{1,1}}], $(\beta >= 0 \&\& \beta 0 > 0) \| (\beta < 0 \&\& \beta + \beta 0 > 0)$},
{Line[{{0,0},{1,0}}], $(\beta <= 0 \&\& \beta 0 < 0) \| (\beta > 0 \&\& \beta + \beta 0 < 0)$},
{Line[{{0,0},{0,1},{1,1}}], $\beta > 0 \&\& \beta 0 == 0$},
{Line[{{0,0}, {-$\frac{\beta 0}{\beta}$, 0}, {$-\frac{\beta 0}{\beta}$,1},{1,1}}], $\beta > 0 \&\& \beta 0 < 0 \&\& \beta + \beta 0 > 0$},
{Line[{{0,0},{1,0},{1,1}}], $\beta > 0 \&\& \beta + \beta 0 == 0$},
{Line[{{0,1},{0,0},{1,0}}], $\beta < 0 \&\& \beta 0 == 0$},
{Line[{{0,1},{-$\frac{\beta 0}{\beta}$, 1}, {$-\frac{\beta 0}{\beta}$, 0}, {1, 0}}], $\beta < 0 \&\& \beta 0 > 0 \&\& \beta + \beta 0 < 0$},
{Line[{{0,1},{1,1},{1,0}}], $\beta < 0 \&\& \beta + \beta 0 == 0$}
}]

nes[$\alpha\_, \alpha 0\_, \beta\_, \beta 0\_$]:=Piecewise[{

(*1*) { {Point[{{0,0},{1,0},{1,1},{0,1}}],Rectangle[{0,0},{1,1}]},
$(\alpha == 0 \&\& \alpha 0 == 0) \&\& (\beta == 0 \&\& \beta 0 == 0)$ },

(*2*) { {Point[{{0,1},{1,1}}],Line[{{0,1},{1,1}}]},
$(\alpha == 0 \&\& \alpha 0 == 0) \&\& ((\beta \geq 0 \&\& \beta 0 > 0) \|$
$(\beta < 0 \&\& \beta + \beta 0 > 0))) \| ((\alpha > 0 \&\& \alpha + \alpha 0 == 0) \&\& ((\beta \geq 0 \&\& \beta 0 > 0) \|$
$(\beta < 0 \&\& \beta + \beta 0 > 0))) \|$
$((\alpha > 0 \&\& \alpha + \alpha 0 == 0) \&\& (\beta < 0 \&\& \beta + \beta 0 == 0)) \|$
$((\alpha < 0 \&\& \alpha + \alpha 0 == 0) \&\& (\beta > 0 \&\& \beta 0 == 0)) \|$
$((\alpha < 0 \&\& \alpha + \alpha 0 == 0) \&\& ((\beta \geq 0 \&\& \beta 0 > 0) \|$
$(\beta < 0 \&\& \beta + \beta 0 > 0)))$ },

(*3*) { {Point[{{0,0},{1,0}}],Line[{{0,0},{1,0}}]},
$((\alpha == 0 \&\& \alpha 0 == 0) \&\& ((\beta \leq 0 \&\& \beta 0 < 0) \|$
$(\beta > 0 \&\& \beta + \beta 0 < 0))) \|$
$((\alpha > 0 \&\& \alpha 0 == 0) \&\& ((\beta \leq 0 \&\& \beta 0 < 0) \| (\beta > 0 \&\& \beta + \beta 0 < 0))) \|$
$((\alpha > 0 \&\& \alpha 0 == 0) \&\& (\beta < 0 \&\& \beta 0 == 0)) \|$
$((\alpha < 0 \&\& \alpha 0 == 0) \&\& (\beta > 0 \&\& \beta + \beta 0 == 0)) \|$
$((\alpha < 0 \&\& \alpha 0 == 0) \&\& ((\beta \leq 0 \&\& \beta 0 < 0) \| (\beta > 0 \&\& \beta + \beta 0 < 0)))$ },

(*4*) { {Point[{{0,0},{0,1},{1,1}}],Line[{{0,0},{0,1},{1,1}}]},
$((\alpha == 0 \&\& \alpha 0 == 0) \&\& (\beta > 0 \&\& \beta 0 == 0)) \|$
$((\alpha > 0 \&\& \alpha + \alpha 0 == 0) \&\& (\beta == 0 \&\& \beta 0 == 0)) \|$
$((\alpha > 0 \&\& \alpha + \alpha 0 == 0) \&\& (\beta > 0 \&\& \beta 0 == 0))$ },

(*5*) { {Point[g2[$\beta, \beta 0$][[1]]], $g2[\beta, \beta 0]$},

$((\alpha == 0\&\&\alpha 0 == 0)\&\&(\beta > 0\&\&\beta 0 < 0\&\&\beta + \beta 0 > 0))\|$
$((\alpha == 0\&\&\alpha 0 == 0)\&\&(\beta < 0\&\&\beta 0 > 0\&\&\beta + \beta 0 < 0))$ },

(*6*) { {Point[{{0,0},{1,0},{1,1}}],Line[{{0,0},{1,0},{1,1}}]},
$((\alpha == 0\&\&\alpha 0 == 0)\&\&(\beta > 0\&\&\beta + \beta 0 == 0))\|$
$((\alpha > 0\&\&\alpha 0 == 0)\&\&(\beta == 0\&\&\beta 0 == 0))\|$
$((\alpha > 0\&\&\alpha 0 == 0)\&\&(\beta > 0\&\&\beta + \beta 0 == 0))$ },

(*7*) { {Point[{{0,1},{0,0},{1,0}}],Line[{{0,1},{0,0},{1,0}}]},
$((\alpha == 0\&\&\alpha 0 == 0)\&\&(\beta < 0\&\&\beta 0 == 0))\|$
$((\alpha < 0\&\&\alpha 0 == 0)\&\&(\beta == 0\&\&\beta 0 == 0))\|$
$((\alpha < 0\&\&\alpha 0 == 0)\&\&(\beta < 0\&\&\beta 0 == 0))$ },

(*8*) { {Point[{{0,1},{1,1},{1,0}}],Line[{{0,1},{1,1},{1,0}}]},
$((\alpha == 0\&\&\alpha 0 == 0)\&\&(\beta < 0\&\&\beta + \beta 0 == 0))\|$
$((\alpha < 0\&\&\alpha + \alpha 0 == 0)\&\&(\beta == 0\&\&\beta 0 == 0))\|$
$((\alpha < 0\&\&\alpha + \alpha 0 == 0)\&\&(\beta < 0\&\&\beta + \beta 0 == 0))$ },

(*9*) { {Point[{{1,0},{1,1}}],Line[{{1,0},{1,1}}]},
$(((\alpha \geq 0\&\&\alpha 0 > 0)\|$
$(\alpha < 0\&\&\alpha + \alpha 0 > 0))\&\&(\beta == 0\&\&\beta 0 == 0))\|$
$(((\alpha \geq 0\&\&\alpha 0 > 0)\|(\alpha < 0\&\&\alpha + \alpha 0 > 0))\&\&(\beta > 0\&\&\beta + \beta 0 == 0))\|$
$(((\alpha \geq 0\&\&\alpha 0 > 0)\|(\alpha < 0\&\&\alpha + \alpha 0 > 0))\&\&(\beta < 0\&\&\beta + \beta 0 == 0))\|$
$((\alpha > 0\&\&\alpha 0 == 0)\&\&(\beta < 0\&\&\beta + \beta 0 == 0))\|$
$((\alpha < 0\&\&\alpha + \alpha 0 == 0)\&\&(\beta > 0\&\&\beta + \beta 0 == 0))$ },

(*10*) { {Point[{1,1}]},
$(((\alpha \geq 0\&\&\alpha 0 > 0)\|(\alpha < 0\&\&\alpha + \alpha 0 > 0))\&\&((\beta \geq 0\&\&\beta 0 > 0)\|$
$(\beta < 0\&\&\beta + \beta 0 > 0)))\|(((\alpha \geq 0\&\&\alpha 0 > 0)\|$
$(\alpha < 0\&\&\alpha + \alpha 0 > 0))\&\&(\beta > 0\&\&\beta 0 == 0))\|(((\alpha \geq 0\&\&\alpha 0 > 0)\|$
$(\alpha < 0\&\&\alpha + \alpha 0 > 0))\&\&(\beta > 0\&\&\beta 0 < 0\&\&\beta + \beta 0 > 0))\|$
$((\alpha > 0\&\&\alpha 0 == 0)\&\&((\beta \geq 0\&\&\beta 0 > 0)\|(\beta < 0\&\&\beta + \beta 0 > 0)))\|$
$((\alpha > 0\&\&\alpha 0 < 0\&\&\alpha + \alpha 0 > 0)\&\&((\beta \geq 0\&\&\beta 0 > 0)\|$
$(\beta < 0\&\&\beta + \beta 0 > 0)))$ },

(*11*) { {Point[{1,0}]},
$(((\alpha \geq 0\&\&\alpha 0 > 0)\|(\alpha < 0\&\&\alpha + \alpha 0 > 0))\&\&((\beta \leq 0\&\&\beta 0 < 0)\|$
$(\beta > 0\&\&\beta + \beta 0 < 0)))\|(((\alpha \geq 0\&\&\alpha 0 > 0)\|$
$(\alpha < 0\&\&\alpha + \alpha 0 > 0))\&\&(\beta < 0\&\&\beta 0 == 0))\|(((\alpha \geq 0\&\&\alpha 0 > 0)\|$
$(\alpha < 0\&\&\alpha + \alpha 0 > 0))\&\&(\beta < 0\&\&\beta 0 > 0\&\&\beta + \beta 0 < 0))\|$
$((\alpha < 0\&\&\alpha + \alpha 0 == 0)\&\&((\beta \leq 0\&\&\beta 0 < 0)\|(\beta > 0\&\&\beta + \beta 0 < 0)))\|$

$((\alpha < 0\&\&\alpha0 > 0\&\&\alpha + \alpha0 < 0)\&\&((\beta \leq 0\&\&\beta0 < 0)\|$
$(\beta > 0\&\&\beta + \beta0 < 0)))$ },

(*12*) { {Point[{{0,0},{0,1}}],Line[{{0,0},{0,1}}]},
$(((\alpha \leq 0\&\&\alpha0 < 0)\|$
$(\alpha > 0\&\&\alpha + \alpha0 < 0))\&\&(\beta == 0\&\&\beta0 == 0))\|$
$(((\alpha \leq 0\&\&\alpha0 < 0)\|(\alpha > 0\&\&\alpha + \alpha0 < 0))\&\&(\beta > 0\&\&\beta0 == 0))\|$
$(((\alpha \leq 0\&\&\alpha0 < 0)\|(\alpha > 0\&\&\alpha + \alpha0 < 0))\&\&(\beta < 0\&\&\beta0 == 0))\|$
$((\alpha > 0\&\&\alpha + \alpha0 == 0)\&\&(\beta < 0\&\&\beta0 == 0))\|$
$((\alpha < 0\&\&\alpha0 == 0)\&\&(\beta > 0\&\&\beta0 == 0))$ },

(*13*) { {Point[{0,1}]},
$(((\alpha \leq 0\&\&\alpha0 < 0)\|(\alpha > 0\&\&\alpha + \alpha0 < 0))\&\&((\beta \geq 0\&\&\beta0 > 0)\|$
$(\beta < 0\&\&\beta + \beta0 > 0)))\|(((\alpha \leq 0\&\&\alpha0 < 0)\|$
$(\alpha > 0\&\&\alpha + \alpha0 < 0))\&\&(\beta < 0\&\&\beta0 > 0\&\&\beta + \beta0 < 0))\|$
$(((\alpha \leq 0\&\&\alpha0 < 0)\|(\alpha > 0\&\&\alpha + \alpha0 < 0))\&\&(\beta < 0\&\&\beta + \beta0 == 0))\|$
$((\alpha < 0\&\&\alpha0 == 0)\&\&((\beta \geq 0\&\&\beta0 > 0)\|(\beta < 0\&\&\beta + \beta0 > 0)))\|$
$((\alpha < 0\&\&\alpha0 > 0\&\&\alpha + \alpha0 < 0)\&\&((\beta \geq 0\&\&\beta0 > 0)\|$
$(\beta < 0\&\&\beta + \beta0 > 0)))$ },

(*14*) { {Point[{0,0}]},
$(((\alpha \leq 0\&\&\alpha0 < 0)\|$
$(\alpha > 0\&\&\alpha + \alpha0 < 0))\&\&(\beta > 0\&\&\beta + \beta0 == 0))\|$
$(((\alpha \leq 0\&\&\alpha0 < 0)\|(\alpha > 0\&\&\alpha + \alpha0 < 0))\&\&((\beta \leq 0\&\&\beta0 < 0)\|$
$(\beta > 0\&\&\beta + \beta0 < 0)))\|(((\alpha \leq 0\&\&\alpha0 < 0)\|$
$(\alpha > 0\&\&\alpha + \alpha0 < 0))\&\&(\beta > 0\&\&\beta0 < 0\&\&\beta + \beta0 > 0))\|$
$((\alpha > 0\&\&\alpha0 < 0\&\&\alpha + \alpha0 > 0)\&\&((\beta \leq 0\&\&\beta0 < 0)\|$
$(\beta > 0\&\&\beta + \beta0 < 0)))\|((\alpha > 0\&\&\alpha + \alpha0 == 0)\&\&((\beta \leq 0\&\&\beta0 < 0)\|$
$(\beta > 0\&\&\beta + \beta0 < 0)))$ },

(*15*) { {Point[{{0,0},{1,1}}]},
$((\alpha > 0\&\&\alpha0 == 0)\&\&(\beta > 0\&\&\beta0 == 0))\|$
$((\alpha > 0\&\&\alpha + \alpha0 == 0)\&\&(\beta > 0\&\&\beta + \beta0 == 0))$ },

(*16*) { {Point[{{0,0},$\{-\frac{\beta0}{\beta}, 0\}, \{1, 1\}\}$], $Line[\{\{0, 0\}, \{-\frac{\beta0}{\beta}, 0\}\}]$},
$(\alpha > 0\&\&\alpha0 == 0)\&\&(\beta > 0\&\&\beta0 < 0\&\&\beta + \beta0 > 0)$ },

(*17*) { {Point[$\{\{-\frac{\beta0}{\beta}, 0\}, \{1, 0\}\}$], Line[$\{\{-\frac{\beta0}{\beta}, 0\}, \{1, 0\}\}$]},
$(\alpha > 0\&\&\alpha0 == 0)\&\&(\beta < 0\&\&\beta0 > 0\&\&\beta + \beta0 < 0)$ },

(*18*) { {Point[g1[$\alpha, \alpha0$][[1]]], g1[$\alpha, \alpha0$]},

$((\alpha > 0\&\&\alpha0 < 0\&\&\alpha + \alpha0 > 0)\&\&(\beta == 0\&\&\beta0 == 0))\|$
$((\alpha < 0\&\&\alpha0 > 0\&\&\alpha + \alpha0 < 0)\&\&(\beta == 0\&\&\beta0 == 0))$ },

(*19*) { {Point[{{0,0},{0,$-\frac{\alpha0}{\alpha}$},{1,1}}], Line[{{0,0},{0,$-\frac{\alpha0}{\alpha}$}}]},
$(\alpha > 0\&\&\alpha0 < 0\&\&\alpha + \alpha0 > 0)\&\&(\beta > 0\&\&\beta0 == 0)$ },

(*20*) { {Point[{{0,0},{$-\frac{\beta0}{\beta}, -\frac{\alpha0}{\alpha}$},{1,1}}]},
$(\alpha > 0\&\&\alpha0 < 0\&\&\alpha + \alpha0 > 0)\&\&$
$(\beta > 0\&\&\beta0 < 0\&\&\beta + \beta0 > 0)$ },

(*21*) { {Point[{{0,0},{1,$-\frac{\alpha0}{\alpha}$},{1,1}}], Line[{{1,$-\frac{\alpha0}{\alpha}$},{1,1}}]},
$(\alpha > 0\&\&\alpha0 < 0\&\&\alpha + \alpha0 > 0)\&\&(\beta > 0\&\&\beta + \beta0 == 0)$ },

(*22*) { {Point[{{0,0},{0,$-\frac{\alpha0}{\alpha}$}}], Line[{{0,0},{0,$-\frac{\alpha0}{\alpha}$}}]},
$(\alpha > 0\&\&\alpha0 < 0\&\&\alpha + \alpha0 > 0)\&\&(\beta < 0\&\&\beta0 == 0)$ },

(*23*) { {Point[{{$-\frac{\beta0}{\beta}, -\frac{\alpha0}{\alpha}$}}]},
$((\alpha > 0\&\&\alpha0 < 0\&\&\alpha + \alpha0 > 0)\&\&$
$(\beta < 0\&\&\beta0 > 0\&\&\beta + \beta0 < 0))\|$
$((\alpha < 0\&\&\alpha0 > 0\&\&\alpha + \alpha0 < 0)\&\&$
$(\beta > 0\&\&\beta0 < 0\&\&\beta + \beta0 > 0))$ },

(*24*) { {Point[{{1,-$\frac{\alpha0}{\alpha}$},{1,1}}], Line[{{1,$-\frac{\alpha0}{\alpha}$},{1,1}}]},
$(\alpha > 0\&\&\alpha0 < 0\&\&\alpha + \alpha0 > 0)\&\&(\beta < 0\&\&\beta + \beta0 == 0)$ },

(*25*) { {Point[{{0,0},{$-\frac{\beta0}{\beta}$,1},{1,1}}], Line[{{$-\frac{\beta0}{\beta}$,1},{1,1}}]},
$(\alpha > 0\&\&\alpha + \alpha0 == 0)\&\&(\beta > 0\&\&\beta0 < 0\&\&\beta + \beta0 > 0)$ },

(*26*) { {Point[{{0,1},{$-\frac{\beta0}{\beta}$,1}}], Line[{{0,1},{$-\frac{\beta0}{\beta}$,1}}]},
$(\alpha > 0\&\&\alpha + \alpha0 == 0)\&\&(\beta < 0\&\&\beta0 > 0\&\&\beta + \beta0 < 0)$ },

(*27*) { {Point[{{0,0},{$-\frac{\beta0}{\beta}$,0}}], Line[{{0,0},{$-\frac{\beta0}{\beta}$,0}}]},
$(\alpha < 0\&\&\alpha0 == 0)\&\&(\beta > 0\&\&\beta0 < 0\&\&\beta + \beta0 > 0)$ },

(*28*) { {Point[{{$-\frac{\beta0}{\beta}$,0}, {1,0},{0,1}}], Line[{{$-\frac{\beta0}{\beta}$,0}, {1,0}}]},
$(\alpha < 0\&\&\alpha0 == 0)\&\&(\beta < 0\&\&\beta0 > 0\&\&\beta + \beta0 < 0)$ },

(*29*) { {Point[{{1,0},{0,1}}]},
$((\alpha < 0\&\&\alpha0 == 0)\&\&(\beta < 0\&\&\beta + \beta0 == 0))\|$
$((\alpha < 0\&\&\alpha + \alpha0 == 0)\&\&(\beta < 0\&\&\beta0 == 0))$ },

(*30*) { {Point[{{0,$-\frac{\alpha0}{\alpha}$},{0,1}}], Line[{{0,$-\frac{\alpha0}{\alpha}$},{0,1}}]},

$(\alpha < 0 \&\& \alpha 0 > 0 \&\& \alpha + \alpha 0 < 0) \&\& (\beta > 0 \&\& \beta 0 == 0)$ },

(*31*) { {Point[{{1,$-\frac{\alpha 0}{\alpha}$},{1,0}}], Line[{{1,$-\frac{\alpha 0}{\alpha}$},{1,0}}]},
$(\alpha < 0 \&\& \alpha 0 > 0 \&\& \alpha + \alpha 0 < 0) \&\& (\beta > 0 \&\& \beta + \beta 0 == 0)$ },

(*32*) { {Point[{{0,$-\frac{\alpha 0}{\alpha}$},{0,1},{1,0} }], Line[{{0,$-\frac{\alpha 0}{\alpha}$},{0,1}}]},
$(\alpha < 0 \&\& \alpha 0 > 0 \&\& \alpha + \alpha 0 < 0) \&\& (\beta < 0 \&\& \beta 0 == 0)$ },

(*33*) { {Point[{{0,1},{$-\frac{\beta 0}{\beta}, -\frac{\alpha 0}{\alpha}$},{1,0}}]},
$(\alpha < 0 \&\& \alpha 0 > 0 \&\& \alpha + \alpha 0 < 0) \&\&$
$(\beta < 0 \&\& \beta 0 > 0 \&\& \beta + \beta 0 < 0)$ },

(*34*) { {Point[{{1,$-\frac{\alpha 0}{\alpha}$},{1,0},{0,1}}], Line[{{1,$-\frac{\alpha 0}{\alpha}$},{1,0}}]},
$(\alpha < 0 \&\& \alpha 0 > 0 \&\& \alpha + \alpha 0 < 0) \&\& (\beta < 0 \&\& \beta + \beta 0 == 0)$ },

(*35*) { {Point[{{$-\frac{\beta 0}{\beta}$,1},{1,1}}], Line[{{$-\frac{\beta 0}{\beta}$,1},{1,1}}]},
$(\alpha < 0 \&\& \alpha + \alpha 0 == 0) \&\& (\beta > 0 \&\& \beta 0 < 0 \&\& \beta + \beta 0 > 0)$ },

(*36*) { {Point[{{0,1},{$-\frac{\beta 0}{\beta}$,1},{1,0}}], Line[{{0,1},{$-\frac{\beta 0}{\beta}$,1}}]},
$(\alpha < 0 \&\& \alpha + \alpha 0 == 0) \&\& (\beta < 0 \&\& \beta 0 > 0 \&\& \beta + \beta 0 < 0)$ }

}]
Manipulate[
Grid[{{Graphics[{Thick,
Blue,g1[a11-a21+a22-a12,a22-a12],
Green,g2[b11-b12+b22-b21,b22-b21],
Red,PointSize[Large],
nes[a11-a21+a22-a12,a22-a12,b11-b12+b22-b21,b22-b21]},
PlotRange → {{0,1},{0,1}},Axes → True,AxesLabel → {"x_1","y_1"},
ImageSize → {400,400}]},{" "},
{Text@Style["Reference Nash Equilibria",Bold]},
{Text@Style[nes[a11-a21+a22-a12,a22-a12,
b11-b12+b22-b21,b22-b21][[1,1]],
Bold]}},ItemSize → {Automatic,{10,1,1,3}},
Alignment → {Center,Top}], Style["Matrix A",Bold],
{{a11,10,"a_{11}"},-10,10,1,Appearance → "Labeled",ImageSize → Tiny},
{{a12, 1,"a_{12}"},-10,10,1,Appearance → "Labeled",ImageSize → Tiny},
{{a21,-2,"a_{21}"},-10,10,1,Appearance → "Labeled",ImageSize → Tiny},
{{a22,-4,"a_{22}"},-10,10,1,Appearance → "Labeled",ImageSize → Tiny},

Delimiter,{{NonAntagonistic,True},{True,False}},
Delimiter,Style["Matrix B",Bold],
{{b11, 4,"b_{11}"},-10,10,1,Enabled → NonAntagonistic,
Appearance → "Labeled",ImageSize → Tiny},
{{b12,-3,"b_{12}"},-10,10,1,Enabled → NonAntagonistic,
Appearance → "Labeled",ImageSize → Tiny},
{{b21,-1,"b_{21}"},-10,10,1,Enabled → NonAntagonistic,
Appearance → "Labeled",ImageSize → Tiny},
{{b22,-4,"b_{22}"},-10,10,1,Enabled → NonAntagonistic,
Appearance → "Labeled",ImageSize → Tiny},
Delimiter, Style["Matrices A and B",Bold],
Dynamic[TableForm[{{ToString[a11]<>" , "<>
ToString[If[NonAntagonistic,b11,b11=-a11]],ToString[a12]<>" , "<>
ToString[If[NonAntagonistic,b11,b12=-a12]]},{ToString[a21]<>" , "<>
ToString[If[NonAntagonistic,b21,b21=-a21]],ToString[a22]<>" , "<>
ToString[If[NonAntagonistic,b22,b22=-a22]]}},
TableHeadings → {{"1","2"},{" 1"," 2"}}, TableSpacing →{2,2}]],
SaveDefinitions → True]

As the enumerated 36 cases include all the 81 abstractly possible cases of the results of graph intersections, the proof is complete. □

Even though we establish the 36 different forms/cases needed to define exhaustively the Nash equilibrium set function, we can summarize them and can conclude additionally that a Nash equilibrium set may be:

1. **a border point** as one of the vertices of the square (cases: 10, 11, 13, 14),

2. **an interior point** of the square (cases: 5, 23),

3. **two border points** as two opposite vertices of the square (cases: 15, 29),

4. **a unit border segment** as one of the sides of the square (cases: 2, 3, 9, 12),

5. **two unit border segments** as two connected sides of the square (one vertical and one horizontal) (cases: 4, 6, 7, 8),

6. **a union of one point and one non-unit segment** as one vertex of the square and one non-unit segment on opposite side of the square (cases: 16, 19, 21, 25, 28, 32, 34, 36),

7. **one non-unit segment** as a segment on one of the sides of the square (cases: 17, 22, 24, 26, 27, 30, 31, 35),

8. **a graph** of one of the players as a union of three connected segments (case 18),

9. **three distinct points** as two corner opposite vertices of the square and one interior point (cases: 20, 33),

10. **a unit square** (case 1).

Corollary 4.1. *Nash equilibrium set in dyadic mixed strategy games may be formed by*

1. *a point,*

2. *two points,*

3. *three points,*

4. *a segment,*

5. *two connected segments,*

6. *three connected segments,*

7. *union of non-connected one point and one segment,*

8. *the unit square.*

5 Conclusions

For the dyadic mixed strategy games we developed an analytic method for Nash equilibrium set computing as the value of the NES function. The function is defined by a formula in the proof of theorem 4.1 applying a Wolfram language function definition. The corollary summarizes all the results in a simple and useful statement.

A preceding version of the proposed approach/algorithm was realized in the Wolfram language too. It was published in the Wolfram Demonstrations Project [15]. That code may be freely viewed, verified and downloaded from the address [15].

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