Computer Science Journal of Moldova, vol.24, no.2(71), 2016

Numerical solutions of Kendall and Pollaczek-Khintchin equations for exhaustive polling systems with semi-Markov delays

Gheorghe Mishkoy, Diana Bejenari, Lilia Mitev, Ionela R. Ticu

Abstract

Some analytical results for exhaustive polling systems with semi-Markov delays, such as Pollaczek-Khintchin virtual and steady state analog are presented. Numerical solutions for k-busy period, probability of states and queue length distribution are obtained. Numerical examples are presented.

Keywords: Polling systems with semi-Markov delays, Pollaczek-Khintchin formula, Kendall equation, k-busy period, probability of states, queue length, numerical algorithms.

1 Introduction

It is known that wireless networks have developed rapidly last years. For planning regional wireless networks, models and research methods of polling systems are used [1]. A polling model is a system of multiple queues accessed by a single server in a given order. Among important characteristics of these systems are the k-busy period, probability of states and queueing length [2]. We consider a queueing system of polling type with semi-Markov delays. Handling mechanism for this system is given by polling table $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, r\}$, where the function shows that at the stage j, $j = \overline{1, n}$, user number k, $k = \overline{1, r}$, $r \leq n$ is served (more details see in [1]). The items (messages) of the user k, according to Poisson distribution with parameter λ_k arrive. The service time for the items of class k is a random variable B_k with distribution function $B_k(x) = P\{B_k < x\}$.

Duration of the orientation from one user to user k is a random variable C_k with distribution function $C_k(x) = P\{C_k < x\}$. Thus C_k can be interpreted as a loss of time in preparing the service process for user of class k. The main purpose of research of polling systems is to determine the characteristics of systems development. But not always analytical formulas can be used directly, so great care is offered for numerical algorithms. In this paper, using the methodology of generalized priority systems and generalized algorithms elaborated in [3], for this characteristics numerical solutions with necessary required accuracy are obtained. Some examples and numerical results are presented.

2 The *k*-busy period

Definition 1. *The k-busy period is a measure of the time that expires from when a server begins to process, after an empty queue, to when the k-queue becomes empty again for the first time [3].*

Denote by Π_k^δ the length of the k-busy period, and by $\Pi_k^\delta(x) = P\{\Pi_k^\delta < x\}$, -its distribution function. Let consider that $\pi_k^\delta(s) = \int\limits_0^\infty e^{-st} d\Pi_k^\delta(x)$ is the Laplace-Stieltjes transform of distribution function of k-busy period. The proof of the presented below analytical results (Theorem 1-4) can be obtained using method of catastrophes [3].

Theorem 1. *Function $\pi_k^\delta(s)$ is determined from equation*

$$\pi_k^\delta(s) = c_k(s + \lambda_k - \lambda_k \pi_k(s))\pi_k(s), \tag{2.1}$$

where

$$\pi_k(s) = \beta_k(s + \lambda_k - \lambda_k \pi_k(s)), \tag{2.2}$$

and with $c_k(s)$ and $\beta_k(s)$ denoting the Laplace-Stieltjes transforms of distribution functions $C_k(x)$ and $B_k(x)$,

$$c_k(s) = \int\limits_0^\infty e^{-sx} dC_k(x), \ \beta_k(s) = \int\limits_0^\infty e^{-sx} dB_k(x).$$

Remark 2.1 *If $\lambda_k \beta_{k1} < 1$, $\lambda_k c_{k1} < 1$, then first moment of k-busy period is determined from:*

$$\pi_{k1}^{\delta} = \frac{\beta_{k1}}{1 - \lambda_k \beta_{k1}} + \frac{c_{k1}}{1 - \lambda_k c_{k1}}, \tag{2.3}$$

where β_{k1} and c_{k1} are the first moments of $C_k(x)$ and $B_k(x)$.

Remark 2.2 *If we consider that $C_k = 0$ and $k = 1$, then from formula (2.1) it follows that $\pi_k^{\delta}(s) = \pi_k(s)$ and $\pi_1^{\delta} = \pi_1(s) = = \beta_1(s + \lambda - \lambda\pi_1(s))$, respectively.*

Thus, expression (2.1) can be viewed as an analog of Kendall equation obtained for classical $M|G|1$ system [4].

3 Probability of states

Probability of states have a decisive role in determining important parameters such as, for example, blocking probabilities used in network technologies equipped with QoS (Quality of Service) and CoS (Class of Service). At arbitrary time $x \in (0, \infty)$ system can be in one and only one of the following states: or in serving state, or in the state of changing the states, or is free. If the polling system is in serving state, question arises: which user's message is served in that time? If the system is in the state of changing, to which class of messages the exchange occurs?

Denote $P_{B_k}(x), P_{C_k}(x)$ and $P_0(x)$ the probabilities that at instant x the system is busy by service of k-messages, switching to k-messages and system is free, respectively.

Theorem 2. *The Laplace-Stieltjes transforms of $P_{B_k}(x)$, $P_{C_k}(x)$ and $P_0(x)$ are determined from*

$$p_{B_k}(s) = \frac{\lambda_k[1 - \pi_k(s)]}{s[s + \lambda_k - \lambda_k \pi_k^{\delta}(s)]}, \tag{3.1}$$

$$p_{C_k}(s) = \frac{\lambda_k[1 - c_k(s)]}{s[s + \lambda_k - \lambda_k \pi_k^{\delta}(s)]}, \tag{3.2}$$

$$p_0(s) = \frac{1}{s} - [p_{B_k}(s) + p_{C_k}(s)]), \tag{3.3}$$

where $\pi_k^\delta(s)$ *and* $\pi_k(s)$ *is determined from (2.1), (2.2).*

Denote $P_{B_k} = \lim_{x\to\infty} P_{B_k}(x)$, $P_{C_k} = \lim_{x\to\infty} P_{C_k}(x)$, $P_0 = \lim_{x\to\infty} P_0(x)$.

Remark 3.1 *If* $\lambda_k\beta_{k1} < 1$, $\lambda_k c_{k1} < 1$, *then*

$$P_{B_k} = \lim_{s\downarrow 0} sp_{B_k}(s), \tag{3.4}$$

$$P_{C_k} = \lim_{s\downarrow 0} sp_{C_k}(s), \tag{3.5}$$

$$P_0 = \lim_{s\downarrow 0} sp_0(s), \tag{3.6}$$

and

$$P_{B_k} = \frac{\lambda_k \pi_{k1}}{1 + \lambda_k \pi_{k1}^\delta}, \tag{3.7}$$

$$P_{C_k} = \frac{\lambda_k c_{k1}}{1 + \lambda_k \pi_{k1}^\delta}, \tag{3.8}$$

$$P_0 = 1 - \frac{\lambda_k (c_{k1} + \pi_{k1})}{1 + \lambda_k \pi_{k1}^\delta}. \tag{3.9}$$

4 The Pollaczek-Khintchin virtual analog

Let $P_m(x)$ be the probability that at the instant x there are m messages in the k-queue. Denote

$$P_k(z,x) = \sum_{m=1}^{\infty} P_m(x) z^m, 0 \le z \le 1,$$

the generating function of queue length distribution and

$$p_k(z,s) = \int_0^\infty e^{-sx} P_k(z,x)dx,$$

the Laplace transform of function $P_k(z,s)$.

Theorem 3.

$$p_k(z,s) = \frac{1+\lambda_k \pi_k^\delta(z,s)}{s+\lambda_k-\lambda_k z}, \tag{4.1}$$

$$\pi_k^\delta(z,s) = \frac{1-c_k(s+\lambda_k-\lambda_k z)}{s+\lambda_k-\lambda_k z} + \frac{\beta_k(z,s)}{z-\beta_k(s+\lambda_k-\lambda_k z)}\times$$

$$\times[zc_k(s+\lambda_k-\lambda_k z)-\pi_k^\delta(s)], \tag{4.2}$$

$$\beta_k(z,s) = \frac{1-\beta_k(s+\lambda_k-\lambda_k z)}{s+\lambda_k-\lambda_k z}. \tag{4.3}$$

Remark 4.1 *In the next section it is shown that from the Theorem 3 it follows Theorem 4, where formula (5.2) results from. Thus, the result from Theorem 3 can be viewed as a virtual analogue of the Pollaczek-Khintchin equation.*

5 The Pollaczek-Khintchin steady state analog

Theorem 4. *If* $\lambda_k\beta_{k1} < 1$, $\lambda_k c_{k1} < 1$, *then*

$$P_k(z) = \lim_{s\downarrow 0} sp_k(z,s),$$

and

$$P_k(z) = \frac{1+\lambda_k\pi_k^\delta(z,0)}{1+\lambda_k\pi_{k1}^\delta}. \tag{5.1}$$

Remark 5.1 *If* $C_k = 0$ *and* $k = 1$, *then*

$$P_{k1}(z) = P_k(z) = \frac{\beta(\lambda-\lambda z)(z-1)(1-\lambda\beta_1)}{z-\beta(\lambda-\lambda z)}, \tag{5.2}$$

where $\beta_1(\cdot) = \beta(\cdot)$ *and* $\beta_{11} = \beta_1$.

Formula (5.2) is referred to in most text-books on queueing analysis and it is known as the Pollaczek-Khintchin transform equation (Pollaczek (1961); Khintchin (1963)).

6 The first moment of virtual distribution

Let $L_k(x)$ be the expectation of non stationary (virtual) distribution of k-queue length

$$l_k(s) = \int_0^\infty e^{-sx} L_k(x)dx.$$

It is easy to see that $-l_k'(s)|_{s=0} = L_k(x)$. Thus, after deriving by s the expression from Theorem 3, changing the sign and placing $s = 0$, it is obtained the analytical expression for $l_k(s)$.

Remark 6.1.

$$l_k(s) = \frac{\lambda_k}{s + \lambda_k - \lambda_k \pi_k^\delta(s)} \times$$

$$\times \{ \frac{c_{k1}\lambda_k s + \lambda_k(1 - c_k(s))}{s^2} + \frac{[1 - \beta_k(s)(s - \lambda_k) + \lambda_k s \beta_{k1}](c_k(s) - \pi_k^\delta(s))}{s^2(1 - \beta_k(s))} -$$

$$- \frac{(1 + \lambda_k \beta_{k1})(c_k(s) - \pi_k^\delta(s))}{s(1 - \beta_k(s))^2} \}. \tag{6.1}$$

7 Numerical algorithms

The analytical results formulated above, although they are of interest from fundamental theoretical point of view, are quite complicated for numerical modelling. Indeed, for example, $\pi_k^\delta(s)$ given by the Theorem 1 occurs in most of the following expressions (Theorem 2, 3, etc.). But for determining this function it is necessary to solve the functional equation (2.2), which does not have the exact analytical solution, but which effectively can be solved numerically [3, 5]. As an example, we will present a numerical algorithm of successive approximations.

Algorithm 1.

Input: $\{\lambda_k\}_{k=1}^r; \{b_k\}_{k=1}^r; \{c_k\}_{k=1}^r; s; r; \varepsilon > 0.$
Output: $k; \{\pi_k(s)\}_{k=1}^r; \{\pi_k^\delta(s)\}_{k=1}^r.$
Descriptions:
1. Laplace-Stieltjes transforms of exponential distribution functions $B_k(x)$ and $C_k(x)$, are determined:
$\beta_k(s) = \frac{b_k}{s+b_k}$; $\overline{c}_k = \frac{c_k}{s+c_k}$.

2. Distribution function for k-busy period is determined, using Theorem 1, for $k = 1, ..., r$. For $n = 0$,
$\pi_k^{(0)}(s) = 0$, $\pi_k{}^{(n)}(s) = \beta_k(s + \lambda_k - \lambda_k \pi_k{}^{(n-1)}(s))$,
$\pi_k^{\delta(n)}(s) = \overline{c}_k(s + \lambda_k - \lambda_k \pi_k{}^{(n)}(s))\pi_k{}^{(n)}(s)$.

Stop condition: $|\pi_k^{(n)}(s) - \pi_k^{(n-1)(s)}| < \varepsilon$.

We will further mention that the presented examples from the Section 8 are not suggested from any concrete practical examples. Their mission is to demonstrate the efficiency of the algorithms and that the elaborated algorithms are invariant to the type of the distribution function (it does not depend on concrete expression of the distribution function). We will also observe that the numerical results obtained, presented in Tables, are in full accordance and do not contradict the analytical results.

8 Numerical examples

In this section, numerical results are presented. The results of each example are obtained from the analogous algorithm, which is presented above.

- ***k*-busy period**
  Example 8.1 The type of distribution function taken by $B_k(x)$ and $C_k(x)$ is the *Exponential distribution*, so

$$B_k(x) = 1 - e^{b_k x}, x > 0 \text{ and } C_k(x) = 1 - e^{c_k x}, x > 0,$$

with the following parameters:

λ_k ={2, 3, 5, 6, 3, 8, 5, 4, 7, 3, 4, 2, 9, 7, 6, 4, 6, 3, 5, 2},

b_k ={7, 5, 9, 4, 6, 2, 5, 4, 8, 6, 5, 3, 4, 5, 7, 6, 2, 4, 8, 6},

c_k ={7, 5, 9, 4, 6, 2, 5, 4, 8, 6, 5, 3, 4, 5, 7, 6, 2, 4, 8, 6},

s =0.2.

The results of the program are presented in Table 1.

Table 1. Numerical results for LST of distribution function for k-busy period $\pi_k^\delta(s)$

k	$\pi_k(s)$	$\pi_k^\delta(s)$	k	$\pi_k(s)$	$\pi_k^\delta(s)$
1	0.499831	0.181751	11	0.369369	0.096169
2	0.389415	0.151739	12	0.287740	0.096169
3	0.515323	0.098878	13	0.236751	0.075282
4	0.278846	0.121721	14	0.316054	0.085492
5	0.438095	0.149959	15	0.424283	0.180499
6	0.133333	0.042254	16	0.418269	0.110781
7	0.350425	0.122974	17	0.152174	0.047297
8	0.313589	0.058366	18	0.333156	0.155468
9	0.445795	0.199467	19	0.480989	0.197734
10	0.438095	0.172673	20	0.458472	0.165468

Example 8.2 The type of distribution function taken by $B_k(x)$ is the *Erlang distribution* and by $C_k(x)$ is the *Exponential distribution*, so

$$B_k(x) = \begin{cases} 0, & x < 0 \\ \int\limits_0^x \lambda_k \frac{(\lambda_k u)^{k-1}}{(k-1)!} e^{-\lambda_k u} du, & x \geq 0 \end{cases}$$

and

$$C_k(x) = 1 - e^{c_k x}, x > 0,$$

with the following parameters:

λ_k ={3, 5, 7, 6, 8, 4, 5, 7, 3, 6, 8, 5, 9, 7, 6, 2, 4, 1, 1, 1},

b_k ={4, 4, 6, 7, 4, 8, 8, 3, 2, 5, 5, 7, 6, 4, 8, 6, 4, 2, 4, 6},

c_k ={6, 7, 4, 3, 7, 6, 9, 7, 4, 5, 3, 2, 5, 6, 7, 5, 4, 3, 2, 8},

p_k ={7, 5, 4, 3, 6, 5, 2, 5, 5, 7, 6, 5, 9, 8, 6, 5, 4, 6, 7, 6},

s =0.2.

The results of the program are presented in Table 2.

Table 2. Numerical results for LST of distribution function for k-busy period $\pi_k^\delta(s)$

k	$\pi_k(s)$	$\pi_k^\delta(s)$	k	$\pi_k(s)$	$\pi_k^\delta(s)$
1	0.000155	5.828645e-05	11	0.000244	4.069452e-05
2	0.000977	0.000360	12	0.006788	0.000972
3	0.008100	0.001806	13	8.347513e-06	1.98751e-06
4	0.044573	0.008499	14	5.947027e-06	1.784112e-06
5	8.706395e-05	2.770304e-05	15	0.424283	0.001071
6	0.013420	0.004751	16	0.010310	0.003688
7	0.174848	0.078190	17	0.0050	0.001350
8	0.000171	5.705125e-05	18	6.4e-05	1.745465e-05
9	0.000129	3.67441e-05	19	0.000457	9.145366e-05
10	0.000128	3.544855e-05	20	0.006196	0.003099

Example 8.3 The type of distribution function taken by $B_k(x)$ is the *Erlang distribution* and by $C_k(x)$ is *Normal distribution*, so

$$B_k(x) = \begin{cases} 0, & x < 0 \\ \int\limits_0^x \lambda_k \frac{(\lambda_k u)^{k-1}}{(k-1)!} e^{-\lambda_k u} du, & x \geq 0 \end{cases}$$

and

$$C_k(x) = \frac{1}{\sigma_k}\sqrt{2\pi} \int\limits_{-\infty}^{x} e^{-\frac{(u-a)^2}{2\sigma_k{}^2}} du, \; -\infty < x < \infty,$$

with the following parameters:

λ_k ={0.6, 0.4, 0.6, 0.7, 0.3, 0.4, 0.6, 0.2, 0.6, 0.4, 0.7, 0.1, 0.3, 0.5, 0.7, 0.5},

b_k ={0.2, 0.4, 0.7, 0.4, 0.6, 0.8, 0.6, 0.3, 0.4, 0.5, 0.3, 0.2, 0.5, 0.4, 0.6, 0.7},

c_k ={0.5, 0.3, 0.5, 0.7, 0.5, 0.3, 0.4, 0.8, 0.5, 0.4, 0.2, 0.3, 0.4, 0.5, 0.3, 0.5},

p_k ={2, 4, 3, 5, 4, 6, 5, 4, 3, 2, 4, 6, 5, 4, 7, 6},

σ_k ={2, 4, 3, 5, 4, 6, 5, 4, 3, 2, 4, 6, 5, 4, 7, 6}, s =0.2.

The results are presented in the Table 3.

Table 3. Numerical results for LST of distribution function for k-busy period $\pi_k^\delta(s)$

k	$\pi_k(s)$	$\pi_k^\delta(s)$	k	$\pi_k(s)$	$\pi_k^\delta(s)$
1	0.028566	0.019760	9	0.024037	0.015351
2	0.012551	0.011409	10	0.163897	0.154544
3	0.075241	0.058980	11	0.002108	0.001942
4	0.001348	0.000656	12	0.000544	0.000473
5	0.047328	0.034053	13	0.012758	0.010210
6	0.015996	0.015313	14	0.008963	0.007629
7	0.007416	0.00564	15	0.000682	0.000563
8	0.012482	0.008618	16	0.007012	0.005264

Example 8.4 The types of distribution function for $B_k(x)$ and $C_k(x)$ are given in the Table 5. We used the following notations:

– If the distribution function is *Uniform* we denote it by the letter **U**,

– If the distribution function is *Erlang* – **I**,

– If the distribution function is *Exponential* – **E**,

– If the distribution function is *Normal* – **N**.

λ_k ={0.2, 0.4, 0.7, 0.5, 0.6, 0.3, 0.4, 0.5, 0.6, 0.9},

s =0.2.

Required parameters for each distribution function are given in the Table 4.

Table 4. Parameters for distribution functions

U	N	U	I	N
$a = 4$ $b = 1$	$\sigma = 0.6$ $a = 0.2$	$a = 5$ $b = 2$	$k = 2$	$\sigma = 0.9$ $a = 0.2$
U	I	N	I	U
$a = 10$	$k = 3$	$\sigma = 0.4$ $b = 2$	$k = 4$ $a = 0.1$	$a = 14$ $b = 7$
I	U	I	U	I
$k = 1$	$a = 9$ $b = 2$	$k = 2$	$a = 3$	$k = 2$ $b = 1$

The results of the program are given in the Table 5.

Table 5. Numerical results for LST of distribution function for k-busy period $\pi_k^\delta(s)$

k	$B_k(x)$	$C_k(x)$	$\pi_k(s)$	$\pi_k^\delta(s)$
1	U	N	0.367026	0.363908
2	E	U	0.846724	0.250955
3	I	N	0.599745	0.675492
4	E	U	0.894586	0.147271
5	I	N	0.608076	0.601602
6	I	U	0.68603	0.014749
7	E	I	0.704387	0.496661
8	E	U	0.920868	0.178205
9	I	E	0.624615	0.565239
10	U	I	0.139467	0.059018

For the case when $B_k(x)$ and $C_k(x)$ are PH distribution functions, it is necessary to obtain the matrix form of Kendall and generalized Kendall equations. These analitical results are obtained in [5].

Example 8.5 The types of distribution functions taken by $B_k(x)$ and $C_k(x)$ are PH distributions with representation (α^t, T_k), (α^t, P_k), so

$$B_k(x) = 1 - \alpha_t e^{T_k x} e, x > 0,$$

$$C_k(x) = 1 - \alpha_t e^{P_k x} e, x > 0,$$

with the following parameters:

$\lambda_k = \{0.4; 0.4; 0.3; 0.6; 0.5\}$,

$\tilde{\lambda_k} = \{0.4; 0.2; 0.4; 0.6; 0.6\}$,

$\delta_k = \{0.6; 0.3; 0.4; 0.5; 0.2\}$,

$s = 0.2$.

The results of the program are presented in Table 6.

Table 6. Numerical results for LST of distribution function for k-busy period $\pi_k^{\delta}(s)$

k	$\pi_k(s)$	$\pi_k^{\delta}(s)$
1	0.026725	0.901550
2	0.066859	0.718482
3	0.012627	0.928276
4	0.035894	0.804515
5	0.022830	0.709058

Example 8.6 The types of distribution functions taken by $B_k(x)$ and $C_k(x)$ are PH distributions with representation (α^t, T_k), (α^t, P_k), so

$$B_k(x) = 1 - \alpha_t e^{T_k x} e, x > 0,$$

$$C_k(x) = 1 - \alpha_t e^{P_k x} e, x > 0,$$

with the following parameters:

$\lambda_k = \{0.5; 0.6; 0.3; 0.4; 0.5; 0.2; 0.6; 0.6; 0.2; 0.1\}$,

$\tilde{\lambda_k} = \{0.2; 0.3; 0.4; 0.2; 0.6; 0.7; 0.8; 0.4; 0.2; 0.3\}$,

$\delta_k = \{0.3; 0.4; 0.1; 0.2; 0.6; 0.8; 0.5; 0.4; 0.4; 0.8\}$,

$s = 0.5$.

The results of the program are presented in Table 7.

Table 7. Numerical results for LST of distribution function for k-busy period $\pi_k^\delta(s)$

k	$\pi_k(s)$	$\pi_k^\delta(s)$	k	$\pi_k(s)$	$\pi_k^\delta(s)$
1	0.012692	0.864672	6	0.000060	0.999554
2	0.014688	0.865907	7	0.003161	0.959972
3	0.000978	0.957140	8	0.010383	0.891422
4	0.006395	0.895401	9	0.000542	0.995270
5	0.002997	0.973018	10	0.000017	0.999915

Example 8.7 The types of distribution functions taken by $B_k(x)$ and $C_k(x)$ are PH distributions with representation (α^t, T_k), (α^t, P_k), so

$$B_k(x) = 1 - \alpha_t e^{T_k x} e, x > 0,$$

$$C_k(x) = 1 - \alpha_t e^{P_k x} e, x > 0,$$

with the following parameters:

$\lambda_k = \{0.2; 0.3; 0.1; 0.5; 0.6; 0.7; 0.4; 0.8; 0.4; 0.5; 0.3; 0.7; 0.8; 0.4;$
$0.6; 0.9; 0.3; 0.4; 0.5; 0.4\}$,

$\tilde{\lambda_k} = \{0.2; 0.3; 0.5; 0.2; 0.3; 0.7; 0.8; 0.4; 0.3; 0.5; 0.1; 0.5; 0.8; 0.4;$
$0.3; 0.6; 0.4; 0.9; 0.4; 0.2\}$,

$\delta_k = \{0.5; 0.4; 0.8; 0.4; 0.4; 0.7; 0.4; 0.3; 0.8; 0.2; 0.1; 0.5; 0.4; 0.7;$
$0.5; 0.4; 0.7; 0.9; 0.4; 0.3\}$,

$s = 0.5$.

The results of the program are presented in Table 8.

Table 8. Numerical results for LST of distribution function for k-busy period $\pi_k^\delta(s)$

k	$\pi_k(s)$	$\pi_k^\delta(s)$	k	$\pi_k(s)$	$\pi_k^\delta(s)$
1	0.002444	0.986440	11	0.012414	0.750670
2	0.005566	0.956771	12	0.029777	0.796091
3	0.000068	0.999659	13	0.021775	0.763410
4	0.030767	0.812228	14	0.009064	0.954890
5	0.034760	0.766861	15	0.034760	0.807550
6	0.018947	0.881203	16	0.043203	0.644660
7	0.003083	0.960978	17	0.003927	0.980092
8	0.051492	0.569886	18	0.002451	0.984919
9	0.012501	0.951740	19	0.016581	0.864629
10	0.012555	0.785024	20	0.017712	0.851134

- **Probability of states**

Example 8.8 $k = 4$, $\lambda_4 = 1.456$. Time of serving is Erlang with parameters $\alpha = 1$ and $k_e = 2$, $Erl(1,2)$. Time of exchange for user k is Exponential with parameter $c = 0.268, E(0.268)$. For different values of t we have the probabilities $P_{B_4}(t)$ and $P_{C_4}(t)$. The results are presented in Table 9 and Table 10.

Table 9. Numerical results for probabilities of states for the 4^{th} user, $\lambda_4 = 1.456$, in terms of LST and different values of t

t	$P_{B_4}(t)$	$P_{C_4}(t)$
0.01	0.833	0.0193
0.2	0.07599	0.00125
0.5	0.06629	0.00116
1.5	0.04472	0.00049
2.9	0.02874	0.00019
5.8	0.01448	0.00005
10.4	0.00680	0.00001

Table 10. Numerical results for probabilities of states for the 4^{th} user, $\lambda_4 = 1.456$, in terms of LST

t	$P_{B_4}(t)$	$P_{C_4}(t)$
1	0.05389	0.00074
2	0.037774	0.00034
3	0.02794	0.00018
4	0.2153	0.0010
5	0.01711	0.00006
6	0.01396	0.00009
7	0.01155	0.00003
8	0.00974	0.00002
9	0.00832	0.00001
10	0.00719	0.00001

Example 8.9 We have the same parameters like in Example 8.8 with the exception of flow requirements parameter for user k: $\lambda_k = 2.987465$. The results are presented in Table 11 and Table 12.

Table 11. Numerical results for probabilities of states for the 4^{th} user, $\lambda_4 = 2.987465$, in terms of LST and different values of t

t	$P_{B_4}(t)$	$P_{C_4}(t)$
0.01	0.06466	0.03934
0.2	0.06200	0.03217
0.5	0.05781	0.02413
1.5	0.04553	0.01101
2.9	0.03336	0.00457
5.8	0.01957	0.00133
10.4	0.01037	0.00033

Table 12. Numerical results for probabilities of states for the 4^{th} user, $\lambda_4 = 2.987465$, in terms of LST

t	$P_{B_4}(t)$	$P_{C_4}(t)$
1	0.05128	0.01587
2	0.04057	0.00795
3	0.03268	0.00451
4	0.02682	0.00277
5	0.02238	0.00181
6	0.01895	0.00124
7	0.01625	0.00088
8	0.014909	0.00064
9	0.01233	0.00048
10	0.01088	0.00036

Remark 8.1 *Theorem 2 gives us the possibility for modeling the probability of states for an arbitrary* $k, 1 \leq k \leq r$.

- **Queue length**

Example 8.10 Let $n = 9$, where n is the number of users and parameters λ_k, b_k and c_k are presented in the form of vectors:

$\lambda_k = \{0.2, 0.4, 0.3, 0.1, 0.5, 0.6, 0.4, 0.1, 0.2\}$,

$b_k = \{0.1, 0.4, 0.5, 0.1, 0.2, 0.4, 0.1, 0.5, 0.3\}$,

$c_k = \{0.4, 0.6, 0.4, 0.2, 0.1, 0.2, 0.3, 0.4, 0.1\}, s = 0.2$.

The results are presented in Table 13.

Table 13. Numerical results for queue length distribution $l_k(s)$, in terms of LST

k	$\pi_k(s)$	$\pi_k^\delta(s)$	$l_k(s)$
1	0.21904	0.11586	5.52940
2	0.49853	0.29894	3.86539
3	0.61134	0.34125	1.63709
4	0.26785	0.11320	2.87008
5	0.25865	0.03856	3.01735
6	0.42105	0.11267	1.91095
7	0.15679	0.05618	5.32859
8	0.68287	0.43239	1.04412
9	0.49806	0.12439	2.17897

References

[1] V. M. Vishnevsky and O. V. Semenova, *Polling Systems: The theory and applications in the broadband wireless networks.* Moscow, Russia: Texnocfera, 2007, 312 p. (in Russian)

[2] V. V. Rycov and Gh. K. Mishkoy, "A new approach for analysis of polling systems," in *Proc. of the Int. Conf. on Control Problems*, Moscow, 2009, pp. 1749–1758.

[3] Gh. K. Mishkoy, *Generalized Priority Systems.* Chisinau, Republic of Moldova: Stiinta, 2009, 200 p. (in Russian)

[4] D. G. Kendall, "Some problems in the theory of queues," *J. Roy. Statist. Soc. (B).*, vol. 13, no. 2, pp. 151–185, 1951.

[5] Gh. Mishkoy, Udo R. Krieger and D. Bejenari, "Matrix algorithm for Polling models with PH distribution," *Buletinul Academiei de Stiinte a Republicii Moldova. Matematica*, vol. 68, no. 1, pp. 70–80, 2012.

Gh. Mishkoy, D. Bejenari,
L. Mitev, I. R. Ticu

Received June 01, 2016

Gh. Mishkoy
Academy of Science of Moldova, Free International University of Moldova
Chisinau, Republic of Moldova
E–mail: gmiscoi@ulim.md

D. Bejenari
Free International University of Moldova
Chisinau, Republic of Moldova
E–mail: artemis85@mail.ru

L. Mitev
Institute of Mathematics and Computer Science of Academy of Science of Moldova,
Free International University of Moldova
Chisinau, Republic of Moldova
E–mail: liliausate@yahoo.com

I. R. Ticu
Constanta Maritime University
Constanta, Romania
E–mail: ionela.ticu@yahoo.com