Computer Science Journal of Moldova, vol.**24**, no.2(71), 2016

Hat problem on graphs with exactly three cycles

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Abstract

This paper is devoted to investigation of the hat problem on graphs with exactly three cycles. In the hat problem, each of n players is randomly fitted with a blue or red hat. Everybody can try to guess simultaneously his own hat color by looking at the hat colors of the other players. The team wins if at least one player guesses his hat color correctly, and no one guesses his hat color wrong; otherwise the team loses. The aim is to maximize the probability of winning. Note that every player can see everybody excluding himself. This problem has been considered on a graph, where the vertices correspond to the players, and a player can see each player to whom he is connected by an edge. We show that the hat number of a graph with exactly three cycles is $\frac{3}{4}$ if it contains a triangle, and $\frac{1}{2}$ otherwise.

Keywords: Hat problem, Strategy.
2010 Mathematics Subject Classification: 05Cxx.

1 Introduction

In the hat problem there are n players who may coordinate a strategy before the game begins. Each player gets a hat whose color is selected randomly and independently to be blue with probability 1/2 and red otherwise. Each player can see the colors of all other hats but not of his own. Simultaneously, each player may guess a color or pass. The players win if at least one player guesses correctly the color of his own hat, and no player guesses wrong. The goal is to find a strategy that maximizes the probability of winning. This maximum probability is called the value of the game. This problem was formulated by Ebert [2], and further considered for example in [4], [5], [11].

The hat problem on a graph was considered by Krzywkowski [6]. The players are placed on the vertices of a graph, and a player can only see the colors of hats of his neighbors. The requirement for winning remains the same. If the graph is a complete graph, this is exactly Ebert's original problem. Krzywkowski in [6] showed that if the graph is a tree, the value of the corresponding game is 1/2. In [7] the same result is shown when the graph is a cycle on four vertices. The hat problem on bipartite graphs, cycles, unicyclic graphs, and graphs with exactly two cycles are studied in [1], [3], [7]–[10], [12]. In this paper we study the hat problem in graphs with exactly three cycles. Let $h(G)$ denotes the value of the hat problem on a graph G. We shall prove the following.

Theorem 1 *Let G be a graph with exactly three cycles. Then $h(G) = \frac{3}{4}$ if G contains a triangle, and $h(G) = \frac{1}{2}$ otherwise.*

2 Notations

For notation and terminology not given here we refer to [13]. Let $G = (V(G), E(G))$ be a graph. For a vertex $v \in V(G)$, the *open neighborhood* of v, is $N_G(v) = \{x \in V(G) : vx \in E(G)\}$. The *degree* of a vertex v is $\deg_G(v) = \deg(v) = |N_G(v)|$. We say that a vertex v is *neighborhood-dominated* if there is some other vertex u such that $N_G(v) \subseteq N_G(u)$. If H is a subgraph of G, then we write $H \subseteq G$.

Let $V(G) = \{v_1, v_2, ..., v_n\}$. A function $c : V(G) \to \{b, r\}$ is a vertex coloring, where b refers to the blue color and r refers to the red color. If $v_i \in V(G)$, then $c(v_i)$ is the color of v_i. By a case for the graph G we mean a sequence $(c(v_1), c(v_2), ..., c(v_n))$. We denote the set of all cases for the graph G by $C(G)$. Note that $|C(G)| = 2^{|V(G)|}$. If $v_i \in V(G)$, then by s_i we denote a function $s_i : V(G) \to \{b, r, *\}$, where $s_i(v_j)$ is the first letter of the color of v_j if v_i sees v_j, and mark $*$ otherwise, that is, $s_i(v_j) = c(v_j)$ if $v_j \in N_G(v_i)$, while $s_i(v_j) = *$ if

$v_j \in V(G) - N_G(v_i)$. By a situation of the vertex v_i in the graph G we mean the sequence $(s_i(v_1), s_i(v_2), ..., s_i(v_n))$. The set of all possible situations of v_i in the graph G is denoted by $St_i(G)$. Observe that $|St_i(G)| = 2^{|N_G(v_i)|}$. If $v_i \in V(G)$, then we say that a case $(c_1, c_2, ..., c_n)$ for the graph G corresponds to a situation $(t_1, t_2, ..., t_n)$ of the vertex v_i in the graph G if it is created from this situation only by changing every mark $*$ to the letter b or r. So, a case corresponds to a situation of v_i if every vertex adjacent to v_i, in that case has the same color as in that situation. To every situation of the vertex v_i in the graph G correspond $2^{|V(G)|-\deg_G(v_i)}$ cases, because every situation of v_i has $|V(G)| - \deg_G(v_i)$ mark $*$.

By a statement of a vertex we mean its declaration about the color it guesses it is. By the effect of a case we mean a win or a loss. According to the definition of the hat problem, the effect of a case is a win if at least one vertex states its color correctly and no vertex states its color wrong. The effect of a case is a loss if no vertex states its color or somebody states its color wrong. By a guessing instruction for the vertex $v_i \in V(G)$ (denote by g_i) we mean a function $g_i : St_i(G) \to \{b, r, p\}$ which, for a given situation, gives the first letter of the color v_i guesses it is or a letter p if v_i passes. Thus a guessing instruction is a rule which determines the conduct of the vertex v_i in every situation. By a strategy for the graph G we mean a sequence $(g_1, g_2, ..., g_n)$. By $F(G)$ we denote the family of all strategies for the graph G.

Let $v_i \in V(G)$ and $S \in F(G)$. We say that v_i never states its color in the strategy S if v_i passes in every situation. We say that v_i always states its color in strategy S if v_i states its color in every situation, that is, for every $T \in St_i(G)$ we have $g_i(T) \in \{b, r\}$ ($g_i(T) \neq p$, equivalently). If $S \in F(G)$, then by $Cw(S)$ and $Cl(S)$ we denote the sets of cases for the graph G in which the team wins or loses, respectively. Observe that $|Cw(S)| + |Cl(S)| = |C(G)|$. Consequently, by the chance of success of the strategy S we mean the number $p(S) = \frac{|Cw(S)|}{|C(G)|}$. By the hat number of the graph G we mean the number $h(G) = max\{p(S) : S \in F(G)\}$. Note that $p(S) \leq h(G)$. We say that the strategy S is optimal for the graph G if $p(S) = h(G)$. By $F^0(G)$ we denote the family of all optimal strategies for the graph G.

3 Known results

In this section we state some known results that we need to prove our main result. We denote by P_n, C_n and K_n the path, the cycle and the complete graph with n vertices, respectively. We begin with the following theorem.

Theorem 2 (Krzywkowski, [6]) *If H is a subgraph of G, then $h(H) \leq h(G)$.*

Corollary 1 (Krzywkowski, [6]) *For every graph G, $h(G) \geq \frac{1}{2}$.*

Let $\omega(G)$ denotes the *clique number* of a graph G, i.e. the maximum number of vertices that each pair of them are adjacent. Also let $\chi(G)$ denotes the *chromatic number* of G, i.e. the minimum number of colors in a vertex coloring such that adjacent vertices receive different colors. Feige, [3] presented the following important results.

Theorem 3 (Feige, [3]) *For every graph, $h(G) = h(K_{\omega(G)})$, if $\chi(G) = \omega(G)$.*

Theorem 4 (Feige, [3]) *If $\omega(G) + 1$ is a power of 2, then $h(G) = \frac{\omega(G)}{\omega(G)+1}$.*

Lemma 1 (Feige, [3]) *If v is a neighborhood-dominated vertex of a graph G, then $h(G) = h(G - v)$.*

Lemma 2 (Feige, [3]) *If a graph G is a disjoint union of two graphs G_1 and G_2, then $h(G) = \max\{h(G_1), h(G_2)\}$.*

We denote by $G_1 \cup G_2$ the disjoint union of two graphs G_1 and G_2. The hat number of several classes of graphs including paths, cycles, unicyclic graphs, and graphs with precisely two cycles are determined as follows.

Theorem 5 (Krzywkowski, [6]) *For every path P_n we have $h(P_n) = \frac{1}{2}$.*

Theorem 6 (Feige, [3], Krzywkowski, [7], [8]) *For every cycle C_n with $n > 3$, $h(C_n) = \frac{1}{2}$.*

Lemma 3 (Krzywkowski, [10]) *If G is a unicyclic graph with no triangle, then $h(G) = \frac{1}{2}$.*

Theorem 7 (Balegh, Jafari Rad, [1]) *If G is a graph with no triangle and exactly two cycles, then $h(G) = \frac{1}{2}$.*

The next two theorems consider optimal strategies such that some vertex always (never, respectively) states its color.

Theorem 8 (Krzywkowski, [6]) *Let v be a vertex of a graph G. If $S \in F^0(G)$ is a strategy such that v always states its color, then $h(G) = \frac{1}{2}$.*

Theorem 9 (Krzywkowski, [6]) *Let v be a vertex of a graph G. If $S \in F^0(G)$ is a strategy such that v never state its color, then $h(G) = h(G - v)$.*

Remark 1 *Let the strategy S is optimal for a graph G, then we have $h(G) = p(S)$, we get $p(S) \geq \frac{1}{2}$.*

The next lemma is about the non-necessity of statements of any further vertices in a case in which some vertex already states its color.

Lemma 4 (Krzywkowski, [7]) *Let G be a graph and let S be a strategy for G. Let C be a case in which some vertex states its color. Then a statement of any other vertex cannot improve the effect of the case C.*

4 Proof of Theorem 1

Let G be a graph with exactly three cycles. Assume that $\delta(G) = 1$. Clearly any vertex of degree one is a neighborhood-dominated vertex. If y_1 is a vertex of degree one in G, then by Lemma 1, $h(G - y_1) = h(G)$.

If $\delta(G - y_1) = 1$ and y_2 is a vertex of degree one in $G - y_1$, then by Lemma 1, $h(G-y_1-y_2) = h(G-y_1) = h(G)$. Continuing this process, there is an integer k such that $h(G) = h(G - y_1 - y_2 - ... - y_k)$, and $\delta(G - y_1 - y_2 - ... - y_k) \geq 2$. Thus we may assume that $\delta(G) \geq 2$. Assume G has a triangle. Clearly $\omega(G) = 3$ since G has exactly three cycles. Then by Theorem 4, we have $h(G) = \frac{3}{4}$. Thus for the next we assume that G contains no triangle. The following lemma plays an important role for the next.

Lemma 5 *Suppose* $P = v_1v_2v_3v_4$ *is a path in* G *with* $\deg_G(v_2) = \deg_G(v_3) = 2$*, and* $v_4 \notin N_G(v_1)$*. Let* H *be the graph obtained from* G *by deleting the vertices* v_2 *and* v_3 *and, adding an edge between* v_1 *and* v_4*. Then* $h(G) \leq h(H)$*.*

Proof. Let H_1 be obtained from G by adding the edge v_1v_4. By Theorem 2, $h(G) \leq h(H_1)$. Then v_3 is a neighborhood dominated vertex in H_1, and thus by Theorem 2 and Lemma 1, $h(H_1) = H(H_1 - v_3)$. But v_2 is a neighborhood dominated vertex in $H_1 - v_3$, and thus by Theorem 2 and Lemma 1, $H(H_1 - v_3) = h(H_1 - v_3 - v_1)$. Now $h(G) \leq h(H_1) = h(H_1 - v_3) = h(H_1 - v_3 - v_1) = h(H)$, as desired. □

4.1 G has no cut-vertex

Since G has no cut-vertex, it is obtained from a cycle by adding a path $P = x_0x_1....x_k$ between two non-consecutive vertices u and v, where $u = x_0$ and $v = x_k$. Thus G contains two cycles C_1 and C_2 such that $V(C_1) \cap V(C_2) = \{x_0, ..., x_k\}$. Let $|V(C_1)| = n_1$ and $|V(C_2)| = n_2$. If both n_1 and n_2 are even, then $\chi(G) = \omega(G) = 2$, and so by Theorem 3, $h(G) = \frac{1}{2}$. Thus assume that at least one of n_1 or n_2 is odd. We aim to obtain a graph G^* with $h(G) \leq h(G^*)$ and $h(G^*) = 1/2$, and then the result follows by Theorem 1. We do this in some stages, and in each stage of the proof, without loss of generality, we assume that in each stage G has the properties of the desired G^*.

By applying Lemma 5, we may assume that $k \leq 3$.

Lemma 6 *If* $k = 1$*, then* $h(G) \leq \frac{1}{2}$*.*

Proof. By applying Lemma 5, we may assume that $n_2 = 5$ and $n_1 \in \{4, 5\}$. Assume first that $n_1 = 5$. Let $n_1 = n_2 = 5$, $C_1 = x_0x_1a_1a_2a_3x_0$, and $C_2 = x_0x_1b_1b_2b_3x_0$. Let $G_1 = G + b_1a_2$. Then a_1 is a neighborhood dominated vertex in G_1, and thus by Theorem 2 and Lemma 1, $h(G) \leq h(G_1) = h(G_1 - a_1)$. Let $G_2 = G_1 - a_1$, and $G_3 = G_2 + a_2b_3$. Then b_2 is a neighborhood dominated vertex in G_3, and thus by Theorem 2 and Lemma 1, $h(G_3) = h(G_3 - b_2)$. Let $G_4 = G_3 - b_2$. Then a_3 is a neighborhood dominated vertex in G_4, and thus by Theorem 2 and Lemma 1, $h(G_4) = h(G_4 - a_3)$. But $G_4 - a_3$ is a cycle, and by Theorem 6, $h(G_4 - a_3) = 1/2$. Thus $h(G) \leq h(G_1) \leq h(G_2) \leq h(G_3) \leq h(G_4) \leq 1/2$.

Next assume that $n_1 = 4$. Let $C_1 = abx_1x_0a$, where $N_G(b) = \{a, x_1\}$. Since $N_G(b) \subseteq N_G(x_0)$, by Lemma 1, $h(G) = h(G - b)$. But $G - b$ is a unicyclic graph, and so by Lemma 3, $h(G) = h(G - b) = 1/2$. $\square$

Lemma 7 *If $k = 2$, then $h(G) \leq 1/2$.*

Proof. Assume that $k = 2$. By applying Lemma 5, we may assume that $n_2 = 5$, and $n_1 \in \{4, 5\}$. First assume that $n_1 = 5$. Let $C_1 = x_0x_1x_2a_1a_2x_0$, and $C_2 = x_0x_1x_2b_1b_2x_0$. Let $G_1 = G + b_2a_1$. Then b_1 is a neighborhood dominated vertex in G_1, and thus by Theorem 2 and Lemma 1, $h(G) \leq h(G_1) = h(G_1 - b_1)$. Let $G_2 = G_1 - b_1$. Then b_2 is a neighborhood dominated vertex in G_2, and thus by Theorem 2 and Lemma 1, $h(G_2) = h(G_2 - b_2)$. But $G_2 - b_2$ is a cycle, and by Theorem 6, $h(G_2 - b_2) = 1/2$. Thus $h(G) \leq h(G_1) \leq h(G_2) \leq 1/2$.

Next assume that $n_1 = 4$. Then C_1 has a neighborhood-dominated vertex, say x, which $x \notin \{x_0, x_1, x_2\}$, and thus by Theorem 2 and Lemma 1, we find that $h(G) \leq h(G - x) = h(C_2) = \frac{1}{2}$, implying that $h(G) \leq 1/2$. $\square$

Lemma 8 *If $k = 3$, then $h(G) \leq 1/2$.*

Proof. Assume that $k = 3$. Since G has no triangle, $\{n_1, n_2\} \neq \{4, 5\}$. If $n_1 = 4$, then x_1 is a neighborhood-dominated vertex, and thus by Theorem 2 and Lemma 1, we find that $h(G) \leq h(G - x_1) = 1/2$. Thus

$n_1 > 4$, and similarly $n_2 > 4$. Let n_1 be even. Let $x_1x_0v_1v_2$ be a path on C_1 with $v_1 \neq x_1$, and let H be obtained from G by joining x_1 to v_2. Then v_1 is a neighborhood-dominated vertex in H, and thus by Lemma 1, $h(G) \leq h(H - v_1)$. But $h(H - v_1) \leq 1/2$ by Lemma 7. Thus $h(G) \leq 1/2$. Similarly if n_1 is odd, then $h(G) \leq 1/2$. □

4.2 G has some cut-vertex

Assume that G has precisely one cut-vertex. Then G contains precisely three cycles C_1, C_2 and C_3 with one common vertex, say w. For convenience we denote G by $G_1(n_1, n_2, n_3)$, where $n_i = |V(C_i)|$ for $i = 1, 2, 3$. If n_i is even for all $i = 1, 2, 3$, then by Theorem 3, we have $\chi(G) = \omega(G) = 2$, and so $h(G) = \frac{1}{2}$. Thus without loss of generality assume that n_1 is odd. By applying Lemma 5, we may assume that $n_1 = 5$, $n_2 \in \{4, 5\}$ and $n_3 \in \{4, 5\}$. Assume that $n_2 = 4$. Let $V(C_2) = \{a, b, c, w\}$, where $N_G(b) = \{a, c\}$. Then b is a neighborhood-dominated vertex, and thus by Theorem 2 and Lemma 1, we find that $h(G) = h(G - b)$. But $G - b$ is a graph with exactly two cycles, and by Theorem 7, $h(G) = 1/2$. Thus we assume that $n_2 = n_3 = 5$. Thus $G = G_1(5, 5, 5)$.

Let $V(G) = \{a_1, a_2, a_3, a_4, v, b_1, b_2, b_3, b_4, c_1, c_2, c_3, c_4\}$, where $N(v) = \{a_1, a_4, b_1, b_4\}$, a_i is adjacent to a_{i+1} for $i = 1, 2, 3$, b_j is adjacent to b_{j+1} for $j = 1, 2, 3$, and c_k is adjacent to c_{k+1} for $k = 1, 2, 3$. Let $H_1 = G + a_4b_3$. Then b_4 is a neighborhood-dominated vertex in H_1, and by Theorem 2 and Lemma 1, $h(H_1) = h(H_1 - b_4)$. Let $H_2 = H_1 - b_4$ and $H_3 = H_2 + a_1b_2$. Then b_1 is a neighborhood-dominated vertex in H_3, and by Theorem 2 and Lemma 1, $h(H_3) = h(H_3 - b_1)$. Let $H_4 = H_3 - b_1$ and $H_5 = H_4 + a_3b_2$. Then a_2 is a neighborhood-dominated vertex in H_5, and by Theorem 2 and Lemma 1, $h(H_5) = h(H_5 - a_2)$. Let $H_6 = H_5 - a_2$. We now see that b_3 is a neighborhood-dominated vertex in H_6, and by Theorem 2 and Lemma 1, $h(H_6) = h(H_6 - b_3)$. But $H_6 - b_3$ is a graph with two cycles, and thus by Theorem 7 $h(H_6 - b_3) = 1/2$.

Thus

$$\begin{aligned} h(G) \leq h(H_1) &\leq h(H_2) \leq h(H_3) \\ &\leq h(H_4) \leq h(H_5) \\ &\leq h(H_6) \leq h(H_6 - b_3) = 1/2 \end{aligned}$$

as desired.

Assume now that G has at least two cut-vertices. Assume that G has two cut vertices w_1, w_2 such that $w_1 \in V(C_1)$, $w_2 \in V(C_2)$ and the shortest path from w_1 to w_2 (say P) does not intersect C_3. Let $z_1 \in N(w_2)$ be a vertex on P. Let $v_1v_2w_2z_1$ be a path on C_3, and let $H = G + v_1z_1$. Clearly by Theorem 2, we have $h(G) \leq h(H)$. Observe that v_2 is a dominated vertex. By Lemma 1, we get $h(H) = h(H - v_2)$. If $z_1 \neq w_1$, then we consider a vertex $z_2 \in N(z_1)$ on P, and continue this process. Continuing this process, we obtain a graph H^* with precisely three cycles C_1, C_3 and C_2', where $V(C_1) \cap V(C_2') = \{w_1\}$. A similar argument holds for C_1, C_3, or C_2, C_3. Thus we may assume that G has two cut vertices w_1, w_2 such that $V(C_1) \cap V(C_2) = \{w_1\}$ and $V(C_2) \cap V(C_3) = \{w_2\}$, and $w_1 \notin N(w2)$. As before, we may assume that $|V(C_i)| = n_i$ for $i = 1, 2, 3$. Also for convenience, we denote $G = G_2(n_1, n_2, n_3)$. By applying Lemma 5, we may assume that $n_1, n_3 \in \{4, 5\}$. Assume that $n_1 = 4$. Let $V(C_1) = \{a, b, c, w_1\}$, where $N_G(b) = \{a, c\}$. Since $N_G(b) \subseteq N_G(w_1)$, by Lemma 1, $h(G) = h(G - b)$. Since $G - u$ is a graph with precisely two cycles, by Theorem 7, $h(G) \leq 1/2$. Thus $n_1 = 5$ and similarly $n_2 = 5$. Assume that $n_2 \geq 4$ is even. By applying Lemma 5, we may assume that $n_2 = 4$. Let $V(C_2) = \{w_1, v_1, w_2, v_2\}$, where $w_1 \in V(C_1 \cap C_2)$ and $w_2 \in V(C_2 \cap C_3)$. Without loss of generality, observe $N_G(v_1) = \{w_1, w_2\}$. Clearly b is a neighborhood-dominated vertex, and so by Lemma 1, $h(G) = h(G - b)$. But $G - b$ is a graph with exactly two cycles, and by Theorem 7, $h(G) = 1/2$. Thus assume that $n_2 \geq 5$ is odd. By applying Lemma 5, we may assume that $n_2 = 5$.

Lemma 9 $h(G_2(5, 5, 5) = 1/2$.

Proof. Let S be an optimal strategy for G. Let us assume that some vertices, say v_i, never states its color. Then by Theorem 9, we have

$h(G) = h(G - v_i)$. If $deg(v_i) = 2$, then $G - v_i$ is a graph with precisely two cycles. By Theorem 7, we get $h(G) = h(G - v_i) = \frac{1}{2}$. If $deg(v_i) > 2$, then $G - v_i = P_4 \cup G'$, where G' is a unicyclic graph. Then by Theorems 2, 5 and Lemma 3, we get $h(G) = h(G - v_i) = max\{h(P_4), h(G')\} = \frac{1}{2}$. Thus we assume that every vertex guesses its color. If there exists a vertex that always states its color, then by Theorem 8, $h(G) = \frac{1}{2}$. Thus assume that no vertex in G always states its color. Now let us assume that every vertex states its color in at least one situation. We consider the following two possibilities.

(1) Every vertex states its color in exactly one situation.

Every statement of every vertex in any situation is wrong in exactly $2^{|V(G)| - d_G(v_i) - 1}$ cases, because every situation of any vertex v_i is corresponded to $2^{|V(G)| - |N_G(v_i)|}$ cases, and in half of them the vertex v_i has the color it states it. Since every vertex states its color in exactly one situation, there are exactly 2^{12} correct statements, and then the team can win in at most 2^{12} cases, even if every of the 2^{12} correct statements is in another cases. This implies that $p(S) = \frac{|Cw(S)|}{|C(G)|} \leq \frac{1}{2}$. Since $S \in F^0(G)$, we have $h(G) \leq \frac{1}{2}$. Since by Corollary 1, we have $h(G) \geq \frac{1}{2}$, we get $h(G) = \frac{1}{2}$.

(2) There is a vertex that states its color in more than one situation.

Since we seek minimal number of cases with wrong statements, let us assume that there is a vertex, say v_i, that states its color in exactly two situations. This vertex states its color when views an even number of blue or red colors. Without loss of generality, let v_i and v_j state their colors if it view an even number of blue colors. Let S' be an optimal strategy different from S such that for any pair of vertices v_i and v_j, one of v_i or v_j states its color when views an even number of blue colors, and the other one does not state its color when views an even number of blue colors. Let v_j does not state its color when views an even number of blue colors. Then clearly the other vertex, v_i, states its color when views an even number of blue colors. By Lemma 4 the statement of v_j cannot improve the result of any of these cases. Therefore, $p(S') \leq p(S)$. Since $S' \in F^0(G)$, then strategy S is also optimal for G. Note that if v_j never states its color in the strategy S', then $S' = S$ and we have a possibility already considered.

$\square$

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