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Bell Numbers of Complete Multipartite Graphs

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Abstract

The *Stirling number* $S(G;k)$ is the number of partitions of the vertices of a graph G into k nonempty independent sets and the number of all partitions of G is its *Bell number*, $B(G)$. We find $S(G;k)$ and $B(G)$ when G is any complete multipartite graph, giving the upper bounds of these parameters for any graph.

Keywords: Bell number, Bell polynomial, Partition, Stirling numbers.

1 Introduction

Throughout this paper, the graph $G = (V, E)$ will be a finite simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The *join* of two graphs G_1 and G_2, denoted by $G_1 \vee G_2$, is the graph G whose vertex set is $V(G) = V(G_1) \cup V(G_2)$, a disjoint union, and whose edge set is $E(G) = E(G_1) \cup E(G_2) \cup \{u_1u_2 \mid u_1 \in V(G_1),\ u_2 \in V(G_2)\}$. For example, $\overline{K}_{n_1} \vee \overline{K}_{n_2} \vee \ldots \vee \overline{K}_{n_l} = K(n_1, n_2, \ldots, n_l)$, a complete l-partite graph ($l \geq 1$) with parts sizes $n_1, n_2, \ldots, n_l$. The special case when $l = 1$, $G = \overline{K}_{n_1} = E_{n_1}$, the null graph. See Figure 1 for the case when $l = 2$ and $n_1 = n_2 = 3$. A *partition* $\sigma = \sigma(n)$ of an n-set X is a set of nonempty subsets of X such that each element of X is in exactly one of the subsets of X. The elements or parts of σ are often called *blocks*, and the number of blocks of σ is its *rank*. For simplicity, we refer to a partition of rank k as a k-partition. B. Duncan and R. Peele [5] called the number of k-partitions of a graph G the *(graphical) Stirling number* of G and it is denoted by $S(G, k)$; this is the number of (vertex) independent sets of G. So when $G = E_n$, $S(G;k) = \left\{ {n \atop k} \right\}$, the Stirling number of the second kind, which counts the number of

k-partitions of a set of n elements. The (total) number of distinct partitions of G is its *Bell number* which we denote by $B(G)$. In other words, $B(G) = \sum_{k=1}^{n} S(G;k)$ and when $G = E_n$, $B(E_n) = \displaystyle\sum_{i=1}^{n} \left\{ {n \atop i} \right\} = B_n$, where B_n is the n^{th} Bell number. It is well documented that the exponential generating function for Bell numbers is $\exp(e^x - 1)$ i.e., $\displaystyle\sum_{n\geq 0} \frac{B_n}{n!} x^n = e^{e^x - 1}$. We call the rank-generating function $F(G;x) = \displaystyle\sum_{k=1}^{n} S(G;k)x^k$ the *partition polynomial* of the graph G. Some basic properties of this polynomial were first studied by Korfhage [9] and later by Brenti et al. [2], [3], [17]. D. Galvin and D.T. Thanh have recently named this polynomial, the Stirling polynomial [6]. It is worth noting that $F(G;1) = B(G)$. If $S(G;k) = \left\{ {n \atop k} \right\}$, then $F(G;x) = F(x)$ is the Bell polynomial which is a very well studied mathematical tool in combinatorial analysis [4], [13]. When x^i is replaced by the falling factorial $x^{\underline{i}} = x(x-1)(x-2)\ldots(x-i+1)$, the polynomial $F(G;x) = \chi(G;x) = \displaystyle\sum_{k=1}^{n} S(G;k)x^{\underline{k}}$ is the chromatic polynomial, which gives the number of proper colorings of a graph with n vertices, using at most x colors (see for e.g., [1], [12], [14], [15]). Clearly, there are $\chi(E_n;x) = \displaystyle\sum_{k=1}^{n} \left\{ {n \atop k} \right\} x^{\underline{k}} = x^n$ colorings (with no restriction) of E_n. We call the sequence $\Big(S(G;p)\Big)_{c\leq p\leq |V(G)|}$ the *partition sequence*. When $c = \chi$, the chromatic number, this sequence is referred to as chromatic vector by Goldman et al. [7] and as chromatic spectrum by Voloshin [16].

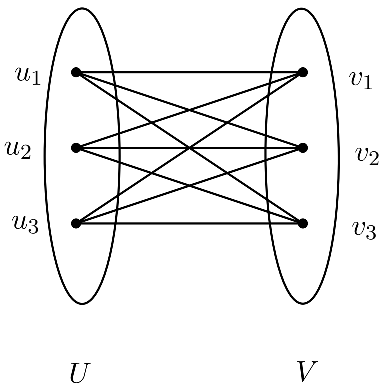

Figure 1: Complete Bipartite $K(3,3)$

The Bell numbers of special graphs have been well researched [4]–[6], [8], [10], [18]. Recently, W. Yan [18] showed that the Bell number of a k-tree on n vertices is B_{n-k}, $k \geq 1$; this is the number of k-nonconsecutive partitions of a set with n elements. This result is a generalization of that of A. O. Munagi [11] for paths, and the work of Duncan and Peele [5] for generalized paths and acyclic graphs. We record these and a few other results for some special graphs G in Table 1; the values found in the table can be obtained through either of the following:

(a) The transformation $S(G;k) = \dfrac{1}{k!}\displaystyle\sum_{x\geq 0}\binom{k}{x}(-1)^{k-x}\chi(G;x)$

(b) The recursion $F(G;x) = F(G-e;x) - F(G/e;x)$, where $G-e$ and G/e are the deletion and contraction graph operations on the edge e in G, respectively.

This paper was primarily inspired by the work of the previously mentioned authors as it adds to the results listed in Table 1, by extending those in rows 1, 2 and 6. In Section 2 we give a basic example of a general case which we present in Section 3.

In Table 1, $E_n, K_n, T_n^m, S_n^{(m)}, P_n^{(m)}, C_n, W_n, K(m,n)$, and $\overline{G}$ denote a null graph, a complete graph, an m-tree, an m-star, an m-path, a cycle, a wheel, a complete bipartite graph, and the complement of G, respectively. For basic notions of these graphs see [19]. This table is adapted from the one produced by Z. Kereskényi-Balogh and G. Nyul [6].

Table 1. Partition sequences and Bell numbers of some graphs

G	**Partition sequence**	$B(G)$
E_n	$\left(\left\{ {n \atop p} \right\}\right)_{p\geq 1}$	B_n
K_n	$\left(1\right)_{p=n}$	1
$T_n^m, S_n^{(m)}, P_n^{(m)}$	$\left(\left\{ {n-m \atop p-m} \right\}\right)_{p\geq m}$	B_{n-m}
C_n	$\left(\sum_{j=p-1}^{n-1}(-1)^{n-1-j}\left\{ {j \atop p-1} \right\}\right)_{p\geq 2}$	$\sum_{j=1}^{n-1}(-1)^{n-1-j}B_j$
W_n, $n\geq 4$	$\left(\sum_{j=p-2}^{n-2}(-1)^{n-2-j}\left\{ {j \atop p-1} \right\}\right)_{p\geq 3}$	$\sum_{j=1}^{n-2}(-1)^{n-2-j}B_j$
$K(m,n)$	$\left(\sum_{j=1}^{p}\left\{ {m \atop j} \right\}\left\{ {n \atop p-j} \right\}\right)_{p\geq 2}$	$B_m \cdot B_n$
$\overline{S_n}$, $n\geq 2$	$\left(n-1\right)_{p=n-1}$ and $\left(1\right)_{p=n}$	n
$\overline{P_n}$, $n\geq 2$	$\left(\binom{p}{n-p}\right)_{p\geq\lceil\frac{n}{2}\rceil}$	F_{n+1} (Fibonacci number)
$\overline{C_n}$, $n\geq 4$	$\left(\frac{n}{p}\binom{p}{n-p}\right)_{p\geq\lceil\frac{n}{2}\rceil}$	L_n (Lucas number)

2 Example

Consider the bipartite graph $G = K(3,3)$ with parts $U = \{u_1, u_2, u_3\}$ and $V = \{v_1, v_2, v_3\}$ in Figure 1. Because $U = V = E_3$, it follows that each set has the partition sequence $(\left\{{3 \atop 1}\right\}, \left\{{3 \atop 2}\right\}, \left\{{3 \atop 3}\right\})$ which is $(1, 3, 1)$ and the distinct partitions of, say U are:

- rank 1: $u_1u_2u_3$
- rank 2: $u_1|u_2u_3$; $u_2|u_1u_3$; $u_3|u_1u_2$
- rank 3: $u_1|u_2|u_3$

Hence the partition polynomial $F(U;x) = F(V;x) = 1x^1 + 3x^2 + 1x^3$. Since no element $x \in U$ can be in the same block as an element $y \in V$, a q-partition of G is therefore composed of all the i-partitions of U and all the j-partitions of V such that $i + j = q$. If we denote by a_i and b_j the terms of the partition sequences of U and V respectively, for each $1 \le i, j \le 3$, then the q-partitions of G form the 3×3 array (Table 2).

Table 2. An array of q-partitions of $G = K(3,3)$ with $q = 2, \ldots, 6$

	b_1	b_2	b_3
a_1	a_1b_1	a_1b_2	a_1b_3
a_2	a_2b_1	a_2b_2	a_2b_3
a_3	a_3b_1	a_3b_2	a_3b_3

Observe that the indices of the off-diagonal entries add up to q, the power of x in the polynomial $F(G;x) = a_1b_1x^2 + (a_1b_2 + a_2b_1)x^3 + (a_1b_3 + a_2b_2 + a_3b_1)x^4 + (a_2b_3 + a_3b_2)x^5 + a_3b_3x^6$. Since $a_1 = a_3 = b_1 = b_3 = 1$ and $a_2 = b_2 = 3$, it follows that $F(G;x) = 1x^2 + 6x^3 + 11x^4 + 6x^5 + 1x^6 = F(U;x) \cdot F(V;x)$. The corresponding partition sequence is $(0, 1, 6, 11, 6, 1)$ with $S(G;1) = 0$, $S(G;2) = 1$, $S(G;3) = 6$, $S(G;4) = 11$, $S(G;5) = 6$, $S(G;6) = 1$ and the Bell number is $B(G) = 25 = B(V) \cdot B(U)$.

3 Bell numbers of complete multipartite graphs

Theorem 1. *Suppose $G_1, \ldots, G_l$ are graphs, each with a partition vector $(a_k^1, \ldots, a_k^{n_k})$, $1 \le k \le l$. If $G = G_1 \vee \ldots \vee G_l$, then the partition polynomial*

$$F(G;x) = \sum_{q=l}^{n_1+\ldots+n_l} \Big(\sum_{\substack{(j_1,\ldots,j_l) \\ j_1+\ldots+j_l=q}} a_1^{j_1} \ldots a_l^{j_l} \Big) x^q \text{ for all } l \ge 1.$$

Proof. When $l = 1$, $F(G;x) = \displaystyle\sum_{\substack{q=1 \\ j_1=q}}^{n_1} a^{j_1} x^q$ is the partition polynomial of $G = G_1$. For $1 \le k \le l$, $F(G_k;x) = \sum_{i=1}^{n_k} a_k^i x^i$, by definition. Now suppose each of the following k columns represents the terms of each partition polynomial, $F(G_k;x)$.

$$\begin{array}{cccc} a_1^1 x & a_2^1 x & \ldots & a_l^1 x^1 \\ a_1^2 x^2 & a_2^2 x^2 & \ldots & a_l^2 x^2 \\ \vdots & \vdots & & \vdots \\ a_1^{n_1} x^{n_1} & a_2^{n_2} x^{n_2} & \ldots & a_l^{n_l} x^{n_l} \end{array} \tag{1}$$

Since, for any partition of $V(G)$, no element $u \in V(G_i)$ can be in the same block with an element $w \in V(G_j)$, $i \ne j$, this implies that $F(G;x) = \displaystyle\prod_{k=1}^{l} F(G_k;x)$. Moreover, a term of $F(G;x)$ that involves, say x^q, is obtained by taking $a_k^{j_k} x^{j_k}$ from the k^{th} column and forming the product $\displaystyle\prod_{k=1}^{l} a_k^{j_k} x^{j_k}$, with the exponents of x satisfying $\sum_{k=1}^{l} j_k = q$. This implies that all the terms of x^q are $\displaystyle\sum_{\substack{(j_1,\ldots,j_l) \\ \sum j_k = n}} \prod_{k=1}^{l} a_k^{j_k} x^{j_k}$. Because a sum over all the terms of x^q for $l \le q \le \sum_{i=1}^{n_k}$ is the polynomial $F(G;x)$, this gives the result.

□

Corollary 1. *The partition polynomial of a complete l-partite graph with part sizes n_i is*

$$F(G;x)=\sum_{q=l}^{n_1+\ldots+n_l}\Big(\sum_{\substack{(j_1,\ldots,j_l)\\ j_1+\ldots+j_l=q}}\left\{{n_1 \atop j_1}\right\}\cdots\left\{{n_l \atop j_l}\right\}\Big)x^q.$$

Proof. Because $G=\overline{K}_{n_1}\vee\overline{K}_{n_2}\vee\ldots\vee\overline{K}_{n_l}$ and $a_k^j=\left\{{n_k \atop j}\right\}$ for $1\le j,k\le l$, the result follows from Theorem 1. □

Corollary 2. *The partition sequence of a complete l-partite graph with part sizes $n_i\ge 1$ is* $\Big(\sum_{\substack{(j_1,\ldots,j_l)\\ \sum j_k=p}}\left\{{n_1 \atop j_1}\right\}\cdots\left\{{n_l \atop j_l}\right\}\Big)_{p\ge l}$, $l\ge 1$.

Since each $F(\overline{K}_{n_k};1)=B_{n_k}$ and $F(G;1)=\prod_{k=1}^{l}F(\overline{K}_{n_k};1)$, the next result follows.

Corollary 3. *The Bell number of a complete l-partite graph with parts sizes n_i is* $B(G)=\prod_{i=1}^{l}B_{n_i}$.

Remarks. Observe that Corollaries 2 and 3 generalize the result on row 6 of Table 1, which extends those of rows 1 and 2. As mentioned in the introduction, the lower bound for the Stirling numbers of any l-colorable graph H is its chromatic number $\chi(H)=l$. Because every l-colorable graph H is a subgraph of some complete l-partite graph G, the previous two results give the upper bounds for $S(H;k)$ and $B(H)$.

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