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The Chromatic Spectrum of a Ramsey Mixed Hypergraph

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Abstract

We extend known structural theorems, primarily a result of Axenovich and Iverson, for the strict edge colorings of the complete graph K_n which avoid monochromatic and rainbow triangles to discover recursive relationships between the chromatic spectra of the bihypergraphs modeling this coloring problem. In so doing, we begin a systematic study of coloring properties of mixed hypergraphs derived from coloring the edges of a complete graph K_n in such a way that there are no rainbow copies of K_r and no monochromatic copies of K_m, where $n \geqslant r \geqslant 3$, $n \geqslant m \geqslant 3$. We present the chromatic spectra of the bihypergraph models of K_n for $4 \leqslant n \leqslant 12$ and $r = m = 3$. This study fits in the larger context of investigating mixed hypergraph structures that realize given spectral values, as well as investigations of the sufficiency of the spectral coefficients in obtaining recursive relationships without the need to subdivide them further into terms that count finer distinctions in the feasible partitions of the hypergraph. The bihypergraphs arising in this simplest case where $r = m = 3$ have spectra that are gap free and which do allow a recursive relationship, albeit a complicated one. The continuation of this project in future work will examine if both of these facts remain true for derived Ramsey Mixed Hypergraphs corresponding to larger r and m.

Keywords: Ramsey number, antiramsey number, mixed hypergraph coloring, feasible partition, chromatic spectrum.

Mathematics Subject Classification: 05C15, 05C65

1 Definitions

A *hypergraph* $\mathcal{H} = (X, \mathcal{E})$ is a collection of vertices $X = \{x_i | i \in I\}$ and a collection of hyperedges $\mathcal{E} = \{e_j \subseteq X | j \in J\}$. A *coloring* of $\mathcal{H}$ is a mapping $f : X \to [k]$, where $[k] = \{1, ..., k\}$. The inverse image $f^{-1}(i)$ is a *color set* defined by the coloring f and the collection of the nonempty color sets defined by f gives a partition of X. When f is a surjection, the coloring is said to be *strict*. Below, all colorings are assumed to be strict unless stated otherwise. In *mixed hypergraph coloring* ([14]–[16]) there are subsets $\mathcal{C}$ and $\mathcal{D}$ of the hyperedge set $\mathcal{E}$ which place conditions on colorings. A coloring f is *proper* if it assigns the same color to at least two vertices in each hyperedge in $\mathcal{C}$ and different colors to at least two vertices in each hyperedge in $\mathcal{D}$. Hence, a hyperedge in $\mathcal{C}$ cannot be rainbow (all vertices of distinct colors), and a hyperedge in $\mathcal{D}$ cannot be monochrome (all vertices of the same color). For convenience, we refer to the hyperedges in $\mathcal{C}$ and $\mathcal{D}$ as C-edges and D-edges. The partitions of X corresponding to proper colorings are the *feasible partitions* of $\mathcal{H}$. A mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ is a *bihypergraph* when $\mathcal{E} = \mathcal{C} = \mathcal{D}$, thereby requiring both coloring conditions on every hyperedge of $\mathcal{H}$.

Mixed hypergraph coloring has many diverse applications. The monograph [16] gives an overview of many of these applications, such as list colorings without lists and problems in resource allocation, data base management, and molecular biology. As it is shown in [10], mixed hypergraphs can be used to efficiently model many graph coloring problems including homomorphisms of simple graphs and multigraphs; circular colorings; (H, C, K)-colorings; locally surjective, locally bijective, and locally injective homomorphisms; $L(p, q)$-labelings; the channel assignment problem; and T-colorings and generalized T-colorings. There are also applications of mixed hypergraph coloring to issues regarding cybersecurity. One such reference is a recent PhD thesis [12] giving algorithms for scalable fault tolerance.

It is well-known, see [16], that the chromatic polynomial $P = P(\mathcal{H}) = P(\mathcal{H}, \lambda)$, which is the function that counts the number of, not necessarily strict, proper λ-colorings of $\mathcal{H}$, can be expressed in the form

$$P = \sum_{i=1}^{n} R_i \lambda^{\underline{i}}, \qquad (1)$$

where $|X| = n$ and $\lambda^{\underline{i}} = \lambda(\lambda-1)\cdots(\lambda-i+1)$ is the falling factorial. Note that the falling factorial counts the number of colorings of the complete graph on i vertices. The coefficients R_i in (1) count the number of feasible partitions of $\mathcal{H}$ using i nonempty subsets. Without coloring conditions, given by defining the sets $\mathcal{C}$ and $\mathcal{D}$ in $\mathcal{E}$, R_i is the Stirling number of the second kind $S(n,i)$ and their sum is the n^{th} Bell number B_n counting the number of partitions of a set of order n. See, for example, [7]. With coloring conditions, we have the trivial bound $R_i \leqslant S(n,i)$. The collection of these coefficients $\{R_1, ..., R_n\}$ is called the *chromatic spectrum* of $\mathcal{H}$. The smallest value of i for which R_i is nonzero is the (lower)-chromatic number $\chi(\mathcal{H})$ and the largest value of i for which R_i is nonzero is the upper-chromatic number $\overline{\chi}(\mathcal{H})$. If all R_i are nonzero for $\chi(\mathcal{H}) \leqslant i \leqslant \overline{\chi}(\mathcal{H})$, the spectrum is said to be *gap free.* The set of indices for which the spectral coefficients are nonzero is the *feasible set* of $\mathcal{H}$. Not only are gaps possible, but any finite set of positive integers is the feasible set of some mixed hypergraph if and only if the set omits 1, or includes 1 and is gap free. See [8].

The splitting-contraction algorithm [14]–[16] finds the chromatic polynomial of any mixed hypergraph, but it has a high level of computational complexity that makes it impractical to use in large hypergraphs. In [13], we use an extension to the splitting-contraction algorithm in the special case of complete uniform interval mixed hypergraphs to find recursive relationships between their chromatic polynomials, where the recursion is on the order of their vertex sets. Recursive relationships for the chromatic polynomials naturally include recursive relationships for the chromatic spectrum values. Our main results here regard a collection of bihypergraphs which model a particular question in Ramsey Theory. We find recursive relationships for the individual chromatic spectral values of these bihypergraphs directly, and do not consider further the full chromatic polynomials. By focusing on the chromatic spectra, we are focused on the growth patterns of the collections of feasible partitions. We give two different ideas of equivalence of colorings to distinguish between the objects counted by the chromatic spectral values and coloring patterns that are the same in a weaker sense.

Two colorings f and g are *isomorphic* if there is a permutation σ of the vertex set X so that $f = g \circ \sigma$. Two k-colorings are *equivalent* if there is a permuation θ of the colors $[k]$ so that $f = \theta \circ g$. Equivalent colorings give the same feasible partition, whereas isomorphic colorings only give the same coloring pattern. Since the chromatic spectral values count the number of nonequivalent colorings, or isomorphic colorings with multiplicity, we say nonequivalent colorings are distinct.

2 Background of a Ramsey Problem

Let $f : \mathcal{E}(K_n) \to [k]$ be an edge coloring of the complete graph on n vertices. Let G_m and G_r be two graphs. The coloring f is said to be (G_m, G_r)-good if there is no monochrome subgraph isomorphic to G_m and no rainbow subgraph isomorphic to G_r. Define the derived *Ramsey Mixed Hypergraph* $\mathcal{H}_R = (\mathcal{E}(K_n), \mathcal{C}(G_r), \mathcal{D}(G_m))$ to be the mixed hypergraph with vertex set corresponding to the edge set of K_n, C-edges corresponding to the copies of G_r in K_n, and D-edges corresponding to the copies of G_m in K_n. In a more general setting this concept was first introduced by Voloshin in 2002 in [16, p.157] under the name of "derived mixed hypergraph of a hypergraph $\mathcal{H} = (X, \mathcal{E})$". In this language, for example, the classic graph Ramsey number $R(p, p)$ (the smallest integer n such that any 2-coloring of $\mathcal{E}(K_n)$ contains a monochromatic copy of K_p in color 1 or a monochromatic copy of K_p in color 2) is the smallest integer n such that the lower chromatic number $\chi(\mathcal{H}_R) = \chi(\mathcal{E}(K_n), \emptyset, \mathcal{D}(K_p)) > 2$.

Clearly, a (G_m, G_r)-good edge coloring f of K_n is a proper coloring of the mixed hypergraph $\mathcal{H}_R$. Axenovich and Iverson [1] define $maxR(n; G_m, G_r)$ and $minR(n; G_m, G_r)$ to be the maximum and minimum number of colors, respectively, in a (G_m, G_r)-good edge coloring of K_n. These numbers are the upper and lower chromatic numbers of $\mathcal{H}_R$. Further, let $F(k; G_m, G_r)$ be the largest value of n for which there is a (G_m, G_r)-good edge k-coloring of K_n. These kinds of numbers have been studied by many authors in Ramsey and anti-Ramsey Theory, see for example [3]–[5], [9].

For the remainder we work with the particular case when $G_m \cong G_r \cong K_3$. As such, it should be understood that good colorings in the following are (K_3, K_3)-good edge colorings of a complete graph. By an easy induction argument, it can be seen that $maxR(n; K_3, K_3) = n-1$. In [6] the authors show that every good $(n-1)$-coloring can be obtained as a kind of product of two good colored cliques. In [3], Chung and Graham prove that

$$F(k; K_3, K_3) = \begin{cases} 5^{k/2} & \text{if } k \text{ is even} \\ 2 \cdot 5^{(k-1)/2} & \text{if } k \text{ is odd} \end{cases} \tag{2}$$

by examining the *monochrome neighborhoods* of a fixed vertex x, defined by $N_i(x) = \{y | f(xy) = i\}$. They also remark that a similar analysis shows that the colorings in the extremal cases can be described using a recursive process with two kinds of products on colored cliques with two and five factors. Axenovich and Iverson in [1] give a more detailed account of these extremal colorings by using product structures on sets of colorings that equate to the products described by Chung and Graham. Axenovich and Iverson also give a proof examining monochrome neighborhoods, and use a structural lemma begun in an earlier paper by Axenovich and Jamison [2]. In the next section we state their lemma and show that it, in fact, gives a description of all good edge colorings of K_n using the same kind of product structures, but we need a product with four factors in addition to the products with two and five factors. Note that we favor the perspective of products of colored subgraphs, as described by Chung and Graham, as opposed to the products of sets of colorings in Axenovich and Iverson. These products are built by designating color patterns on the join edges of an underlying join of the factors. Though we refer to these as product structures, they are actually multivalued products that are more clearly described as blow ups of the coloring patterns used on the join edges. That perspective was taken by Axenovich and Iverson in their products of sets of colorings. We explicitly define these products after listing the isomorphism classes of the good colorings for K_2 through K_5.

The structural theorem on the good edge colorings gives recursive relationships between the chromatic spectral values of the corresponding bihypergraphs. The recursion is on n, the size of the underlying complete graphs which generate this family of derived 3-regular bihypergraphs. Many authors have examined complexity issues and robustness of mixed hypergraphs, and it is known that even 3-regular bihypergraphs provide diverse models with a lot of complexity. Even the computation of the chromatic spectrum of 3-regular bihypergraphs is a hard problem. For example, in [13], the bihypergraph case for uniform complete interval hypergraphs, which have a relatively simple structure, is shown to be much harder than the cases where all the hyperedges are $\mathcal{C}$-edges or all $\mathcal{D}$-edges. In the latter cases simple recursive relationships are found for all sizes of the uniform edges, but for bihypergraphs only recursive relationships are shown for bi-edges of size 3 or size 4. Our main result is the recursive relationships that follow from the product structure. However, we can also use the product structure to find an explicit formula for the leading coefficient, and indicate some weak bounds on the growth of the spectral coefficient of degree $n-2$. After commenting on the techniques used to obtain these results, we conclude with the chromatic spectra corresponding to K_2 through K_{12}, a description of the Java program used to compute these data using the recursive relationship, and a description of a brute force sorting algorithm that was also used to count feasible partitions up through the K_{11} case.

3 Structure of Good Colorings

In Figure 1 we list the isomorphic colorings, or coloring patterns, for all of the good colorings of K_2 through K_5. With each figure we list the number of distinct colorings in that isomorphism class.

The coloring patterns in Figure 1 can be obtained by hand, though it can be tedious work to get the patterns for K_5. They also result from

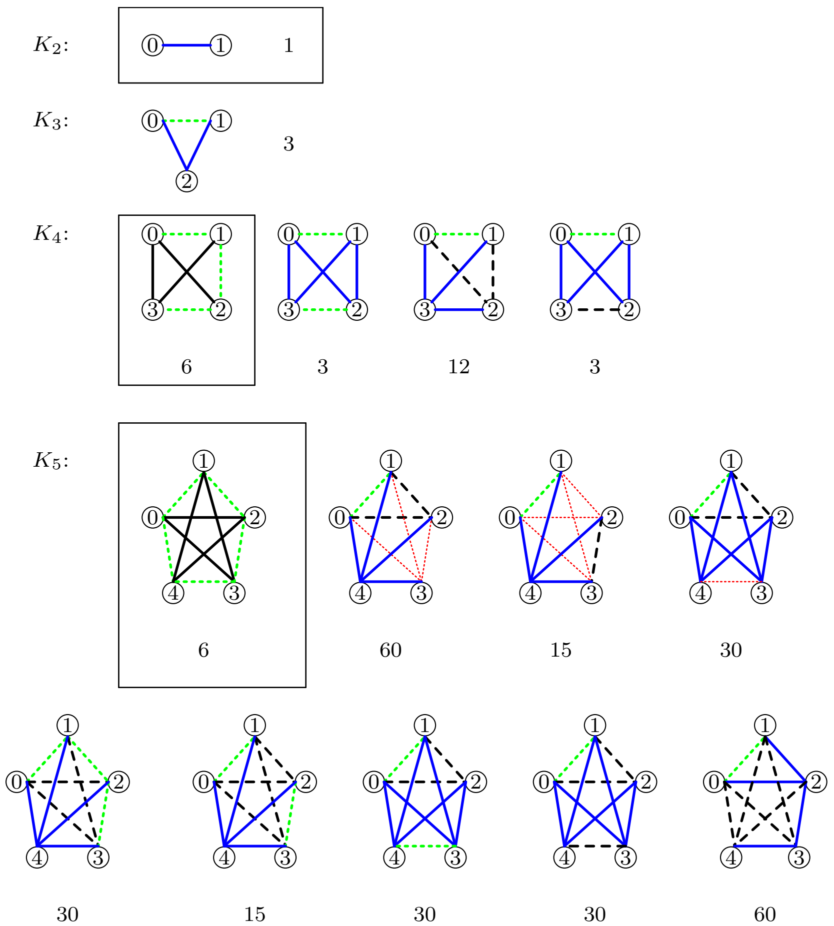

Figure 1. The isomorphism classes of (K_3, K_3)-good coloring patterns for K_2 through K_5 and the number of distinct colorings in each class. The designated patterns are the prime patterns used to define products that generate all good colorings of any K_n.

the forthcoming structural theorem. The three patterns in Figure 1 in the three boxes are the *prime* patterns used to define the necessary product structures.

Let the symbol $[r_0, r_1]$ represent any of the colored complete graphs obtained by replacing the vertices in the prime coloring of K_2 with colored complete graphs K_{r_0} and K_{r_1} where the factors are edge colored with colors different from that used in the original K_2. The resulting complete graph $K_{r_0+r_1}$ is the join of the two underlying factors with each of the join edges colored by the color in the original K_2. This is the operation used in [6] and is equivalent to operations appearing in [3] and [1].

Let the symbol $[r_0, r_1, r_2, r_3]$ represent any of the colored complete graphs obtained by replacing the vertices in any of the six prime colorings of K_4 with colored complete graphs $K_{r_0}, K_{r_1}, K_{r_2}$, and K_{r_3}, where the factors are edge colored with colors different from the two used in the original K_4.

Let the symbol $[r_0, r_1, r_2, r_3, r_4]$ represent any of the colored complete graphs obtained by replacing the vertices in any of the six prime colorings of K_5 with colored complete graphs $K_{r_0}, K_{r_1}, K_{r_2}, K_{r_3}$, and K_{r_4}, where the factors are edge colored with colors different from the two used in the original K_5. This is the same operation as the one defined in [3], but they used just one of the six 2-colorings of K_5 and considered permutations of the factors to obtain the other results of our multivalued product.

For convienence, when the orders of the factors are known, we always order the factors so that $r_0 \geqslant r_1 \geqslant \ldots \geqslant r_4$.

The products of sets of colorings defined in [1] give all of these products and more, but they only use the ones corresponding to these two and five factor products to describe the extremal coloring patterns corresponding to (2).

The three distinct good colorings of K_3 are obtained from the product $[2, 1]$. The nonprime good colorings of K_4 are obtained from $[2, 2]$ and $[3, 1]$. Note that the operation $[2, 2]$ gives 2-colorings when the same color is used on the factors and 3-colorings when different colors are used on the factors. The first two nonprime good colorings of K_5

shown in Figure 1 are 4-colorings obtained from using the two nonisomporphic 3-colorings of K_4 in $[4, 1]$. The last good 4-coloring pattern is obtained from $[3, 2]$. The five good 3-coloring patterns of K_5 are given by $[4, 1], [4, 1], [3, 2], [3, 2]$, and $[2, 1, 1, 1]$, respectively. The two nonisomorphic good 4-coloring patterns given by $[4, 1]$ come from the two nonisomorphic good 2-colorings of K_4. However, the two nonisomorphic good 4-coloring patterns given by $[3, 2]$ come from the two choices of colors on K_2. See Section 4 to see feasible partitions of the labeled edge set of K_5 corresponding to these coloring patterns.

We now show that these three operations are enough to generate all of the good colorings for any K_n.

Following Axenovich and Iverson, we define a *monochromatic pair* and a *mixed pair* of subsets of vertices of an edge colored complete graph. Let f be a good coloring of K_n and let A and B be subsets of $V(K_n)$. Put $c(A, B)$ equal to the set of colors f assigned to edges ab for any $a \in A$ and $b \in B$. A pair (A, B) is monochromatic if $c(A, B) = \{i\}$. The pair is mixed if there are partitions $A = A' \cup A''$ and $B = B' \cup B''$ with $c(A', B') = c(B', B'') = c(B'', A'') = i$ and $c(A', B'') = c(A', A'') = c(A'', B') = \{j\}$. The sets A' and B' may be empty, but not both. See Figure 2.

Lemma 1. *[Axenovich/Iverson] Let $v \in V(K_n)$ and $V_i = N_i(v)$. Order the colors so that $V_i \neq \emptyset$ for $i = 1, \ldots, m$. Then there is a reordering such that:*

a) $c(V_i, V_j) = \{i, j\}$ and the pair (V_i, V_j) is either monochromatic or mixed,

b) If (V_i, V_j) is monochromatic, then $c(V_i, V_j) = \{i\}$,

c) If (V_i, V_j) is mixed, then $j = i + 1$,

d) If there is a mixed pair (V_i, V_j), then neither (V_{i-1}, V_i) or (V_{i+1}, V_{i+2}) is mixed.

Figure 2. A mixed pair A and B.

An edge coloring is *lexical* if there is an ordering of the vertex set so that each edge v_iv_j is assigned color i whenever $i < j$. As Axenovich and Iverson comment, their lemma classifies good colorings as blow ups of lexical colorings, with the possible exceptions of mixed pairs of consecutive sets. A set can be in only one mixed pair. See Figure 3.

Theorem 1. *Any good colored K_n is obtained uniquely as one of the products from $[r_0, r_1]$, $[r_0, r_1, r_2, r_3]$, or $[r_0, r_1, r_2, r_3, r_4]$, with $r_0 \geqslant r_1 \geqslant \ldots \geqslant r_4$ and with good-colorings on each factor.*

Proof Choose a vertex $v \in V(K_n)$ and order the monochrome neighborhoods of v as in the conclusion of Lemma 1.

Suppose $m = 1$. Then $V(K_n) - v = N_1(v)$ and the colored K_n is realized by $[n-1, 1]$ with the join edges colored by color 1.

Suppose $m \geqslant 2$. There are two cases to consider. See Figure 3.

If (V_1, V_2) is a monochromatic pair, then $c(V_1, V_j) = \{1\}$ for all $j > 1$ and the colored K_n is realized by a product with two factors given by the induced subgraphs $K_n[v \cup \bigcup_{i=2}^{m} V_i]$ and $K_n[V_1]$.

If (V_1, V_2) is a mixed pair, then $c(V_1, V_j) = \{1\}$ for all $j > 2$, $c(V_2, V_j) = \{2\}$ for all $j > 2$, and the colored K_n is realized by a product with four or five factors, depending on whether one of the subsets of the partitions of V_1 or V_2 is empty. The factors are the subgraphs induced by V_1', V_1'', V_2', V_2'', and $v \cup \bigcup_{i=3}^{m} V_i$.

The uniqueness follows immediately, since the colors on the join edges of the products are incident to every vertex in each of the three product structures.

□

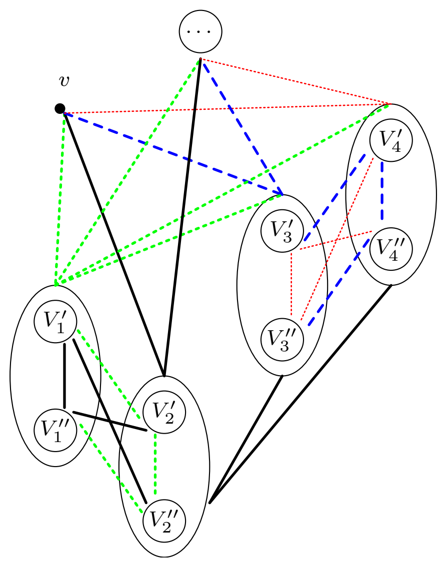

Figure 3. The product structures of any good coloring for Theorem 1. When V_1' and V_2' are both empty, the product has two factors. When V_1' is empty, the product has four factors. Otherwise the product has five factors.

There are no restrictions on the size of the factors in Theorem 1. When each factor is K_1, we get the three prime coloring patterns of Figure 1. Considering the possible arithmetic partitions of 5 of sizes

2, 4, and 5, the patterns shown in Figure 1 form a complete list of the good coloring patterns for K_5.
If we color each factor using one fewer color than its order, we have that each product can have up to $(r_0 - 1) + (r_1 - 1) + (1) = n - 1$, $(r_0-1)+(r_1-1)+(r_2-1)+(r_3-1)+(2) = n-2$, or $(r_0-1)+(r_1-1)+(r_2-1)+(r_3-1)+(r_4-1)+(2) = n-3$ colors. The full structural theorem then immediately gives the result of [6] that the good colorings using the maximum number of colors $n-1$ are only obtained from products with two factors with the maximum number of colors used on each factor and these sets of colors are distinct. The minimum number of colors is the maximum of the lower chromatic numbers corresponding to the factors plus 1 if there are two factors, or plus 2 if there are four or five factors. These minimums clearly cannot decrease as the order increases. Further, it is inductively clear, using products with two factors, that the spectra of these derived bihypergraphs are gap free. We are interested to see if any of the derived Ramsey Mixed Hypergraphs, for some combination of K_r and K_m, realize gaps in their chromatic spectra. We plan to investigate that possibility in future research continuing the project begun in this paper.

Define R_i^n to be the coefficients of the chromatic spectrum of the derived Ramsey Bihypergraph of K_n with bi-edges corresponding to the triangles of K_n.

Theorem 2. *The leading coefficient R_{n-1}^n is $(2n-3)!!$*

Theorem 2 follows immediately from the following lemma.

Lemma 2. *Each distinct good $(n-2)$-coloring of K_{n-1} extends to $(2n-3)$ distinct good $(n-1)$-colorings of K_n.*

Proof It is easy to check that each of the 3 distinct good 2-colorings of K_3 extends to 5 distinct good 3-colorings of K_4. Now assume the statement of the lemma is true for all orders up through $n-1$. Hence, each good $(r-1)$-coloring of K_r extends to $2(r+1)-3 = 2r-1$ good r-colorings of K_{r+1} for $r = 1, \ldots, (n-2)$. Pick any $v \in V(K_n)$ and any good $(n-2)$-coloring of K_{n-1} applied to $K_n - v$. From the structural theorem and comment following it, this coloring can be obtained as a product $[r, s]$ with $r + s = n - 1$ using $r - 1$ colors on the first factor and $s - 1$ distinct colors on the second factor. For convenience let's say the color on these join edges is green. To increase the number of colors used on K_n, at least one of the edges joining v to K_{n-1} must be a new color, say red. To avoid creating rainbow triangles, there are two possibilities: all of these edges can be red, or some of these edges joining v to one of the factors K_r or K_s are red while all of the edges joining v to the other factor are green. From the induction hypothesis we see there are $1+(2r-1)+(2s-1) = 2n-3$ possible good colorings of K_n produced from the selected good $(n-2)$-coloring of K_{n-1}.

□

The reasoning in the proof of Lemma 2 extends to products with four and five factors, however we can only obtain a much weaker result than that of Theorem 2.

Theorem 3. *The coefficient* R^n_{n-2} *is greater than* $(2n-5)R^{n-1}_{n-3}$.

Proof Theorem 3 is equivalent to the statement: each distinct good $(n-3)$-coloring of K_{n-1} extends to at least $(2n-5)$ distinct good $(n-2)$-colorings of K_n.

We begin the induction with K_5. The prime 2-colorings of K_4 each extend to 5 good 3-colorings of K_5 when we join a single vertex to K_4. However, each of the nonprime 2-colorings of K_4 extends to 7 good 3-colorings of K_5. Without subdividing this collection of good colorings, we have the weak relationship that $R^5_3 > 5R^4_2$. Note we are also ignoring any 3-colorings of K_5 that are extensions of good 3-colorings of K_4. Now assume the statement of the claim is true for all orders up through $n-1$. Pick any $v \in V(K_n)$ and any good $(n-3)$-coloring of K_{n-1} applied to $K_n - v$. From the structural theorem and comment following it, this coloring can be obtained as a product with two factors or as a product with four factors.

Treating first the cases when the coloring on K_{n-1} is a product with four factors, we again note that each of the factors must be colored with the maximum number of colors r_i-1, and there must be a new color, say red, used on at least one of the join edges leading to v when we extend this coloring to a coloring of K_n. It could be the case that all of the edges joining v to K_{n-1} are red. Otherwise, let's begin by assuming the red edges lead to only one factor, say K_{r_0}. To avoid rainbow triangles, the edges joining v to another factor, say K_{r_1}, must all be colored using the color on the edges joining the two factors K_{r_0} and K_{r_1}. Since we can replace r_1 with r_2 or r_3 in the previous observation, we also see that these are in fact the only possible patterns; either all of the edges are red, or there are only red edges leading to one factor and the edges leading to the other factors are determined by the product structure. As in the proof of Lemma 2, this good $(n-3)$-coloring on K_{n-1} extends to $2r_0-1+2r_1-1+2r_2-1+2r_3-1+1 = 2n-5$ good $(n-2)$-colorings on K_n.

In the cases when the good coloring on K_{n-1} is a product with two factors, and noting that we are requiring a new color be used on at least one edge joining v to K_{n-1}, there are three subcases possible. Either one of the factors is colored with the maximum number of colors $r_i - 1$ while the other is colored with one fewer than the maximum number, $r_j - 2$, and the colors used on the factors are distinct; or both factors are colored with the maximum number of colors but share one color. In the first subcase we use the inductive hypothesis that each good $(r_i - 2)$-coloring of K_{r_i} extends to at least $(2r_i - 3)$ distinct good (r_i-1)-colorings of K_{r_i+1} to get that this good $(n-3)$-coloring of K_{n-1} extends to at least $1+2r_i-3+2r_j-1 = 2n-5$ good $(n-2)$-colorings of K_n. In the last subcase, we have that the good coloring extends to at least $1 + 2r_0 - 1 + 2r_1 - 1 = 2n - 3$ good colorings. Not only is $2n - 3$ greater than $2n - 5$, but particular colorings that use the shared color on all of the edges incident with one vertex of each of the two factors permit additional extensions with edges joining v to these vertices possibly colored with this shared color. This completes the proof.

□

Though any good coloring of K_n must be an extension of a good coloring of K_{n-1}, Theorem 3 shows the need to analyze the details of a particular coloring pattern on K_{n-1} to determine how many extensions it allows. That analysis is even more complex when we do not increase the number of colors used. Chung and Graham show, in obtaining (2), that if we do not increase the number of colors used, the number of good colorings increases for a while and then decreases to zero. In [13] we first subdivide each spectral coefficient into terms correpsonding to individual patterns and then piece the subdivisions together to find recursive relationships between the spectral coefficients themselves. In this case, with these derived bihypergraphs, the product structure with only three prime patterns immediately produces a recursive relationship between the spectral coefficients. The details of the contributions made by different coloring patterns are inherent in this relationship with the repetition of all of the coefficients throughout the formula. This formula permits relatively efficient computation of the spectral coefficients, but does not easily permit more insight, of the form given above, of the growth patterns of these values.

To state the general recursions we need more terms. Since the recursion is very large, we use a single sigma representing summations over multiple parameters.

Let P_2, P_4, and P_5 be the sets of arithmetic partitions of n of size $2, 4$, and 5 with terms arranged in nonincreasing order. Elements of these sets form the products $[r_0, r_1]$, $[r_0, r_1, r_2, r_3]$, and $[r_0, r_1, r_2, r_3, r_4]$. Let $D_2^{[r_0,r_1]}, D_4^{[r_0,r_1,r_2,r_3]}$, and $D_5^{[r_0,r_1,r_2,r_3,r_4]}$ be the number of ways of decomposing K_n as the join of the factors K_{r_i} in the respective case. The numbers D_i^* are functions of a particular arithmetic partition and are multinomial coefficients divided by factorials corresponding to numbers of repeated terms in the partition.
We use $C_k^n = n!/k!/(n-k)!$ for the binomial coefficient n choose k and $P_k^n = n!/(n-k)!$ for the number of permutations of k objects chosen from n objects.

Theorem 4. *The general coefficient R_c^n is given by*

$$D_2^{[n-1,1]} R_{c-1}^{n-1} + 6D_4^{[n-3,1,1,1]} R_{c-2}^{n-3} + 6D_5^{[n-4,1,1,1,1]} R_{c-2}^{n-4} +$$
$$\sum_{P_2, r_1>1} D_2^{[r_0,r_1]} R_p^{r_0} R_q^{r_1} C_{z_1}^p P_{z_1}^q +$$
$$\sum_{P_4, r_1>1, r_2=1} 6D_4^{[r_0,r_1,1,1]} R_p^{r_0} R_q^{r_1} C_{z_2}^p P_{z_2}^q +$$
$$\sum_{P_4, r_2>1, r_3=1} 6D_4^{[r_0,r_1,r_2,1]} R_p^{r_0} R_q^{r_1} R_r^{r_2} C_u^p P_u^q C_w^{t_1} P_w^r +$$
$$\sum_{P_4, r_3>1} 6D_4^{[r_0,r_1,r_2,r_3]} R_p^{r_0} R_q^{r_1} R_r^{r_2} R_s^{r_3} C_u^p P_u^q C_v^{t_1} P_v^r C_x^{t_2} P_x^s +$$
$$\sum_{P_5, r_1>1, r_2=1} 6D_5^{[r_0,r_1,1,1,1]} R_p^{r_0} R_q^{r_1} C_{z_2}^p P_{z_2}^q +$$
$$\sum_{P_5, r_2>1, r_3=1} 6D_5^{[r_0,r_1,r_2,1,1]} R_p^{r_0} R_q^{r_1} R_r^{r_2} C_u^p P_u^q C_w^{t_1} P_w^r +$$
$$\sum_{P_5, r_3>1, r_4=1} 6D_5^{[r_0,r_1,r_2,r_3,1]} R_p^{r_0} R_q^{r_1} R_r^{r_2} R_s^{r_3} C_u^p P_u^q C_v^{t_1} P_v^r C_x^{t_2} P_x^s +$$
$$\sum_{P_5, r_4>1} 6D_5^{[r_0,r_1,r_2,r_3,r_4]} R_p^{r_0} R_q^{r_1} R_r^{r_2} R_s^{r_3} R_t^{r_4} C_u^p P_u^q C_v^{t_1} P_v^r C_j^{t_2} P_j^s C_y^{t_3} P_y^t,$$

where $t_1 = p+q-u$, $t_2 = p+q-u+r-v$, $t_3 = p+q-u+r-v+s-j$, $z_1 = p+q-c+1$, $z_2 = p+q-c+2$, $w = t_1+r-c+2$, $x = t_2+s-c+2$, $y = t_3+t-c-2$, and each of the sums is over all possible choices of colors on the factors so that the total number of colors used is $c-2$ (except the first sum which has $c-1$ colors) and over each of the possible numbers of intersections of colors between the factors.

Proof We only explain the last summation, as the earlier terms follow from a similar counting argument, where we must separate cases based on how many factors of each product are K_1. The last sum gives the good c-colorings obtained from products with five factors which are all bigger than K_1. The symbol D_5^* is the number of ways of decomposing K_n with these factors, and there are six distinct coloring patterns possible on the edges joining these factors for each decomposition and each of the possible coloring patterns on the individual factors. We must multiply by the numbers of distinct good colorings of the factors with p, q, r, s, and t colors used on each of the factors, where $p + q + r + s + t \geqslant c - 2$. The first two factors may share u colors. If so, we must multiply by the number of ways of choosing the common u colors from the p colors used on the first factor when we color the second factor. Moreover, we get different coloring patterns by permuting these common colors amongst themselves in the coloring of the second factor. The third factor may share v colors with the union of the first two factors. The fourth factor may share j colors with the union of the first three factors. The term y is the number of colors the fifth factor must share with the previous four factors, given the particular choices made on the colors of the previous factors.

□

In the context of finding the chromatic polynomial, or chromatic spectrum, the above recursion is interesting for the complexity of the recursion that can exist even for a 3-regular bihypergraph. Perhaps most interesting are the factors of 6 that represent a subset of the good 2-colorings and not the entire spectral value R_2^4. Are there other examples of ordered families of mixed hypergraphs which exhibit recursive relationships which require a finer decomposition than that which is given by the full chromatic spectrum? In the example given above, that finer level of decomposition only required subdividing one set of feasible partitions. Are there other families that require subdividing more of the spectral values in recursive relationships? In [13] we conjecture that this may be the case for uniform complete interval bihypergraphs with edge size bigger than 4.

4 The spectra for K_2 through K_{12}

Corollary 1. *The chromatic spectrum values for the derived Ramsey Bihypergraphs corresponding to K_2 through K_{12} are as follows:*

K_2: $[1]$
K_3: $[0, 3]$
K_4: $[0, 9, 15]$
K_5: $[0, 6, 165, 105]$
K_6: $[0, 0, 846, 2790, 945]$
K_7: $[0, 0, 3402, 42273, 49770, 10395]$
K_8: $[0, 0, 10836, 557928, 1604925, 961695, 135135]$
K_9: $[0, 0, 19278, 6972966, 44972172, 55829655, 20210715, 2027025]$
K_{10}: $[0, 0, 14742, 81569754, 1201982166, 2778115725, 1906370235, 460971000, 34459425]$
K_{11}: $[0, 0, 0, 875500164, 31404779406, 130507877808, 151891001955, 65874757410, 11365685100, 654729075]$
K_{12}: $[0, 0, 0, 8588423844, 808974051732, 6002106197472, 11277243566820, 7842349288620, 2338009242030, 301618981125, 13749310575]$

The data in Corollary 1 were computed using Java following the recursive relationships in Theorem 4. After first generating the sets of arithmetic partitions P_2, P_4, and P_5 and computing the number of decompositions D_2, D_4, and D_5 for each such partition, one follows the cases of the recursion to compute the number of distinct good colorings contributed by each arithmetic partition. This program can be extended to find the spectra corresponding to larger K_n using big integers to avoid errors created due to loss of memory in Java with long variables after K_{12}. Using longs, Java takes less than 2 seconds (on a Dell desktop with an Intel(R) Core(TM) i7 CPU 860 processor) to compute these spectral values.

We also computed most of the data in Corollary 1 using brute force sorting algorithms to count all good colorings via feasible partitions. Once labels and an ordering of the edges of K_n are chosen, restricted growth strings encode the feasible partitions, see [11] for instance. We choose to label the edges lexically, which means we first label all the edges incident to vertex 0 and then the remaining edges incident to vertex 1 etc. To illustrate this ordering, the distinct good edge colorings of K_5 used as the representatives of the isomorphism classes shown in Figure 1 correspond to the following feasible partitions:

$\{\{0,3,4,7,9\},\{1,2,5,6,8\}\}$

$\{\{0,4,7\},\{1,2,5\},\{3,6,8,9\}\}$, $\{\{0,7\},\{1,2,4,5\},\{3,6,8,9\}\}$, $\{\{0,9\},\{1,4\},\{2,3,5,6,7,8\}\}$, $\{\{0\},\{1,4,9\},\{2,3,5,6,7,8\}\}$, $\{\{0\},\{1,4,7,9\},\{2,3,5,6,8\}\}$

$\{\{0\},\{1,4\},\{2,5,7\},\{3,6,8,9\}\}$, $\{\{0\},\{1,2,4,5\},\{3,6,8,9\},\{7\}\}$, $\{\{0\},\{1,4\},\{2,3,5,6,7,8\},\{9\}\}$

By labeling edges lexically, one can easily determine the labels on all of the triangles. Generate the first $n-1$ entries of the growth string. Upon the generation of the nth entry we encounter a condition on the strings due to a triangle. We then continue with the process only if this condition, that exactly two colors are used, is met. As we generate each additional term of the string we encounter more triangles and only continue the generation of the string if the corresponding conditions are met. The number of triangles encountered at each stage of the string generation goes up by one once the first index on the edge increases. Strings are counted if they make it to the end of the generation process based on their highest entry, and these counts give the chromatic spectral values. Using the same machine as above, it took 3.5 hours to run the sorting algorithm for K_{10}, and it took 75 days to run the sorting algorithm for K_{11}. Comparing the total number of feasible partitions with the total number of possible partitions, counted by Bell numbers, we get approximate densities of $100, 60, 11.82, 0.23, 3.33\times 10^{-4}, 2.23\times 10^{-8}, 5.31\times 10^{-14}, 3.40\times 10^{-21}, 4.64\times 10^{-30}, 1.09\times 10^{-40}$, and 3.68×10^{-53} percent respectively. With a decrease in density in the order of 10^{10} and the corresponding increase in computing time to run the sorting algorithms, and another decrease in density in the order of 10^{13} going from K_{11} to K_{12}, we chose not to run the sorting algorithm for K_{12}.

Our code is available to the reader upon request.

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