Taxonomy of Strategic Games with Information Leaks and Corruption of Simultaneity

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Abstract

We consider pseudo-simultaneous normal form games — strategic games with rules violated by information leaks and simultaneity corruption. We provide classification and construction of a game taxonomy based on applicable solution principles. Existence conditions are highlighted, formulated and analysed.

Keywords: Non-cooperative game, Nash equilibrium, simultaneous and sequential games, Stackelberg equilibrium, knowledge, information leak, corruption, taxonomy.

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1 Introduction

Strategic or normal form game constitutes an abstract mathematical model of decision processes with two or more decision makers (players) [5], [6]. An important supposition of the game is that all the players choose their strategies simultaneously and confidentially, and that everyone determines his gain on the resulting profile. Reality is somewhat diverse. The rules of the games may be broken. Some players may cheat and know the choices of the other players. So, the rule of confidentiality and simultaneity is not respected. Is the essence of initial normal form game change in such games? Is still the Nash equilibrium principle applicable? Do we need other solution principles and other interpretations? How many types of games appear and may they be classified? Can we construct a taxonomy (classification) of these games? Answers to these and other related questions are the objective of presented work.

Usually, the traditional research approach to games of such types relies on consideration of all possible players' best response mappings and analysis of all possible profiles [3]. There is a stable opinion about a high complexity of their analysis and solving [3].

We initiate an approach which sets rules of all possible games with information leaks and highlights their specific characteristics. The approach relies on knowledge vectors of the players and game knowledge net. A taxonomy (classification) of all possible games is done on the bases of the applicable solution principles. The name of every taxon (class) reflects the principle used for including respective games in the same taxon.

As a result of the taxonomy construction and establishing strict characteristics and rules for every taxon, we reveal simplicity of analysis and solving the games. It is an unexpected and impressive result.

For the beginning, let us remember that a Nash equilibrium (NE) sample and the entire Nash equilibrium set (NES) may be determined via intersection of the graphs of best response mappings — a method considered earlier in works [8]–[13]. The approach proves to be expedient for strategic games with information leaks and broken simultaneity as well. Initially, we expose the results for two-matrix games with different levels of knowledge. Then, we expose results for the general multi-matrix game. It is a useful approach both for simplicity of exposition, and for understanding the ideas and results.

1.1 Normal form game and axioms

Consider finite strategic (normal form) game

$$\Gamma = \left\langle N, \{S_p\}_{p\in N}, \{a_{\mathbf{s}}^p = a_{s_1 s_2 \dots s_n}^p\}_{p\in N} \right\rangle,$$

where

$N = \{1, 2, ..., n\}$ is a set of players,

$S_p = \{1, 2, \dots, m_p\}$ is a set of strategies of player $p \in N$,

$s_p \in S_p$ is a strategy of $p \in N$ player,
$\#S_p = m_p < +\infty$, $p \in N$, is a finiteness constraint,
$a^p_{\mathbf{s}} = a^p_{s_1 s_2 \ldots s_n}$ is a player's $p \in N$ pay-off function defined on Cartesian product $S = \underset{p \in N}{\times} S_p$, i.e. for every player $p \in N$ a n dimensional pay-off matrix $A^p[m_1 \times m_2 \times \cdots \times m_n]$ is defined.

For normal form game a system of axioms is stated.

Axiom 1.1. Rationality. *The players behave rationally. The rationality means that every rational player optimizes the value of his pay-off function.*

Axiom 1.2. Knowledge. *The players know the set of players, the strategy sets and the pay-off functions.*

Axiom 1.3. Simultaneity. *The players choose their strategy simultaneously and confidentially in a single-act (single stage) without knowing the chosen strategies of the other players.*

Axiom 1.4. Pay-off. *After all strategy selection the players compute their pay-off as the values of their pay-off functions on the resulting profile.*

Traditionally, simultaneous Nash games [5], [6] are based on these four axioms.

In Stackelberg game the axiom of simultaneity is replaced by the axiom of hierarchy.

Axiom 1.5. Hierarchy. *The players choose their strategies in a known order, e.g. the first player chooses his strategy and communicates it to the second player. The second player (follower) knows the strategy of the leader, chooses his strategy and communicates it to the third player and so on. The last player knows all the strategies of the precedent players and chooses his strategy the last.*

Both the Nash game, and the Stackelberg game, are based commonly on axioms of rationality, knowledge and pay-off. Additionally and distinctively the Nash game [5] is based on axiom of simultaneity, while the Stackelberg game [4], [7] is based on axiom of hierarchy.

Finally, we can deduce that in Nash game all the players choose their strategies simultaneously and every player determines his gain as the value of his pay-off function on the resulting profile. But in Stackelberg game the players choose their strategies sequentially, in a known order and knowing the strategies chosen by the precedent players, and every player determines his gain as the value of his pay-off function on the resulting profile.

1.2 Axiom of simultaneity and its corruption

Both for Nash games, and for Stackelberg games, their own solution principles exist. If the axioms of the games are respected, these solution principles may be applied. Actually, the name of the games are chosen to reflect the solution concept which is applicable.

But, what does it happen when the axioms are violated by the corruption of some of their elements, e.g. some players may know chosen strategies of the other players in Nash games? May such games be examined by applying the same general solution principles (Nash and Stackeberg equilibria) or must new solution concepts be defined and applied?

To respond to these questions, we need to avoid the ambiguity. So, let us examine more exactly the process of decision making in conditions of information leaks, by establishing the axioms of the corrupt games. It is very convenient to consider in such case a manager of the games — a person (or persons) which organizes and manages the decision process in the games. Thereby, we can describe exactly the process of decision making in corrupt games, knowing the source of corruption[1].

At the **first pseudo-stage** of the decision process, the manager declares the players must choose their strategies. After players choose their strategies (intentions), the manager dishonestly from the view-

[1]Corruption: "the abuse of entrusted power to private gain" (Transparency International); "dishonest or fraudulent conduct by those in power, typically involving bribery" (Oxford Dictionary, 2014).

point of the rules of the strategy game may submit to some players information about chosen strategies (corruption and information leaks). With additional information, some of players may want to change their strategies and they can do this at the second pseudo-stage.

At the **second pseudo-stage**, the manager declares the players must submit immediately their choices. At this moment, the players may change their initial decisions. After possible changes in their intentions the players submit definitely their chosen strategies.

For an honest player, the decision process looks like a mono stage process. Only for the dishonest manager and for the players which obtain additional information, the decision process looks like a two stage process.

As an axiom of such a game the axiom of information leak may be stated as

Axiom 1.6. Information leak. *The decision process has two pseudo-stages.*

At the **first pseudo-stage***, the information leak about player chosen strategies may occur.*

At the **second pseudo-stage***, the players choose their strategies, some of them knowing eventually the strategies chosen by the other players.*

Definition 1.1. *Let us define a game with information leak or a corrupt game as a game for which four axioms are fulfilled: rationality, knowledge, pay-off and information leak.*

Remark 1.1. *Let us observe that three axioms of rationality, knowledge and pay-off, are common for Nash game, Stackelberg game and corrupt game (game with information leak).*

Remark 1.2. *Generally, the game with information leak is only a particular case of the corrupt game. We will use interchangeably this name unless we will define a more general context of corrupt game.*

Remark 1.3. *The game with information leak as it is defined above actually includes different types of games.*

To respond to the various questions which appear in the context of corrupt games we consider further the taxonomy (classification) of the all possible types of games, principles of solutions, solution existence conditions and algorithms for solutions determining. The exposition will start with two-matrix games, but firstly we must highlight shortly in this context the essence of so named theory of moves.

1.3 Theory of moves

We must observe that the games we consider in this work may be related to the theory of moves [2]. Nevertheless, it is an important difference — we consider only two pseudo-stages of the decision making process, while the theory of moves does not limit the number of moves to one fixed number. Moreover, theory of moves has initial axioms which are defined in a strict manner as the process of decision making, as the end condition. Additionally, those axioms differ from that accepted for games with information leaks.

The theory of moves is based on the concepts of thinking ahead, stable outcomes, outcomes induced when one player has "moving power", incomplete information, non-myopic concept of equilibrium, etc. The non-myopic equilibrium depends on some parameters, such as, e.g., initial state from which the process of moving starts and who moves the first. It is essential that all games have at least one non-myopic equilibrium. In the games we consider, there are different solution concepts and it is not guaranteed that the solutions exist. These thoughts we expose further.

2 Taxonomy of two-matrix games with information leaks

For two-matrix strategic games, we suppose the process of making decision occurs in two pseudo-stages, because of possible information leaks. At the first pseudo-stage, the players choose their strategies and by corruption, it is possible either for one of them, or for both players, to know the chosen (intention) strategy of the opponent. At the second pseudo-stage, the players use the obtained information, choose their strategies and no more corruption is possible.

First, for such processes of decision making we can distinguish **simultaneous** and **sequential** two-matrix games.

Second, simultaneous two-matrix games may obtain some features of sequential games taking into consideration the obtained information/knowledge ($\gamma\nu\omega\sigma\eta$) possessed by each player in the process of realizing the game.

Remark 2.1. *We suppose initially, when the players begin the strategy selections, they start playing a Nash game, but in the process of strategy selections information leak may occur, the Nash game may degenerate and may change its essence.*

Remark 2.2. *In order to distinguish players without their numbers, we will refer to them as the player and his opponent. So, if the first player is referred to simply as the player, then the second player is referred to as the opponent, and vice versa.*

2.1 Knowledge and types of games

The knowledge of the players is associated with their knowledge vectors γ^A and γ^B.

2.1.1 Knowledge vectors

Essentially, the knowledge vectors have an infinite number of components $\gamma^A = (\gamma_0^A, \gamma_1^A, \dots)$ and $\gamma^B = (\gamma_0^B, \gamma_1^B, \dots)$, with components defined and interpreted as it follows.

- **Player's knowledge of the normal form components.** γ_0^A and γ_0^B are reserved to knowledge about the normal form of the game. Values $\gamma_0^A = 1$ and $\gamma_0^B = 1$ mean that the players have full information about the strategy sets and pay-off functions. It is the case we consider in this work, i.e. mutually $\gamma_0^A = 1$ and $\gamma_0^B = 1$, and these components of the knowledge vectors are simply omitted.

- **Player's knowledge of the opponent's chosen strategy.** Values $\gamma_1^A = 0$ and $\gamma_1^B = 0$ mean for each player, correspondingly, that he doesn't know the opponent's strategy. $\gamma_1^A = 1$ and $\gamma_1^B = 1$ mean for each player, correspondingly, that he knows the opponent's strategy. The combined cases are possible, too.

- **Player's knowledge of the opponent's knowledge of the player's chosen strategy.** $\gamma_2^A = 0$ and $\gamma_2^B = 0$ mean for each player, correspondingly, that he knows that the opponent doesn't know player's strategy. $\gamma_2^A = 1$ and $\gamma_2^B = 1$ mean for each player, correspondingly, that he knows that the opponent knows player's strategy. Evidently, the combined cases are possible, too. Remark that these components may be thought rather as the players beliefs, because such type of knowledge may be as true, as false. In this context it must be observed, that the values of γ_2^A and γ_2^B represent the knowledge/belief about the values of γ_1^B and γ_1^A, correspondingly.

- **The next components $\gamma_3^A, \gamma_4^A, \ldots$ and $\gamma_3^B, \gamma_4^B, \ldots$ of the knowledge vectors are omitted, initially.** Nevertheless, it must be remarked that the values of γ_i^A and γ_i^B represent the knowledge/belief about the values of γ_{i-1}^B and γ_{i-1}^A, correspondingly.

We distinguish the games with l levels of knowledge, for which all components of the knowledge vectors with indices greater than l are equal to 0.

Remark 2.3. *Remark, once again, that there are two pseudo-levels of decision making process. Information leaks may occur only at the first pseudo-level. The knowledge vectors may have any number $l \geq 1$ of components (levels of knowledge).*

2.1.2 Types of games

Depending on the values of the knowledge vectors, different types of games may be considered.

Proposition 2.1. *There are* 4^l *possible types of games* $\Gamma_{\gamma^A\gamma^B}$ *with* l *levels of knowledge.*

Proof. It is enough to emphasize the components of the knowledge vectors $\gamma^A = \left(\gamma_1^A, \dots, \gamma_l^A\right)$ and $\gamma^B = \left(\gamma_1^B, \dots, \gamma_l^B\right)$ and their possible values as 0 or 1. Accordingly, there are 4^i possible pairs of such vectors, i.e. 4^l possible games. □

2.1.3 Knowledge net

Knowledge net is defined as:

$$G = (V, E)\,,$$

where $V = I \cup J \cup \gamma_A \cup \gamma_B, E \subseteq V \times V$.

For the present we will limit ourselves to knowledge vectors.

2.2 Taxonomy Elements

If the information leaks occur only at the first 2 levels, then there are $4^2 = 16$ possible kinds of games with information leaks with $\gamma^A = \left(\gamma_1^A, \gamma_2^A\right)$ and $\gamma^B = \left(\gamma_1^B, \gamma_2^B\right)$, according to the above. From the solution principle perspective, some of them are similar and they may be included in common taxa (classes, families, sets).

Let us highlight the possible kinds of such games by the values of low index, where the first two digits are the values for knowledge vector components of the first player, and the following two digits are the values for knowledge vector components of the second player. We obtain the following taxonomy for two matrix games with information leaks on two levels:

1. Nash taxon: $\mathrm{NT} = \{\Gamma_{00\,00}, \Gamma_{11\,11}, \Gamma_{00\,11}, \Gamma_{11\,00}\}$,

2. Stackelberg taxon: $\mathrm{ST} = \{\Gamma_{01\,10}, \Gamma_{01\,11}, \Gamma_{10\,01}, \Gamma_{11\,01}\}$,

3. Maximin taxon: $\mathrm{MT} = \{\Gamma_{01\,01}\}$,

4. Maximin-Nash taxon: $\mathrm{MNT} = \{\Gamma_{00\,01}, \Gamma_{01\,00}\}$,

5. Optimum taxon: $\text{OT} = \{\Gamma_{10\,10}\}$,
6. Optimum-Nash taxon: $\text{ONT} = \{\Gamma_{00\,10}, \Gamma_{10\ 00}\}$,
7. Optimum-Stackelberg taxon: $\text{OST} = \{\Gamma_{10\,11}, \Gamma_{11\,10}\}$.

The generic name of each taxon is selected on the basis of the correspondent solution principles applied by the players: Nash equilibrium, Stackelberg equilibrium, Maximin principle, Optimum principle or two of them together. Even though the taxon may include some games, the name reflects solution principle or principles applied in all the games of the taxon. If the taxon is formed only by one element, its name is the same as for game.

Remark 2.4. *We choose the term taxon (plural — taxa) to name the set of the related games in order to highlight additionally their acquired pseudo-dynamics [1],[4],[7] and to avoid confusion with mathematically overcharged or too used terms of class, cluster, family, group or set.*

Let us investigate the solution principles for all these taxa.

3 Solution principles of two-matrix games with information leaks on two levels of knowledge

Consider a two-matrix $m \times n$ game Γ with matrices

$$A = (a_{ij}),\ B = (b_{ij}),\ i \in I, j \in J,$$

where $I = \{1, 2, \ldots, m\}$ is the set of strategies of the first player, and $J = \{1, 2, \ldots, n\}$ is the set of strategies of the second player.

We consider the games base on four axioms of rationality, knowledge, pay-off and information leak (axioms 1.1, 1.2, 1.4, 1.6). The players choose simultaneously their strategies and before submitting the results of their selections, the information leaks may occur. One or both of them may know the intention of the opponent. Let us suppose in such case, they may change only once their strategy according to the leaked information. So, the strategic games may be transformed by acquiring additional information into two stage games. At the first stage they choose strategies but do not submit them because of acquiring additional information. At the second stage, according to the leaked information they may change the initial strategies and submit definitely new strategies adjusted to the obtained information. After such submission, the games end and both players determine the values of their pay-off functions.

Evidently, other types of games with information leaks may be considered. Firstly, we will limit ourselves only to such two-pseudo-stage games with information leaks on two levels of knowledge.

For every taxon we will firstly define it, after that we will argue its consistency.

3.1 Nash Taxon

Let us argue that for $\mathrm{NT} = \{\Gamma_{00\,00}, \Gamma_{11\,11}, \Gamma_{00\,11}, \Gamma_{11\,00}\}$ all its elements are Nash games, i.e. axioms 1.1–1.4 are characteristic, too, for these games and for them Nash equilibrium principle may be applied as a common solution principle.

Firstly, let us remember, that the process of decision making in the Nash game, denoted by $\mathrm{N}\Gamma$, is described in the following way. Simultaneously and confidentially, the first player selects the lines i^* of the matrices A and B, and the second player selects columns j^* of the same matrices. The first player gains $a_{i^*j^*}$, and the second player gains $b_{i^*j^*}$.

Evidently, $\Gamma_{00\,00}$ is a pure Nash game, i.e. $\Gamma_{00\,00} = \mathrm{N}\Gamma$. But, it is not difficult to understand that $\Gamma_{11\,11}$, $\Gamma_{00\,11}$, $\Gamma_{11\,00}$, are Nash games, too. So, the taxon (group) is formed by four Nash games, differing only by the knowledge/belief of the players.

If we call the player which applies a Nash equilibrium strategy as an atom (a Nash atom, Nash atomic player) and denote him as N, then the two-player Nash game may be denoted as N_2 (Nash game, Nash molecular game).

Remark 3.1. *We will name and denote in the same manner other types of players and games.*

3.1.1 Nash Equilibrium

The pair of strategies (i^*, j^*) forms a Nash equilibrium if

$$a_{i^*j^*} \geq a_{ij^*}, \forall i \in I,$$
$$b_{i^*j^*} \geq b_{i^*j}, \forall j \in J.$$

3.1.2 Set of Nash Equilibria

An equivalent Nash equilibrium definition may be formulated in terms of graphs of best response (optimal reaction) applications (mappings).

Let

$$\mathrm{Gr_A} = \left\{ (i,j) : \, j \in J, \, i \in \operatorname*{Argmax}_{k \in I} a_{kj} \right\},$$

be the graph of best response application of the first player, and

$$\mathrm{Gr_B} = \left\{ (i,j) : \, i \in I, \, j \in \operatorname*{Argmax}_{k \in J} b_{ik} \right\}.$$

be the graph of best response application of the second player.

$$\mathrm{NE} = \mathrm{Gr_A} \cap \mathrm{Gr_B}$$

forms the set of Nash equilibria.

3.1.3 Nash Equilibrium Existence

Proposition 3.1. *There are Nash games which do not have a Nash equilibrium.*

Proof. Examples of games which do not have a Nash equilibrium are commonly known. □

Remark, the games we consider are pure strategy games. It is a largely known result that every poly-matrix strategic game has Nash equilibria in mixed strategies. In this work we consider only pure strategy games.

3.2 Stackelberg Taxon

Stackelberg Taxon is defined as $\mathrm{ST} = \{\Gamma_{01\,10}, \Gamma_{01\,11}, \Gamma_{10\,01}, \Gamma_{11\,01}\}$. To argue the inclusion of each element in ST, let us remember the decision making process in the Stackelberg game.

Stackelberg two player game has two stages, from the start, and for Stackelberg game the axioms 1.1, 1.2, 1.4 and 1.5 are characteristic. At the first stage, the first player (leader) selects the lines i^* of the matrices A and B, and communicates his choice to the second player (follower). At the second stage, the second player (follower) knows the choice of the first player (leader) and selects columns j^* of the matrices A and B. The first player gains $a_{i^*j^*}$, and the second player gains $b_{i^*j^*}$. If the players change their roles as the leader and the follower, an another Stackelberg game is defined.

The Stackelberg game is denoted by SG_{12} if the first player is the leader and by SG_{21} if the second player is the leader.

$\Gamma_{01\,10}$ is a pure Stackelberg game $\mathrm{S}\Gamma_{12}$, i.e. $\Gamma_{01\,10} = \mathrm{S}\Gamma_{12}$, and $\Gamma_{10\,01}$ is a pure Stackelberg game $\mathrm{S}\Gamma_{21}$, i.e. $\Gamma_{10\,01} = \mathrm{S}\Gamma_{21}$. It is clear that $\Gamma_{01\,11} = \mathrm{S}\Gamma_{12}$ and $\Gamma_{11\,01} = \mathrm{S}\Gamma_{21}$.

3.2.1 Stackelberg Equilibrium

The pair of strategies $(i^*, j^*) \in \mathrm{Gr_B}$ forms a Stackelberg equilibrium if

$$a_{i^*j^*} \geq a_{ij}, \forall (i,j) \in \mathrm{Gr_B}.$$

If the players change their roles and the second player is the leader, then the pair of strategies $(i^*, j^*) \in \mathrm{Gr_A}$ forms a Stackelberg equilibrium if

$$b_{i^*j^*} \geq b_{ij}, \forall (i,j) \in \mathrm{Gr_A}.$$

3.2.2 Set of Stackelberg Equilibria

The sets of Stackelberg equilibria are generally different for Stackelberg games $\mathrm{S}\Gamma_{12}$ and $\mathrm{S}\Gamma_{21}$.

$$\mathrm{SE}_{12} = \underset{(i,j)\in \mathrm{Gr_B}}{\mathrm{Argmax}}\ a_{ij}$$

forms the set of Stackelberg equilibria in a Stackelberg game $\mathrm{S}\Gamma_{12} = \mathrm{S}_{12}$.

$$\mathrm{SE}_{21} = \operatorname*{Argmax}_{(i,j)\in \mathrm{Gr_A}} b_{ij}$$

forms the set of Stackelberg equilibria in a Stackelberg game $\mathrm{S}\Gamma_{21} = \mathrm{S}_{21}$.

It is evident that the notions of Nash and Stackeberg equilibria are not identical. The respective sets of equilibria may have common elements, but the sets generally differ.

3.2.3 Stackelberg Equilibrium Existence

Proposition 3.2. *Every finite Stackelberg game has a Stackelberg equilibrium.*

Proof. The proof follows from the Stackelberg equilibrium definition and the finiteness of the player strategy sets. □

3.3 Maximin Taxon

Maximin Taxon contains only one element $\mathrm{MT} = \{\Gamma_{01\,01}\}$.

The decision making process in the Maximin game $\mathrm{M}\Gamma = \mathrm{M}_2$ follows the axioms 1.1–1.4 as for Nash game. Simultaneously and secretly, as in Nash Game, the first player selects the lines i^* of the matrices A and B, and the second player selects columns j^* of the same matrices. Unlike the Nash game, every player suspects that the opponent may know his choice, i.e. distinction of the Maximin game consists in player attitudes.

3.3.1 Maximin Solution Principle

Players compute the set of their pessimistic strategies.

$$\mathrm{MS_A} = \mathrm{Arg}\max_{i\in I}\min_{j\in J} a_{ij}$$

forms the set of pessimistic strategies of the first player.

$$\mathrm{MS_B} = \mathrm{Arg}\max_{j\in J}\min_{i\in I} b_{ij}$$

forms the set of pessimistic strategies of the second player.

Every element of Cartesian product $\mathrm{MS} = \mathrm{MS_A} \times \mathrm{MS_B}$ forms a maximin solution of Maximin Game $\mathrm{M}\Gamma = \mathrm{M}_2$.

3.3.2 Set of Maximin Solutions

$\mathrm{MS} = \mathrm{M_A} \times \mathrm{M_B}$ is the set of Maximin Solutions of the Maximin Game.

Proposition 3.3. *For matrices A and B the sets* NE, SE_{12}, SE_{21} *and* MS *are generally not identical.*

Proof. It is enough to mention that every Stackelberg game has Stackelberg equilibria and every Maximin game has the maximin solution, but the Nash game with the same matrices may do not have Nash equilibria. Even though the Nash game has equilibria, simple examples may be constructed which illustrate that Nash equilibrium is not identical with the Stackelberg equilibrium and the Maximin solution. □

3.3.3 Maximin Solution Existence

Proposition 3.4. *Every finite Maximin Game has maximin solutions.*

Proof. The proof follows from the finiteness of the strategy sets. □

3.4 Maximin-Nash Taxon

Maximin-Nash Taxon contains two elements:

$$\mathrm{MNT} = \{\Gamma_{00\,01}, \Gamma_{01\,00}\}.$$

Let us suppose, without loss of generality, that the players choose their strategies without knowing the opponent choice. However, one of them (and only one) has the belief that there is information leak about the chosen strategy. Let us denote such a game by $\mathrm{MN}\Gamma = \mathrm{MN}$ or $\mathrm{NM}\Gamma = \mathrm{NM}$.

3.4.1 Maximin-Nash Solution Principle

For defining the solution concept of such games, we can observe firstly that they may be seen as a constrained Nash Game $\Gamma_{00\,00}$, in which additionally must be applied the Maximin principle for the pessimistic player which suspects the corruption. So, for $\Gamma_{00\,01}$ we can define as the solution any element from:

$$\mathrm{NMS} = \mathrm{NE} \cap (\mathrm{I} \times \mathrm{MS_B}),$$

For $\Gamma_{01\,00}$ the solution is any element from:

$$\mathrm{MNS} = \mathrm{NE} \cap (\mathrm{MS_A} \times \mathrm{J}).$$

From the above definitions, it follows that a Maximin-Nash Solution is a Nash Equilibrium for which one of it's components (corresponding to the player which suspects corruption) is a Maximin strategy, too.

3.4.2 Set of Maximin-Nash Solutions

NMS is the set of solutions in game $\mathrm{NM} = \Gamma_{00\,01}$, and MNS is the set of solutions in game $\mathrm{MN} = \Gamma_{01\,00}$.

3.4.3 Maximin-Nash Solution Existence

Proposition 3.5. *If Maximin-Nash Game* MN *has a solution, then the Nash Game has a Nash equilibrium.*

Proof. The proof follows from the definition of the Maximin vs Nash solution. □

Generally, the reciprocal proposition is not true.

3.5 Optimum Taxon

Optimum Taxon is formed only by one element $\mathrm{OT} = \{\Gamma_{10\,10}\}$.

The player strategies are selected as it follows. Let us suppose, that the both players declare they play Nash game, but everyone cheats and (by corruption and information leaks) knows the choices of the opponent. Such a game is denoted by $\mathrm{O}\Gamma = \mathrm{O}_2$.

To formalize this game we must highlight two pseudo-stages of the game. The first pseudo-stage when the players initially choose their strategies (i_0, j_0). And the second pseudo-stage when the players, after knowing (i_0, j_0), may choose their final strategies (i_1, j_1).

3.5.1 Optimum Profile

As everyone do not suspect opponent of cheating, but the both cheat, they play as followers, i.e., in the game O_2 the players act as followers.

The resulting profile is (i_1, j_1), where $i_1 \in \operatorname*{Argmax}_{i \in I} a_{ij_0}$ and $j_1 \in \operatorname*{Argmax}_{j \in J} b_{i_0 j}$.

As both i_1 and j_1 correspond to j_0 and i_0, correspondingly, the pair (i_1, j_1) is not a solution concept. It is a simple profile — an Optimum Profile.

3.5.2 Set of Optimum Profiles

For this game we can define only the set of Optimum Profiles:

$$\mathrm{O}_2\mathrm{P}(i_0, j_0) = \left(\operatorname*{Argmax}_{i \in I} a_{ij_0}, \operatorname*{Argmax}_{j \in J} b_{i_0 j} \right).$$

3.5.3 Optimum Profiles Existence

We mentioned above that the OT taxon is based on Optimum Profile, which is generally not a solution concept. Nevertheless, we may conclude the Optimum Profile exists for every finite game, because of strategy finiteness.

3.6 Optimum-Nash Taxon

This Taxon has two symmetric elements $\mathrm{ONT} = \{\Gamma_{00\,10}, \Gamma_{10\,00}\}$.

Let us suppose, the players declare they play Nash game, but one of them cheats and (by corruption and information leaks) knows the choice of the opponent. We denote this game by $\mathrm{ON}\Gamma = \mathrm{ON}$ or $\mathrm{NO}\Gamma = \mathrm{NO}$.

To formalize this game we highlight two stages of the game, as in the precedent case. The first stage is when the players initially choose their strategies (i_0, j_0). And the second stage is when the cheater changes his strategy as optimal to the opponent strategy. So, at the second stage the strategy (i_0, j_1) or (i_1, j_0) is realised.

3.6.1 Optimum-Nash Profile Principle

As in the case of the Maximin vs Nash Game, for defining the solution concept we can observe firstly that if they play Nash Game $\Gamma_{00\,00}$, i.e., they choose to play a Nash Equilibrium, the cheating is not convenient. For such games, Nash Equilibrium is the solution principle to apply. If the honest player does not play Nash Equilibrium Strategy, he may lose out comparably with the Nash Equilibrium. So, he plays Nash Equilibrium. In such case, for the cheater it is convenient to play a Nash Equilibrium strategy.

As a conclusion, this type of game may be thought as a Nash Game if the game has Nash equilibrium. If the game doesn't have Nash Equilibrium or it has many Nash Equilibria, the principle of the Optimum-Nash profile is applied. One of them chooses his strategy as in Nash game (leader). He can apply the maximin or the Stackelberg strategy of the leader. The opponent chooses his strategy as the last player in Stackelberg game (follower).

3.6.2 Set of Optimum-Nash Profiles

Evidently, if the honest player chooses the Nash Equilibrium Strategy, the set of solutions is identical to NES.

If the honest player chooses maximin strategy, e.g. the first player chooses one of the elements of $\mathrm{MS_A} = \mathrm{Arg}\max\limits_{i\in I}\min\limits_{j\in J} a_{ij}$, the opponent chooses every element from $J^* = \mathrm{Arg}\max\limits_{j\in J} b_{ij}$.

If the honest player chooses Stackelberg leader strategy, the opponent chooses the follower strategy. In such case, the ON Profile is a Stackelberg equilibrium.

3.6.3 Optimum-Nash Profile Existence

Based on the above, ON Profile exists for every ON game. It may be NE, SE, or a simple Maximin-Optimum Profile.

3.7 Optimum-Stackelberg Taxon

Optimum-Stackelberg Taxon contains two symmetric elements:

$$\mathrm{OST} = \{\Gamma_{10\,11}, \Gamma_{11\,10}\}.$$

Let us suppose that each player knows the opponent's chosen strategy, and only one of them knows additionally that the opponent knows his chosen strategy. So, the one which doesn't know that the opponent knows his chosen strategy, will simply select his strategy as optimal response to the opponent's strategy (he will play as an unconscious leader in a Stackelberg game), but the other (which knows additionally that the opponent knows his chosen strategy; player with the value of knowledge vector equal to '11') will know the opponent's reaction and will play as a follower in a Stackelberg game.

Proposition 3.6. *If every player knows a priory what information leaks he will use (he knows the values of his respective knowledge vector), then the player with the value of knowledge vector equal to '11' will play as a leader, and his opponent will play as a follower.*

It is not the case we consider.

3.7.1 Optimum-Stackelberg Solution Principle

If the first player doesn't suspect of information leaks to the second player ($\Gamma_{10\,11}$), but he knows the strategy j selected by the second player, then he chooses his strategy as an optimal response to j, i.e. $i^* \in I^* = \underset{i \in I}{\mathrm{Argmax}}\ a_{ij}$. Let us suppose that $\#I^* = 1$. The second player knows that for his selected strategy j the first player will select i^*. He must select his strategy as an optimal response to the i^*, i.e. $j^* \in J^* = \underset{j \in J}{\text{Argmax}}\, b_{i^*j}$. So, the solution of $\Gamma_{10\,11}$ is (i^*, j^*).

By analogy, we can define the solution concept for $\Gamma_{11\,10}$. If the second player doesn't suspect of information leaks to the first player, but he knows the strategy i selected by the first player, then he chooses his strategy as an optimal response to i, i.e. $j^* \in J^* = \underset{j \in J}{\text{Argmax}}\, b_{ij}$. Let us suppose that $\#J^* = 1$. The first player knows that for his selected strategy i the second player will select j^*. He must select his strategy as an optimal response to the j^*, i.e. $i^* \in I^* = \underset{i \in I}{\text{Argmax}}\, a_{ij^*}$. So, the solution of $\Gamma_{11\,10}$ is (i^*, j^*).

Let us denote such a game by $\text{OS}\Gamma = \text{OS}$. The symmetric one is denoted as $\text{SO}\Gamma = \text{SO}$.

3.7.2 Set of Optimum-Stackelberg Solutions

Let us remember that to define solution concept we impose the cardinality of sets I^* and J^* to be 1. To define the set of solutions we must exclude this supposition. So, for $\Gamma_{10\,11}$ the set $I^* = \underset{i \in I}{\text{Argmax}}\, a_{ij}$ represents all optimal responses to strategy j of the second player. The second player knows/calculates this optimal response set. On its basis, by applying Maximin Principle he defines his set of Maximin Response $J^* = \underset{j \in J}{\text{Argmax}} \underset{i \in I^*}{\min}\, b_{ij}$. So the set of solutions of $\Gamma_{10\,11}$ is $I^* \times J^*$.

Analogically, for $\Gamma_{11\,10}$ the set $J^* = \underset{j \in J}{\text{Argmax}}\, a_{ij}$ represents all optimal responses to strategy i of the first player. The first player knows this optimal response set. On its base, by applying Maximin Principle he defines his set of Maximin Response $I^* = \underset{i \in I}{\text{Argmax}} \underset{j \in J^*}{\min}\, a_{ij}$. So the set of solutions of $\Gamma_{11\,10}$ is $I^* \times J^*$.

3.7.3 Optimum-Stackelberg Solution Existence

Proposition 3.7. *Every finite Optimum-Stackelberg Game* OS *has an Optimum-Stackelberg solution.*

Proof. The proof follows from the definition of the Optimum-Stackelberg Solution and the finiteness of the strategy sets. $\square$

4 Taxonomy of two-matrix games with information leaks and three or more levels of knowledge

According to the above result, there are $4^3 = 64$ possible kinds of games with information leaks $\Gamma_{\gamma^A \gamma^B}$ in the case when the vectors of knowledge have three components $\gamma^A = \left(\gamma_1^A, \gamma_2^A, \gamma_3^A\right)$ and $\gamma^B = \left(\gamma_1^B, \gamma_2^B, \gamma_3^B\right)$ (information leaks may occur on 3 levels).

In this case and in the general case, is it enough to examine only seven taxa of games as for games with two level of knowledge or the number of taxa increases?

Theorem 4.1. *The number of taxa for two-matrix games with information leaks with the number of knowledge levels $l \geq 2$ does not depend on l.*

Proof. Firstly, let us observe that the abstract maximal number of possible taxa depends on number of solution principle applied by two players. In our case, we apply only four solution principle: Nash equilibrium, Stackelberg equilibrium, Maximin principle, and Optimum principle. So, the maximal number of taxa may be equal to 16. But, the rules of the games and knowledge possessed by players in the case of two levels of knowledge make up possible only seven taxa.

By induction, it is provable that this number of taxa remains unchanged for $l \geq 3$. $\square$

5 Repeated two-matrix games with information leaks

If the games described above are considered as molecular games, then we can examine a series of molecular games on every stage of which a molecular game is played. Evidently, such games are a simple consequence of the games of the seven types, corresponding to seven taxa highlighted above.

6 Taxonomy of multi-matrix games with information leaks and three or more levels of knowledge

In the case of three and more players, we can adjust the molecular approach and we can denote the games by their atoms (atom players). Evidently, the number of taxa for such games can increase. Can we present a taxonomy of such games? Can we present a scheme or a table of elementary or molecular games? We are going to answer soon to these questions.

7 Conclusions

Normal form games pretend to be a mathematical model of situations often met in reality. Actually, they formalize an essential part of real decision making situations and processes, but not ultimate. Real decision making situations are influenced by different factors, which may change the essence of the games and the solution principle applicable for their solving. It follows that the initial mathematical models must be modified, at least.

This **work in progress** presents a taxonomy of normal form games with information leaks. Every taxon contains the games solvable on the base of the same solution principle, highlighted in the name.

The games with arbitrary pseudo-levels and levels of knowledge, and the games with bribe are the subject of the work in progress.

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