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Universics: a Theory of Universes of Discourse for Metamathematics and Foundations

Ioachim Drugus

Abstract

A new type of structures called "universes" is introduced to subsume the "von Neumann universe", "Grothendieck universes" and "universes of discourse" of various theories. Theories are also treated as universes, "universes of ideas", where "idea" is a common term for assertions and terms. A dualism between induction and deduction and their treatment on a common basis is provided. The described approach referenced as "universics" is expected to be useful for metamathematical analysis and to serve as a foundation for mathematics. As a motivation for this research served the Harvey Friedman's desideratum to develop a foundational theory based on "induction construction", possibly comprising set theory. This desideratum emerged due to "foundational incompleteness" of set theory. The main results of this paper are an explication of the notion "foundational completeness", and a generalization of well-founded-ness.

Keywords: induction, deduction, reduction, universe

1 Introduction

"Universics" is the term used by philosophers for the approach, which presupposes the treatment of any object from the perspective of the whole Universe, as if "the object was made in the image of the Universe". This is a limit case of the "holistic approach", when the "whole" is chosen to be the "maximal whole", i.e. "the Universe". The term "universics" was coined in the form of plural of modifier "universic" (similar to "mathematics" which comes from modifier "mathematic", "related to knowledge") – a term customarily used for a feature of an object, whereby the Universe "reflects" in the object.

The "reflection principle" in set theory states that in any universe of discourse of a full-fledged set theory there are sets which "reflect" the features of the universe; examples of sets which are in the image of the von Neumann universe are Grothendieck universes. Thus, universics is not new to set theory domain. The goal of this paper is to give to universics the features of a mathematical discipline with its specific methodology, in particular, manifesting as (a) treating both theories and their universes of discourse on equal footing, as "universes", and (b) treating induction and deduction as dual notions.

A term which sounds to be closely correlated with "universics" is "multiverse", used to refer to the multitude of set theories viewed from the perspective of their universes of discourse – a multitude, which emerges due to various methods of constructing universes, among which the best known is "forcing method" [1, 2]. The difference between two approaches is that universics is a theory of separate structures called "universes", whereas in the approach using the term "multiverse", all such structures are treated as parts of a "multiverse".

The Grothendieck universes were introduced as a foundation for category theory – a foundation needed due to the fact, that the main concepts of this theory cannot be expressed in terms of sets, as this is explained next. Really, the "category of categories" is one of the key concepts of category theory, since "functor", "adjointness", "natural equivalence" and other central notions, which determine the value of this theory, are defined proceeding from the supposition that "category of categories" exists. But the existence of such an object presupposes the existence of the "class of classes" – a concept which is "contradictory by definition", since no proper class can be member of another class. Moreover, category theory uses such notions as "functor of functors", which imply existence of "proper classes of proper classes" – objects inadmissible in any set theory or class theory. Thus, since there are mathematical concepts, which cannot be expressed or represented in ZF set theory or NBG (von Neumann-Bernays-Godel) class theory, these theories cannot be said to be "foundations for mathematics".

On the other hand, surprisingly or not, only one of the Grothendieck universes turns out to be sufficiently rich to serve as a "foundation for mathematics" – true, to the same extent as set theory is, that is, as an "incomplete foundation". Merely this fact calls to have clarified the notion "foundational completeness" used by Harvey Friedman in a message to the automated email list "Foundations of Mathematics" (https://cs.nyu.edu/pipermail/fom/1997-November/000143.html). As he wrote, set theory "does not come close to doing everything one might demand of a foundation for mathematics" and it cannot be said to be "foundationally complete" for mathematics. Despite that this is an intuitively clear notion, Friedman mentions that he does not know how to define it. In universics this term will obtain a natural explication and this is one of the main results of this paper.

In addition to the strong conceptual arguments mentioned above to support the statement that set theory is foundationally incomplete for mathematics, there are reasons supporting the thesis that set theory is also foundationally incomplete for informatics (here, the term "informatics" is used to reference mainly the "data structures" used in computer science). Namely, set theory is too poor for representing the data structures used in informatics. On the other hand, *no* arguments were found in favor of the thesis that category theory, founded on Grothendieck universes, or just these universes alone may be foundationally incomplete. Therefore, a theory of universes, among which are also the Grothendieck universes, can be expected to be a foundationally complete theory. Based on this thesis, the axiomatic system of universics introduced in this paper is expected to be of good service to both mathematicians and computer scientists.

As an informal theory, universics was developed in [3, 4], where "universes" were treated as the largest structures, similarly to how the proper classes called "universes" are treated in set theory. There are, though, two major differences between set theory and universics, namely: (a) set theory studies conceptions which are obtained by abstraction from any kind of order, but the "universes", about which universics discusses, are *structures*; universics is a "structuralist" theory, (b) set theory studies "small scale" universes (see section 2.), but universics studies any structures, even though it could be said to be focused on "large scale" universes. The first attempt to present universics as a framework of axiomatic theories was undertaken in [5].

In papers [3] and [4], any structure was considered built by repetitive application of three operations, called "aggregation", "association", "atomification" – operations for building sets, ordered pairs and atoms, respectively. The reason for the choice of these notions as a starting point in building a foundational theory is the belief that the notions "set", "ordered pair" and "atom" are sufficient to serve as a "conceptual orthogonal basis" for a universe of concepts. This conceptuality is intended to describe the "fabric" of a universe, and could be called "small scale universics". The current paper presents a theory of structures called "universes", which can be called "large scale universics". Since the "fabric" of a universe is irrelevant here, any knowledge of [3, 4] is not required for understanding the current paper.

2 On the terminology and conceptuality used in metamathematics

The terms used in a meta-discourse necessarily contain an amount of ambiguity, but one of the goals of universics is to serve as a language of metamathematics. Therefore, this section is intended to sort out some of the terms used in metamathematics and contribute to their precise use. Several other terms used in metamathematics will get a precise treatment later, when these terms will be explicated. In order that a term needing clarification can be easily found while reading the main text, the terms explained in this section are italicized.

A *universe of discourse* is correlated with a theory, which is said to "discuss about" the entities populating this universe, but to "describe" the universe (as a whole). The distinction between to "discuss about" and to "describe", is that one *discusses* necessarily about objects *within* a "universe of discourse", but one can *describe* something, which is not in a "universe of discourse" – say, one can describe the (whole) "universe of discourse", informally or by (the axioms of) a theory.

There are clearly specified universes (like Grothendieck universes), for which no theory describing them have been presented; about these they discuss informally. Also, the von Neumann universe was invented as a view upon the totality of sets and only later it was found that ZF theory extended with terms for ranks of the sets can describe it. Finally, two theories may have the same universe of discourse. These arguments support the treatment of the notion "universe" as a notion on its own, prior for it to be correlated with a theory with this universe as its "universe of discourse". Thus, in this paper the term "universe" will be used without necessarily being followed by the phrase "of discourse".

The term "universe of discourse" of a theory was introduced by George Boole (the inventor of the "algebra of logic") and this term turned out to be very useful in the early days of metamathematics, but later, this term created difficulties, for example, when it was used for a theory treated (itself) as a universe. Currently, the term "domain of discourse" is preferred (say, a search in Wiki of the phrase "universe of discourse" will result in an article titled "Domain of discourse"), even though it did not replace the term "universe of discourse", which remains as its synonym.

In universics, the notion "universe" is treated on the same footing as the notion "theory", and the phrase "universe of discourse" may create even more difficulties in forming correct expressions. But, the term "domain of discourse" will not be used in this paper, and the term "universe of discourse" will *not* create problems, since it will be used only in proper contexts. Also, another phrase starting with particle "of" will follow the word "universe" according the "pattern" of forming the term *universe of sets* customarily used interchangeably with the expression "the universe of discourse of (a) set theory". Similarly, the terms *universe of ideas* (as a generalization of "theory"), *universe of objects* (for the universe described by a theory), *universe of structures* (for any of the previous two) will be here preferred to the longer expressions using the phrase "of discourse".

Fraenkel used the term "object" for the entities in a universe, and since anything populating a universe is called "object", the expression "universe of objects" does not sound to be always convenient. There is, though, a special case, when this term is the most appropriate one, and only this case occurs in this paper – this is when one opposes a “universe of ideas” to a “universe of objects”. Thus, the term *universe of objects* will be used here to express a meaning opposed to that of the term *universe of ideas*. The totality of relationships between elements of one of these universes with those of the other universe is called here *reflexion* – a generic term for both the “interpretation” of elements of the universe of ideas into the universe of objects, and for the inverse relations of “naming” and “describing” the latter through the former ones.

An example of a “universe of objects” is the totality of all the things called “sets”, “classes”, “classes-as-many” (a term introduced by Russel to refer to collections which reflect the plural of a noun; the regular classes were shown to incorrectly reflect it), “multi-sets” , or “aggregates” (a generic term introduced also by Russel seemingly for any kind of set-like objects). Here, the term “aggregation” instead of “aggregate“ is preferred for any “set-likes” – both because this term is widely used in computer science, and in order to avoid confusions with many “theories of aggregates”. Accordingly, the “universe of aggregations” refers to a universe containing any set-likes.

The term *collection* is commonly informally used for a notion generalizing the notions of set and of class. A collection is presupposed to have no repeating elements, i.e. to be really a set or a class, so that the “universe of collections” satisfies the extensionality axiom – an axiom, which occurs in all mainstream set theories. Mathematics does not seem to be same focused on multi-sets as it is on collections. Here, the “universe of collections” is considered as the most important universe for the metamathematics of the foundations of mathematics. On the other hand, the set-likes used in building data structures of informatics, obviously, do not satisfy the extensionality axiom. Therefore, here, the “universe of aggregations” is considered as the most important universe for the foundations of informatics.

The term “collection” is convenient in metamathematical analysis, especially, when the notion of “size” (see below) is irrelevant to the topic of discussion. In set theory, a set is treated as a class which is a member of another class. There are also classes which are not members of other classes, and these are called “proper classes”. Similar to category theory, in metamathematics, the modifiers “small” and “big” are appropriate to modify the noun “collection” in order to distinguish between sets and proper classes in this manner: “small collection” as synonym for “set”, “big collection” as synonym for “proper class”.

The modifiers “small” and “big” reference two values on a scale called “size” expressed in terms of membership relation “$\in$”. Namely, a “big collection” is a maximal collection within the universe of collections governed by the membership relation, and a “small collection” is not maximal in this universe. This dimension can also be referenced as “height/depth” (depending on the perspective from which the membership relation is viewed). Accordingly, in category theory, a category with a big collection of objects and morphisms is said to be a “big category”, and one with such a collection small, is called “small category”.

The universe of discourse of a set theory, let this be ZF, is a proper class (i.e. a “big collection”), but there are also other proper classes, and a question arises: “what singularizes the universe of discourse of ZF among other proper classes making it a ‘universe’ ”. This question cannot be answered in terms of “size”, as this term is used today. To answer it, yet another dimension needs to be considered – one which is referenced here as “extension” in this manner: a collection C will be said to have a *smaller extension* than a collection D, if C $\subseteq$ D. Notice, that in addition to being “big” in size, *the universe of discourse of ZF has the largest extension*, and this answers the question regarding what singularizes the universe of discourse among other classes. To account for both “size” and “extension” dimensions, in universics the terms *small scale* and *large scale* will be used.

Finally, notice that the word “idea” is treated here as a term. Notice, that logicians consider the things used in a theory to be of two kinds – “notions” and “assertions” and they use the generic term *idea* for them (here, the “notions” can be – “properties”, “relations”, “functions”, “operations”, etc.). A theory is also an “idea” which can be treated as an inhabitant of a *universe of ideas*, and if a universe of ideas is populated only by theories, the universe is called “universe of theories". In universics, this is an important universe, without which such notions as "foundational completeness" cannot be explicated in precise terms.

Due to the special role of the universe of theories, certain terms like "foundation", "foundational", "fundamental" which will be explicated here, will be borrowed from it for describing (any) structure called "universe". In particular, the term "foundation of a theory" will be preferred to the term "basis of a theory". This borrowing of terms "from above", considering them as "more general", will provide for a uniform treatment of any universes and will help discover (maybe, totally unexpectedly) similarities between apparently distant notions.

3 What is a universe?

Since the notion "universe" originates in logic (if only set theory is considered as a chapter of logic), one can get a hint on what kind of mathematical structure might be a universe exactly from logic, and namely from the definition of the notion "axiomatic theory", or shorter, "theory". Logicians define a theory as having a "basis" consisting of ideas of two sorts – "basic notions" and "axioms" and consider the other ideas of the theory as obtained either by definition or deduction. A common term used here for basic notions and axioms is the term "axiom". Also, since the process of definition and the process of deduction are similar, the term *reduction* is customarily used for both these processes. According to definition of "theory", a theory uses two sorts of entities – notions and assertions. But the notions can be missing from a theory, and this can be a "theory of assertions". Similarly, by admitting that the assertions can also miss from the theory, one gets a "theory of notions". Customarily, the logicians do not use such "theories", but nothing in this definition prevents these from being "theories". Any of these structures is referenced as 'universe of ideas".

Proceeding from the definition of the notion "theory", and recalling that we prefer the term "foundation" to the term "basis", the notion "universe" is defined here as a triple (U, F, R), where U is a collection called *support*, F is a subcollection of U called *foundation*, and R a binary relation on U called *reduction* relation. As per usual practice for other types of structures, a universe and its support will be denoted and referenced by the same name or notation.

The notion defined next is not primitive, i.e. it does not occur in definition of "universe", but it is the same important as the notion "foundation", since it can be treated as its dual and called "co-foundation". But this notion is intended to serve as an explication of a notion informallly treated in [7] and its name is taken from there. Namely, the collection $U \backslash F$ (complement of F in U) is called *super-structure* of the foundation F in universe U.

In this paper, the meaning of symbol turnstile "⊢" used in sequent calculi is extended to denote the reduction relation correlating any ideas, assertions or notions. A "universe of ideas" is treated as a simple mathematical structure denoted as a triple $(U, R, \text{“} \vdash \text{”})$. Moreover, the use of turnstile symbol will be further extended to stand for arbitrary binary relations. This complies with its use in an increasingly "generic" manner in so called "sub-structural sequent calculi", where this relation is not even supposed to necessarily be reflexive and transitive.

Reduction is to be treated as a "generalization" of many relations between different types of objects. Here, "generalization" is used within quotation marks to emphasize that various other kinds of relations are not really "partial cases" of reduction – they are "reducible to reduction". This means that reduction is a fundamental binary relation and even "generalization", whatever this is, must be treated in terms of reduction.

The ideas about objects and the objects (themselves) abide in a relationship called here "reflection", and the supposition is made that this relation, similar to a "homomorphism", "projects" the same structure "universe" from ideas onto objects. Therefore, a "universe of objects" will be considered here to be the same type of structure as a "universe of ideas". In this manner we came to a definition of the notion "universe", applicable both to theories and to (their) universes of discourse.

One may wonder why the notion of theory is "replaced" with the notion "universe of ideas". Aside from getting to treat theories and their universes of discourse on the same footing, there are also other benefits from this “replacement”. To illustrate this, notice that alongside many alternative “set theories”, there is also one discipline called “set theory” and this kind of ”theory” defies any definitions given by logicians. The theories like “set theory” turn out to be “universes of ideas”, with a foundation consisting of assertions generally accepted as being about the conception “set”. Such theories are referenced here as “informal theories”.

A remark is also in place regarding the treatment of reduction as a *binary* relation. Notice, that in case of deduction, in a correlation like “$x \vdash y$”, x is a list of assertions and y is one assertion, things of different sorts. This forces considering both the lists of assertions and the assertions as things of one sort, “object”. Similarly, a notion is generally reducible to a set of other notions and a “set of notions” is of a sort different from “one notion”. This forces considering reduction of notions also as a relation between things of the same sort, “ideas”. The reason why reduction is treated as a relation between two things, and not between one thing of a sort and many things of the same sort, is that the reduction relation *implicitly presupposes* the existence of a multitude of things to which one thing is reducible. It is exactly this implicit presupposition which made possible development of an alternative set theory to be presented in a future paper.

The symbol “$\dashv$” symmetric to “$\vdash$” will be also used. The formula “$x \dashv y$” is read “x is reducible to y” or “x is generated from/by y”, and is called “direct presentation” of reduction (“direct”, because the order of arguments in formula and in English expression coincides). This “presentation” is more convenient in many cases. The formula “$x \vdash y$” is read “x reduces to y” or “x generates y and is called “inverse presentation”.

As a standard notation for the property “to be a member of foundation” can serve the symbol of reduction “$\vdash$” used as a *unary* predicate symbol as in expression “$\vdash x$” (which is regularly used in sequent calculi), and symmetrically, the symbol “$\dashv$” like this: “x $\dashv$”. Notice, that the notations “$x \vdash$”, “ $\dashv$ x” have another meaning, and in order to avoid confusions, remember for this meaning to place the argument against the "dash", and not the "bar", of the turnstile symbol. Such use of the symbol of reduction for a unary predicate symbol shows that the property "to be in foundation" can be intuitively treated as a "rudiment" of the relation "to be reducible to".

The reason why we focused in so much detail on the two presentations is due to our treatment of induction and deduction as two principles "dual" to one another, i.e. as *de facto* one principle governing the two *symmetrical* universes – the universe of ideas and the universe of objects. It sounds like a good practice to use the symbol "$\vdash$" for deduction and its generalizations, and the symbol "$\dashv$" for all other cases. Also, it is not easy to remember which presentation is "direct" and which is "inverse", but the terms "deductive presentation" and "inductive presentation" are more suggestive and will be preferred (it will become clear later, why "inductive"). This notation practice suggests that membership relation "$\in$" is treated as "dual" to deduction "in a certain sense", and this sense will become clear after precise terms are introduced in the next section. Meanwhile, here are two important classes of universes, dual to each other.

1. $(V, F,$ "$\in$"$)$, where V is the universe of discourse of a "collection theory" (a set theory or a class theory), and F is the set of its atoms. Call this kind of universes "collection universes".

2. $(U, F,$ "$\vdash$"$)$, where U is the set of formulas in a predicate language, F is a subset of closed formulas from U considered as axioms, and "$\vdash$" is the symbol of deduction. Call this kind of universes "deduction universes".

The intuitive meaning of the notion "foundation of universe" is "the totality of all objects from which all objects in the universe can be "built" or "deducted", where "to build" and "to deduct" are described as "dual" actions to one another. The set of atoms is the foundation of a collection universe, and "the set of 'possible axioms', i.e. of formulas from which the axioms for axiomatic theories can be selected" is the foundation of the deduction universe. From this treatement it follows that the notion of deduction universe can be treated as an explication of an "informal theory", since an informal theory is said to be "axiomatized" by an axiomatic theory, and such axiomatization is nothing else, but selecting a set of axioms from among the assertions considered as "fundamental" for the conception of "set".

Notice, that the foundation of a collection universe without atoms is the empty set. Thus, the empty set the existence of which is required in any set theory is to be treated as a foundation, which is "mandatory". This shows that even though the "universe of discourse" of set theory is customarily said to be (just) a "class", it is actually treated by set theorists as a full-fledged "universe" as this notion is defined in this paper! Here is a statement of Friedman, which sounds relevant to this: "The viewpoint is that the empty set of set theory has a unique unequivocal meaning independently of context".

4 Basic notions related with universes

The first "basic notion" defined here is easier to be understood if the reduction is denoted in its "inductive presentation", i.e. as "$\dashv$".

Definition. A sub-collection X of a universe $(U, F,$ "$\dashv$"$)$ is called *transitive collection* in universe U, if the following condition, called *fundamentality principle*, is satisfied:

$$(\forall u, v \in U)((v \in X)\&(u \dashv v) \rightarrow u \in X). \tag{1}$$

The inductive presentation of reduction was preferred, because if reduction is membership, then what was defined above is exactly the well-known property for a sub-collection of a universe of sets to be called "transitive". In a dual presentation, this condition is a kind of "dual *modus ponens*" law for a sub-collection:

$$(\forall u, v \in U)((v \in X)\&(u \vdash v) \rightarrow u \in X). \tag{2}$$

A universe is called *fundamental*, if its foundation is transitive. The mainstream collection universes, where reduction is membership, are trivially fundamental by definition of atoms (see section 6). The non-mainstream collection universes, like the universe of discourse of Aczel's set theory with the "anti-foundation" axiom, are also fundamental. Basically, all theories of sets are fundamental, trivially or non-trivially.

The deduction universes are "fundamental" only if condition (2) is satisfied. Even though *modus ponens* is the main law in the axiomatic theories, the "dual *modus ponens*" law (2) might not hold for a deduction universe. At this point, it is appropriate to explain this term. Notice, that unlike the *modus ponens* law (also called "detachment law"), where the consequent is "detached", in (2) the antecedent is "detached", and this explains the use of the word "dual". For comparison, notice that if the subcollection X is a theory, then the (regular) modus ponens law, which holds for the theory, looks like this:

$$(\forall u, v \in U)((u \in X)\&(u \vdash v) \rightarrow v \in X). \tag{3}$$

The intuitive meaning of "fundamentality principle" is expressed by this reading: "if an idea is fundamental, then another idea to which it can be reduced is also fundamental". This principle makes little sense for axiomatic theories, where the axioms are already chosen as the "fundamental ideas", but it makes a lot of sense for informal theories. So, the intuitive set theory is an informal theory – it is a universe of statements among which some statements are considered as mandatory for describing the conception "set", i.e. are regarded as "fundamental". If a statement B is regarded as fundamental, and later another statement A was found, such that $A \vdash B$, then the fundamentality principle prescribes to consider "fundamental" also the statement A. Thus, fundamentality principle can be used in development of axiomatic theories, which "axiomatize" an informal theory.

The intersection of all transitive subcollections of U containing the collection X is called *transitive closure* of X in U and is denoted as "$[X]$". This is a simple notion for collection universes, but it permits to discuss also about deduction, and in a rather concise manner. So, if X is the axiom set of theory, then "$[X]$" is the set of its theorems.

The *co-universe* of a universe $(U, F,$ " $\dashv$ "$)$ is defined as universe (U^c, F^c, " $\dashv^c$ "), where $U^c = U$, $F^c = U \backslash F$ (the superstructure in U), and "$\dashv^c$" graphically coincides with "$\vdash$". The universe and its co-universe are said to be "dual" to each other. Notice, that a universe dual to a universe U is not just a universe with the symmetric presentation, but also with the superstructure in U as its foundation.

In a customary manner (as for example, in category theory), a "dual notion" with prefix "co-" added to its name is defined for each notion. The superscript "c" will be used in the denotation of a dual notion as above in the definition of the notion "co-universe" or, for example, in notation $[X]^c$ for the co-transitive closure of a subcollection X.

Obviously, the correlation $[F]^c = [F^c]$ takes place, which in words sounds like this: "the co-transitive closure of the foundation is equal to the transitive closure of the superstructure". Therefore, $[F]^c = U$ and $[F^c] = U$ are two equivalent conditions. In words, both have the meaning: "the foundation generates the universe".

This is the right place to explain why the term "foundation" is better than the term "basis" for a universe. In mathematics, the term "basis" is customarily used as a synonym for "generating basis" or "basis of generators". Therefore, mathematicians would consider as implied the statement that "the basis of a universe generates the universe" – a statement which in general case might be wrong! On the other hand, the term "foundation" does not have this connotation of "generation" and this is why it was preferred. Still, when the foundation really generates the universe, like in case of an *axiomatic* theory, or dually, a set theory with atoms, there is no reason to avoid using the term "basis" for the foundation of a universe.

The notions as direct product, homomorphic image, etc. for universes are defined in the customary manner, but since these are not used in this paper, they will not be formulated here. The only notion in addition to those already introduced which is needed here is that of a "sub-universe" and this is defined here like this: a universe U is said to be a sub-universe of the universe V, if the support, the foundation, and the reduction relationship of U (treated as a collection of ordered pairs) are included in the support, foundation and reduction relationship of V, respectively.

5 The universe of structures

The universes of ideas and the universes of objects are structures of the same type, and the distinction between them can be made only in terms of various relations between them. These two kinds of universes are "structures", and one would like to look into the reduction relation *between structures*, before anything else.

If x and y are two *structures*, then the expression "x $\dashv$ y" is conveniently read as "x is a *reduct* of y". The term "reduct" used here comes from universal algebra, where an algebra A is said to be a reduct of an algebra B, if the signature of A is a subset of the signature of B. This can be imagined as "reducing" the B to A by "neglecting" or "ignoring" some of its operations. An intuitive synonym for "reduct" is "rudiment" or "rudimentary structure". In some cases, the expression "x $\dashv$ y" is conveniently read as "x is more *elementary* than y", and in one of such cases, when reduction is membership, even more conveniently: "x is an *element* of y".

The collections can be treated as "final reducts" of mathematical structures, since these have a collection as their support. Next, an explanation follows why the term "universe" is customarily used both for a collection and a structure which has this collection as its support. When the set theorists consider a universe of sets as an "internal model" of a set theory, no doubt they also take into account the membership relation which governs the sets in that universe. Thus, they treat such a "universe" as a structure and not as a "class". But they refer to such a structure as "class" by making use of a linguistic device called "metonymy" – naming a whole by the name of a part. Thus, the reference to a structure by its support, which is the "*final* reduct" of the structure can be treated as a result of applying a "conceptual metonymy" device.

It is by applying the conceptual metonymy device, that other reducts of a universe are also called "universes". One of such reducts is of the kind (U, "$\dashv$") obtainable by ignoring the foundation. Such a universe will be said to be "foundation-free" or (in some cases) "base-free". The most representative example of such a universe is the universe (V, " $\in$ ") of discourse of ZF, where "V" is the standard notation of the class of all pure sets (i.e. sets built out of the empty set). One cannot just "identify" (consider "the same") the pair (U, " $\dashv$ ") and the triple $(U, \emptyset,$ " $\dashv$ "), since "*without* foundation" is not the same as "*with* an empty foundation". Instead, one can use the conceptual metonymy device and call "universe", or more precisely "foundation-free universe", the pair $(U, \text{“} \dashv \text{”})$.

There is yet another kind of reduct of a universe – a reduct obtained by discarding the reduction relation to obtain the ordered pair $P = (U, F)$. Such a universe can be treated as a "problem", where U is the collection of "possible solutions", and F – the set of "actual solutions", of the problem P. This type of "universes-problems" was proposed by Kolmogorov as an alternative interpretation of intuitionism. Finally, the reduct obtained by discarding both the reduction and the basis is a collection – thus, by using the conceptual metonymy device, the collections will be also referenced as "universes".

A *proper* definition of universes as structures, a definition accounting for the reducts of universes so that the metonymy device is *not* needed, *cannot* be formulated in the language of set theory other than by re-defining the notion of relation in a complicated manner. But such a "re-definition" can create risks of ontological and terminological inconsistency. There is, though, an approach, which offers a convenient device for the presentation of universes as structures – the approach presented in [6] which uses the notion "quasiary relation". Roughly, a quasiary relation is a relation with optional correlates. It also sounds plausible, that the notion of "conceptual metonymy" can be explicated in terms of quasiary relations.

An important question is whether any type of structures is reducible to structures of type "universe". The author did not research this, but there is a result of Quine which sounds to give an affirmative reply to this question. Namely, Quine showed that the combinatory logic of Moses Schoenfinkel can be interpreted as a logic of relations (rather than functions) [8]. This result can serve as a basis for the belief, that all possible kinds of structures can be represented as universes.

The final remark in this section is for terminologic purposes. Since there are two objects, which are more "rudimentary" than a binary relation, then the relation together with these two objects can be treated as a "generalized relation" and this is exactly a "universe". Thus, we obtain an important intuitive definition: *A universe is a "generalized* *binary relation*". This treatment is needed for unification of terminology (say, the notion "well-founded" is commonly defined for relations, and here – for universes).

6 Atoms and axioms – dual irreducible objects

Recall that in this paper, the term "axiom" is considered as generalizing the terms "basic notions" and "axioms" of an axiomatic theory. In universics, such "axioms" and the set-theoretic "atoms" are treated as dual, which permits "projecting" the properties of ones to the others. The atoms and the axioms can be called "marginal objects", the atoms – "initial objects", and the axioms – "final objects". Next, the atoms are discussed and the aquired knowledge is applied to axioms.

A set theory with atoms uses a predicate symbol, customarily "$Atom(x)$", in addition to the membership symbol, and it postulates that the atoms make up a set (not a class). In such a theory the atoms are objects of a sort different from that of the sets, the atoms need to be considered as making up the foundation of the universe of discourse of the theory, since there is no other way to distinguish between different sorts. Here, as "atoms" the "regular atoms", those also called "urelements", are referenced. The urelements do not have elements, and are different from the empty set.

A "Quine atom" is an object q which equals to its singleton $\{q\}$, or in other words, a Quine atom is a set (!) "on which the membership relation is reflexive". Thus, a Quine atom is not a proper "atom" – it is a singleton, a special kind of set. Hence, a set theory whose all atoms are Quine atoms is a "pure set theory". If q is a Quine atom in a set theory, then there exists an infinite chain in the universe of discourse of this theory: $q \in q \in \ldots$. A Quine atom is a non-well-founded object, and the transfinite induction principle cannot be proved in a set theory with Quine atoms.

By analogy with set theory, the definitions for two kinds of atoms in arbitrary universes are these: $Urelement(x) \stackrel{\text{def}}{=} \neg\exists y(y \dashv x)$ and $QuineAtom(x) \stackrel{\text{def}}{=} \forall x((y \dashv x) \leftrightarrow y = x)$. In any mainstream "set theory with atoms", the atoms are always of exactly one of these two sorts, and there is no need to introduce a general term for them. But in universics, the following definition of "atoms", which are either "urelements" or a "Quine atoms", makes sense:

$$Atom(x) \stackrel{\text{def}}{=} \forall y((y \dashv x) \to y = x).$$

There is no requirement that the foundation of a universe consists only of atoms, but if this is the case, then the elements of the foundation are "pairwise incomparable", i.e. the following condition is satisfied:

$$(\forall x, y \in F)((x \dashv y)\&(y \dashv x) \to x = y).$$

Such a "foundation" is called here "basis", or "orthogonal basis", and this complies with general mathematical terminologic practices (here the modifier "orthogonal" is used rather for "emphasis"). The foundation of universes of discourse of set theories with atoms is a "basis".

And now, these notions will be used in the "dual presentation" to clarify the terminology regarding the axioms of an axiomatic theory. The logical terminology is not the same "brushed up" one as that of set theory. Really, a statement, which can be deduced from "axioms", cannot be said to be "axiom" – at most it can be called "intended axiom". But in practice, expressions like "set of independent axioms" are often used, and "axioms", which are deducible from other axioms are accepted for an axiomatic theory. A better term for them is "fundamental statements".

In order to treat the "axioms" of an axiomatic theory and the "atoms" as duals, the term "axiomatic set theory" needs to be treated as "axiomatic set theory with 'independent axiom set' " and to maintain the terminology of universics consistent, this convention is here adopted.

7 What is foundational completeness?

Foundationally complete can be the "axiomatic theories" and "$\vdash$" is the convenient symbol for reduction of the theories treated as universes.

The expression "$x \vdash y$" will be read here as "x is more fundamental than y", and the formula "$F(x)$" as "x is fundamental".

Definition. Suppose $(T, A, \text{“} \vdash \text{”})$ is an axiomatic theory and this theory is a sub-universe of the universe $(U, F, \text{“} \vdash \text{”})$. Then the theory T is said to be *foundationally complete* in the universe U, if $[A] = F$.

If in this definition, "U" is a well-founded universe (see below), then the equality $[A] = U$ is definitely true, but using this condition in the definition would limit it to only well-founded universes. Thus, as it was formulated, this definition provides for the most large applicability of this notion, including, to the *non*-well-founded universes.

To get a better perception of this notion, a couple of examples of "wordings", which are close in meaning with the expression "foundationally complete", is in place (to give "precise" examples is impossible, since this is an "explication" and not a definition of a precise notion):

(a) if U is the collection of all assertions considered as true in intuitive set theory, and T is an axiomatic theory, then T is "foundationally complete in U", if T is said to be an "axiomatic theory *of sets*";

(b) if U is the collection of all "mathematical theorems", and T is an axiomatic theory, then T is "foundationally complete in U", if the theory T is said to be "foundations for mathematics".

8 Axiomatic universe theories

Similarly to set theory which is both an informal theory and a collection of "axiomatic set theories" focused on various explications of the conception "set", universics is an informal theory of universes and a collection of axiomatic "universe theories". Various axiomatic theories of Grothendieck universes can serve as examples of the latter kind. In order that universics obtains a practical use, alongside serving as a framework of "universe theories", it must also take over from concrete theories strict treatment of some of the most general subjects. In this paper, "fundamentality" and "well-founded-ness" are considered to be among such subjects, and these are treated here in terms of axiomatic theories. These theories describe the most general features of universes, and their axioms and axiom schemes are referenced as "principles" to be distinguished from the axioms and axioms schemes of more "special" theories based on them.

The language of universe theories introduced in this paper will use a 1^{st} order predicate language with a unary predicate symbol "F", where "$F(x)$" is interpreted in a universe with a foundation F' as "$x \in F'$" – a formula read as "x is fundamental" (or as "x is foundational" for a universe *of theories*, where some theories can be "foundational"), as well as the binary predicate symbol "$\dashv$" and its "symmetrical" symbol "$\vdash$". Obviously, the theory in this language of all universes – denote it as "**U**" – cannot contain any non-logical axioms, since for any such "axiom" there exists a universe which falsifies it. Thus, only theories describing a class of universes narrower than U are interesting.

Universics explicates the property "to be fundamental' of objects and ideas (notions, assertions, but also *theories*), via the theory **F** with the principle (F) below as its only axiom:

$$(\forall \text{ x, y}) \ ((\text{x} \dashv \text{y}) \ \& \ \text{F(y)} \ \rightarrow \ \text{F(x)}) \quad (F).$$

The theory **F**, obviously, describes all fundamental universes, in particular, the universes of all mainstream set theories, where the "foundation axiom" in any form is either postulated or deducible, but also the universe of discourse of non-well-founded theories, including Aczel's set theory with its "anti-foundation" axiom. Thus, the "fundamentality" is treated in universics in so wide manner that most (or maybe all) set theories proposed for the foundation of mathematics are "fundamental". Obviously, of particular interest is the fundamentality principle for the universe *of theories*, where some theories are "fundamental" whereas others are not.

In universics, "well-founded-ness" property is explicated via the theory **R** with the following axiom scheme (R) as its sole axioms:

$$(\forall \text{x}(\text{F(x)} \rightarrow \text{P(x)}) \ \& \ \forall \text{x}(\forall \text{y}((\text{y} \dashv \text{x}) \rightarrow \text{P(x)}) \rightarrow \text{P(x)})) \rightarrow \forall \text{xP(x)} \ (R).$$

In this "principle", P is a formula and "x" is a variable which may or may not enter in P, together forming a "property". The theory **R** describes the universes said to be "*reductive*". To get the first idea about these universes, one can analyze "how they look" in particular cases.

If the reduction relation is membership, then by replacing the symbol "$\dashv$" with the symbol "ϵ" in (R), one can easily discover that the reduction principle is actually the well known epsilon-induction principle generalized also to describe universes with atoms or other non-well founded objects residing in its foundation.

If the reduction is deduction, then in the particular case, when $F(x)$ is the property "x is an axiom" and $P(x)$ is the property "x is true", the reduction principle states a semantic characteristics of axiomatic theories. In the most general setting, the meanings of reduction principle are much wider, and other meanings can be obtained as the result of profound research.

9 Well-founded universes

The notion of well-foundedness in most general treatment is a property of relations (https://en.wikipedia.org/wiki/Well-founded_relation), where the collection, on which a relation is defined, is also taken into account. One can say that this property is currently defined only for the foundation-free universes, and they would obviously expect that this property can be extended to arbitrary universes.

For the foundation-free universes, the induction (epsilon-induction) cannot start from the "induction basis", which is empty, and one would naturally also want to generalize induction to its most general form with "induction basis" as foundation of any universe. This would be the generalization of induction to arbitrary "generalized relations", i.e. to arbitrary universes. In universics, the conjunction $(F)\&(R)$ stands for the "generalized induction principle", and the theory with these two principles as its sole axioms describes the class of all reductive fundamental universes. This class is treated here as the "largest span" of induction principle.

Alongside the axiomatic characterization of reductive fundamental universes, one would expect that these can also be characterized in terms of their structure. Such a characterization is known for the well-founded relations: a relation R on a collection U is well-founded, if and only if, it contains no countable infinite descending chains, that is, there is no infinite sequence $x_0, x_1, x_2, ...$ of elements of U such that $x_{n+1} R\, x_n$ for every natural number n. Considering that this is a characterization of well-founded foundation-free universes, and terming a foundation-free universe with this "chain condition" a "Noetherian universe", one can expect a similar characterization for any reductive fundamental universe, and this will be given below. One may wonder weather the term "Noetherian" was correctly used here, or the term "Artinian" should have been chosen. Actually, the term "Noetherian" is correctly chosen, because this definition can be equivalently formulated in terms of *encreasing* chains of "*ideals*" like in ring theory. This was not done because the "ideals" in arbitrary universes would have no other uses in current paper.

A universe $(U, F, \text{“} \dashv \text{”})$ is called *quasi-Noetherian* if any countable infinite descending chain in U continues from some point in F, that is, if $x_0, x_1, x_2, ...$ is a sequence of elements of U, such that $x_{n+1} \dashv x_n$, for every natural number n, then there exists a natural number n such that $x_m \,\epsilon\, F$, for any m, where $m \geq n$. Obviously, any quasi-Noetherian foundation-free universe is Noetherian.

Theorem. A universe U is both fundamental and reductive, if and only if, U is a quasi-Noetherian universe.

The proof of this theorem, even though formulated in slightly different terms, can be found in [5], published in the Proceeding of the the first "Mathematical Foundations of Informatics", which can be downloaded at this link: http://www.mfoi.eu/workshop2015/proceedings.pdf.

10 Conclusions

The following below conclusions about the use of universics impose themselves from the foregoing presentation.

(1) Since the deduction is an apparatus of metamathematics, and the language of set theory is generally considered as the language of

metamathematics – two “duals” in universics, this approach sounds to offer a convenient language for metamathematical analysis.

(2) An important application of universics can be the “projection” of the results from one domain of research to another.

(3) Usiversics can serve as a foundational theory, presumably on par with category theory. Both these theories use maximal collections, i.e. proper classes, but, unlike category theory, universics uses only one sort of entities based on such collections, the “universes”, and does not invoke (or rather rarely invokes) any second sort of entities (like “morphisms” of category theory). Due to using only one sort of entities, and due to the similarity of its methods with those used in the intuitive set theory, universics can appear to be more intuitive than category theory to those “working mathematicians”, who think that the diagrams of category theory “hide the intuition”.

11 Future research and an open problem

Various directions of research can be easily indicated proceeding from the conclusions in previous section. But also, there is an “open problem”, which prominently imposes itself – one, the solution of which would indicate how important for the foundations are the fundamental reductive universes, outline the ”validity span” of principles (F) and (R), and show whether or not their validity is correlated with the predicativity of definition of set via comprehension principle.

Open Problem. Is there an axiomatic set theory with an impredicative comprehensition in which the principle $(F)\&(R)$ is not deducible?

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Ioachim Drugus
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Institute of Mathematics and Computer Science
Academy of Sciences of Moldova
5 Academiei str., Chisinau, MD-2028, Moldova
E–mail: `ioachim.drugus@math.md`