Computer Science Journal of Moldova, vol.24, no.1(70), 2016

Many-Sorted First-Order Composition-Nominative Logic as Institution

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Abstract

In the paper the institution for many-sorted first-order composition-nominative logic (CNL) is considered. The difference from the author's previous paper on this topic is richer logical system in question due to addition of operations and sorts, and also a slightly weakened constraint on signature morphisms regarding the set of names. The satisfaction condition is proven. Some directions for further research are outlined.

Keywords: Institution theory, many-sorted nominative data, irrefutability.

1 Introduction

Composition-nominative logics (CNL) are program-oriented algebra-based logics [1]–[3]. Many-sorted algebras of partial mappings form a semantic base of CNL. Mappings are defined over classes of nominative data considered in integrity of their intensional and extensional components [2]. The hierarchy of nominative data induces a hierarchy of CNLs. Properties of composition-nominative logics are quite well-studied [1], [3], [4]. Still there is a need to relate the results obtained for these logics to other logics. This can be achieved using such theoretical tools as institutions [5], [6].

Institutions are a unified framework that allows studying properties of logical systems in abstract way independently of notation [5], [7]. Institutions capture a lot of common features of different logics. So considering the logical system one is interested in presenting it as institution and finding out what specificity the obtained institution has.

This paper continues work started in [8], [9]. It aims to construct the institution for many-sorted first-order CNL. This is done in usual fashion when all necessary elements of the corresponding institution are gradually defined starting from category of signatures and ending with checking of satisfaction condition. The difference from one-sorted case is additional structure of sorts. It primarily affects variables and terms. Most compositions remain intact. However, some sort-awareness yet should be considered.

2 Indexed families of sets

In order to identify sorts in the system, we use indexed families of sets and functions. There are two approaches to the definition of the indexed families. The first one is conventional and most commonly used in the literature. Its systematic account can be found in [6]. The second approach is based on fibers. The reasoning behind it is presented in [10]. Some results concerning the connection between the approaches are listed in [11]. In this section we only recall necessary concepts and work out notation convention.

Definition 1. *Given a set of sorts S, an S-*sorted set *$\mathcal{B}$ is an object of the category $\mathbb{S}et^S$. Usually it is denoted $(B_s)_{s\in S}$. An S-*sorted map *is a morphism of the category $\mathbb{S}et^S$.*

It is known that the category $\mathbb{S}et^S$ is equivalent to the slice category $\mathbb{S}et/S$ [12], [13, sec. 7.9]. Where convenient, we use slice category constructions. It is stylistically closer to single-sorted case (provides easy transition by forgetting the sorts). It also allows to save writing by avoiding subscripts. The difference between two categories is that fibers of the object of $\mathbb{S}et/S$ are always disjoint while sets in the indexed family have no such restriction.

Consider an S-sorted set $\mathcal{A} = (A_s)_{s\in S}$. If sets A_s are pairwise disjoint, then there is a total function $T_A\colon A \to S$, where $A = \dot{\bigcup}_{s\in S} A_s$. Dot in the middle of symbol for union emphasizes that arguments are pairwise disjoint. Thus pair (A, T_A) determines indexed family. If the disjointness condition does not hold we can use coproduct $A =$

$\coprod_{s\in S} A_s$. There is a canonical map $T_A\colon A \to S$ such that $T_A(i_s(a)) = s$ for all $s \in S$, $a \in A_s$, where i_s is a coproduct injection. That is the following diagram commutes

$$\begin{tikzcd} A_s \arrow[r, "i_s"] \arrow[d, "!_{A_s}"'] & \coprod_{s\in S} A_s \arrow[d, "T_A"] \\ 1 \arrow[r, "s"] & S \end{tikzcd}$$

In both cases, we use slice category to represent $\mathcal{A}$. We write $\mathcal{A} = (A, T_A)$. In this representation S-sorted map from (A, T_A) to (B, T_B) is a function $f\colon A \to B$ such that the following diagram commutes.

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "T_A"'] & & B \arrow[dl, "T_B"] \\ & S & \end{tikzcd}$$

It is usually quite straightforward to recover presentation in $\mathbb{S}et^S$ from $\mathbb{S}et/S$ representation.

For a given *reindexing* $\varphi\colon S \to S'$, there are reindexing of S-sorted and S'-sorted sets described as "change of base" [13, sec. 9.7]. For any S-sorted set $\mathcal{A} = (A, T_A) = (A_s)_{s\in S}$, the corresponding S'-sorted set is $\varphi(\mathcal{A}) = (A, \varphi \circ T_A)$. Its fibers are defined as follows:

$$\varphi(\mathcal{A}) = \left(\coprod_{\varphi(s)=s'} A_s \right)_{s'\in S'}.$$

If $\mathcal{A}' = (A, T'_A)$ is an S'-sorted set, then we have an S-sorted set $\varphi^*(\mathcal{A}') = (A_{\varphi(s)})_{s\in S}$ defined by pullback along φ. This transition can be demonstrated by the following pullback diagram:

$$\begin{tikzcd} \coprod_{s\in S} A_{\varphi(s)} \arrow[r] \arrow[d, "T_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "T'_A"] \\ S \arrow[r, "\varphi"'] & S' \end{tikzcd}$$

The transitions have functorial behavior and can be applied to S-sorted maps as well.

3 Syntactic part

3.1 Language

Definition 2. *A* signature *of many-sorted first-order composition-nominative logic is a tuple* $\Sigma = (S, \mathcal{V}, P, \mathcal{F})$*, where* S *is a set of sorts,* $\mathcal{V}$ *an* S*-sorted set of names,* P *a set of predicate symbols and* $\mathcal{F}$ *an* S*-sorted set of operation symbols.*

Sentences of the language, called formulas, are constructed using symbols from the signature and a number of special *composition* symbols. Composition symbols form a tuple

$$C = \big(\vee, \neg, \{\exists x\}_{x \in V}, \{S^{v_1 \ldots v_n} \mid \bar{v} \in V^n, v_i \neq v_j \text{ for } i \neq j\}, \\ \{`x\}_{x \in V}, \{S_F^{v_1 \ldots v_n} \mid \bar{v} \in V^n, v_i \neq v_j \text{ for } i \neq j\}\big) .$$

Here traditional compositions: $\neg$ – negation, $\vee$ – disjunction, $\exists x$ – existential quantifier. Composition $`x$ is called *denomination.* Compositions $S^{v_1 \ldots v_n}$, $S_F^{v_1 \ldots v_n}$ are *substitutions in formula* and *in term* respectively. $\bar{v}$ denotes sequence $v_1 \ldots v_n$. There is a uniqueness constraint on names v_i in substitution: $v_i = v_j$ only if $i = j$. Usually composition symbols are not explicitly included into signature because they are fixed and fully determined by $\mathcal{V}$.

First, we define the S-sorted set of terms $\mathcal{T}(\Sigma) = (\mathrm{Ter}, T)$. The definition is mutually inductive for terms and their typing (here we use notation similar to [14])

$$\begin{array}{rcll} \tau & ::= & \alpha & : \ T_F(\alpha) \\ & & `x & : \ T_V(x) \\ & & S_F^{v_1 \ldots v_n}(t; t_1 \ldots t_n) & : \ T(t). \end{array} \tag{1}$$

Here $\alpha \in F$, $x, v_i \in V$, $i = \overline{1,n}$, t, t_i are terms. The terms t_i satisfy condition $T_V(v_i) = T(t_i)$ for all $i = \overline{1,n}$. Sorts after semicolons determine $T(\tau)$. The following notation for substitution is used:

$$[v_1 \mapsto t_1, \ldots, v_n \mapsto t_n]\, t = [\,\bar{v} \mapsto \bar{t}\,]\, t = S_F^{v_1 \ldots v_n}(t; t_1 \ldots t_n).$$

The class of Σ-*sentences* is based on class of terms and defined inductively:

$$\begin{array}{lll} \Phi & ::= & \pi \\ & & \neg\Psi \\ & & \Psi \vee \Psi' \\ & & \exists x \Psi \\ & & S^{v_1 \ldots v_n}(\Psi; t_1 \ldots t_n), \end{array} \tag{2}$$

where $\pi \in P$, $x, v_i \in V$; Ψ and Ψ' are formulas, $S^{v_1 \ldots v_n}$ – substitution in formula. Once again there are typing constraints: $T(t_i) = T_V(v_i)$ for all $i = \overline{1, n}$. We use notation

$$[v_1 \mapsto t_1, \ldots, v_n \mapsto t_n]\, \Phi = [\,\bar{v} \mapsto \bar{t}\,]\, \Phi = S^{v_1 \ldots v_n}(\Phi; t_1 \ldots t_n).$$

Implication, conjunction and universal quantifier are defined conventionally as follows

$$\begin{aligned} \Phi \wedge \Psi &= \neg(\neg\Phi \vee \neg\Psi) \\ \Phi \to \Psi &= \neg\Phi \vee \Psi \\ \forall x \Phi &= \neg\exists x \neg\Phi \\ R_{x_1 \ldots x_n}^{v_1 \ldots v_n} \Phi &= [v_1 \mapsto {}^{\backprime}x_1, \ldots, v_n \mapsto {}^{\backprime}x_n]\, \Phi. \end{aligned}$$

Composition $R_{x_1 \ldots x_n}^{v_1 \ldots v_n}$ is called *renomination* and usually abbreviated as $R_{\bar{x}}^{\bar{v}}$. Uniqueness constraint transfers to the set of *upper* names v_i of renomination.

3.2 Signature morphisms and sentence translation

Definition 3. *A* morphism *of signatures is*

$$\varphi = (\varphi_S, \varphi_V, \varphi_P, \varphi_F) \colon (S, \mathcal{V}, P, \mathcal{F}) \to (S', \mathcal{V}', P', \mathcal{F}'),$$

where $\varphi_P\colon P \to P'$, $\varphi_S\colon S \to S'$ is a reindexing, $\varphi_F\colon \varphi_S(\mathcal{F}) \to \mathcal{F}'$ an S'-sorted map, $\varphi_V\colon \varphi_S(\mathcal{V}) \to \mathcal{V}'$ an injective S'-sorted map. In other words the following diagrams commute:

$$\begin{tikzcd} V \arrow[r, "\varphi_V"] \arrow[d, "T_V"'] & V' \arrow[d, "T'_V"] \\ S \arrow[r, "\varphi_S"'] & S' \end{tikzcd} \qquad \begin{tikzcd} F \arrow[r, "\varphi_F"] \arrow[d, "T_F"'] & F' \arrow[d, "T'_F"] \\ S \arrow[r, "\varphi_S"'] & S' \end{tikzcd}$$

Name component φ_V of signature morphism is restricted to 1-1 mapping to avoid name clashes in substitution (renomination) composition and to be able to extend Mod to a functor.

Our category $\mathbb{S}ig$ is simply a category of signatures and signature morphisms defined above.

Now we can extend action of signature morphism to the Σ-sentences defined in (2), i.e. define $\mathrm{Sen}(\varphi)\colon\ \mathrm{Sen}(S, \mathcal{V}, P, \mathcal{F}) \to \mathrm{Sen}(S', \mathcal{V}', P', \mathcal{F}')$ inductively on structure of the sentence as follows

$$\begin{aligned}
\mathrm{Sen}(\varphi)(\alpha) &= \varphi_F(\alpha) \\
\mathrm{Sen}(\varphi)(`x) &= `\varphi_V(x) \\
\mathrm{Sen}(\varphi)([\,\bar v \mapsto \bar t\,]\, t') &= [\varphi_V(\bar v) \mapsto \mathrm{Sen}(\varphi)(\bar t)]\, \mathrm{Sen}(\varphi)(t') \\
\mathrm{Sen}(\varphi)(\pi) &= \varphi_P(\pi) \\
\mathrm{Sen}(\varphi)(\Phi \vee \Psi) &= \mathrm{Sen}(\varphi)(\Phi) \vee \mathrm{Sen}(\varphi)(\Psi) \\
\mathrm{Sen}(\varphi)(\neg\Phi) &= \neg\, \mathrm{Sen}(\varphi)(\Phi) \\
\mathrm{Sen}(\varphi)(\exists x \Phi) &= \exists \varphi_V(x)\, \mathrm{Sen}(\varphi)(\Phi) \\
\mathrm{Sen}(\varphi)([\,\bar v \mapsto \bar t\,]\, \Phi) &= [\varphi_V(\bar v) \mapsto \mathrm{Sen}(\varphi)(\bar t)]\, \mathrm{Sen}(\varphi)(\Phi).
\end{aligned}$$

Here $\xi(\bar l)$ denotes componentwise application of a function ξ to a list $\bar l = l_1, \ldots, l_n$, i.e. the list $\xi(l_1), \ldots, \xi(l_n)$.

Proposition 1. *The following diagram commutes:*

$$\begin{tikzcd} \mathrm{Ter} \arrow[r, "\mathrm{Sen}(\varphi)"] \arrow[d, "T"'] & \mathrm{Ter}' \arrow[d, "T'"] \\ S \arrow[r, "\varphi_S"'] & S' \end{tikzcd}$$

Proof. By induction on term structure. Let us check congruence rule.

$$\begin{aligned} T'\left(\mathrm{Sen}(\varphi)([\,\bar v \mapsto \bar t\,]\,t')\right) &= T'\left([\varphi_V(\bar v) \mapsto \mathrm{Sen}(\varphi)(\bar t)]\,\mathrm{Sen}(\varphi)(t')\right) \\ &= T'\left(\mathrm{Sen}(\varphi)(t')\right) = \varphi_S(T(t')) \\ &= \varphi_S(T([\,\bar v \mapsto \bar t\,]\,t')). \end{aligned}$$

Here we used induction hypothesis for t', and assumed typing constraint for $[\varphi_V(\bar v) \mapsto \mathrm{Sen}(\varphi)(\bar t)]\,\mathrm{Sen}(\varphi)(t')$. Let us prove the latter. By induction hypothesis, the properties of signature morphism and correctness of original term we have

$$T'(\mathrm{Sen}(\varphi)(\bar t)) = \varphi_S(T(\bar t)) = \varphi_S(T_V(\bar v)) = T'_V(\varphi_V(\bar v)). \qquad \square$$

As a result, $\mathrm{Sen}(\varphi)$ is correctly defined w.r.t. sorts.

Proposition 2. *$\mathbb{S}ig$ is a category.* Sen *is a functor $\mathbb{S}ig \to \mathbb{S}et$.*

In a context where Sen is known, expression $\mathrm{Sen}(\varphi)(\Phi)$ is usually abbreviated as simply $\varphi(\Phi)$.

4 Models and model homomorphisms

4.1 Many-sorted nominative data

The basis for semantics of various composition-nominative logics is formed by nominative sets, quasiary predicates and operations. Let $A \neq \emptyset$ be some set, V be the set of names. A (*partial*) *nominative set* is a partial mapping from V to A, the class of all such mappings is denoted ${}^V\!A$. In this context the set A is called the set of values, ${}^V\!A$ – the set of nominative sets or set of *states*. Nominative sets can be also called *nominative data*. By analogy with single-sorted case, we define many-sorted nominative data.

Definition 4. *A* partial S-sorted map $f\colon (A, T_A) \nrightarrow (B, T_B)$ *is a partial map $f\colon A \nrightarrow B$ such that the following diagram commutes in a* *weak sense*

$$\begin{tikzcd} A \arrow[rr, "f", "\shortmid"{marking}] \arrow[dr, "T_A"'] & & B \arrow[dl, "T_B"] \\ & S & \end{tikzcd}$$

i.e. $T_B \circ f = T_A\big|_{\mathrm{dom}\, f}$.

S-sorted sets and S-sorted partial maps form a category S-$\mathbb{S}et_{part}$.

Definition 5. *Let $\mathcal{V}$, $\mathcal{A}$ be S-sorted sets. An S*-sorted $\mathcal{V}$-nominative set *is a partial S-sorted map $d\colon \mathcal{V} \nrightarrow \mathcal{A}$.*

Let $\mathcal{V} = (V_s)_{s\in S}$, $\mathcal{A} = (A_s)_{s\in S}$. The class of all S-sorted $\mathcal{V}$-nominative sets is also an S-sorted set ${}^{\mathcal{V}}\!\mathcal{A}$ defined as follows:

$${}^{\mathcal{V}}\!\mathcal{A} = ({}^{V_s}\!A_s)_{s\in S}.$$

If we ignore sorts, then S-sorted nominative set d becomes simply a partial function $d\colon V \nrightarrow A$, where $V = \dot\bigcup_{s\in S} V_s$, $A = \coprod_{s\in S} A_s$. In this sense there is an embedding

$${}^{\mathcal{V}}\!\mathcal{A} \hookrightarrow {}^{V}\!A\ .$$

Sometimes we prefer to work with such representation of $d \in {}^{\mathcal{V}}\!\mathcal{A}$ rather than $(d_s)_{s\in S}$.

We use the following notation in regard to partiality. Let $f\colon A \nrightarrow B$, $a, a' \in A$, $b \in B$. We write $f(a)\!\uparrow$ if $a \notin \mathrm{dom}\, f$, otherwise (if $a \in \mathrm{dom}\, f$) we write $f(a)\!\downarrow$. Here $\mathrm{dom}\, f = f^{-1}(B) = \{x \mid (x,y) \in f \text{ for some } y\}$ is the *domain of definition* of f. In the latter case $f(a)\!\downarrow$ can be used as well as the value of f on a, e.g. $f(a)\!\downarrow = b$. Also we use symbol $\cong$ for *strong equality* that makes allowance for undefined value, namely

$$f(a) \cong f(a') \text{ if } f(a)\!\downarrow = f(a')\!\downarrow \text{ or } (f(a)\!\uparrow \text{ and } f(a')\!\uparrow).$$

Two partial functions f and g are equal if and only if $f(x) \cong g(x)$ for all x.

The elements of nominative data are pairs of the form $v \mapsto a$. Expression $v \mapsto a \in_n d$ denotes $d(v)\!\downarrow = a$. Given $v \in V_s$, $a \in A_s$

for some $s \in S$, expression $v \mapsto a$ in the context of ${}^{V}\!A$ means $v \mapsto i_s(a)$, where $i_s \colon A_s \to A$ is a canonical injection. Nominative sets are constructed using set-builder notation with square brackets.

Let us introduce the unary operation $r^{v_1 \ldots v_n}_{x_1 \ldots x_n} \colon {}^{\mathcal{V}}\!\mathcal{A} \to {}^{\mathcal{V}}\!\mathcal{A}$ of finite renomination of nominative set, where $T_V(v_i) = T_V(x_i)$ for all $i = \overline{1, n}$. First, we specify an S-sorted map $\sigma^{v_1 \ldots v_n}_{x_1 \ldots x_n} \colon \mathcal{V} \to \mathcal{V}$ associated with it:

$$\sigma^{v_1 \ldots v_n}_{x_1 \ldots x_n}(v) = \begin{cases} x_i & \text{if } v = v_i. \\ v & \text{otherwise.} \end{cases}$$

Then $r^{v_1 \ldots v_n}_{x_1 \ldots x_n} d = d \circ \sigma^{v_1 \ldots v_n}_{x_1 \ldots x_n}$, where $\circ$ denotes the composition of partial functions.

We require three more operations, *single name binding*, for $d \in {}^{\mathcal{V}}\!\mathcal{A}$, $u \in V_s$, $a \in A_s$, $s \in S$

$$d \triangledown u \mapsto a = d\big|_{V \setminus \{u\}} \mathbin{\dot\cup} [u \mapsto a].$$

Here $\big|_W$ denotes conventional restriction of function domain to W and dot in $\dot\cup$ emphasizes that the union is disjoint. *Finite name binding*, for $d \in {}^{\mathcal{V}}\!\mathcal{A}$, distinct names $v_i \in V_{s_i}$, $a_i \in A_{s_i}$, $s_i \in S$ for $i = \overline{1, n}$

$$d \triangledown [v_i \mapsto a_i \mid i = \overline{1, n}] = d\big|_{V \setminus \{v_i\}_{i=\overline{1,n}}} \mathbin{\dot\cup} [v_i \mapsto a_i \mid i = \overline{1, n}].$$

Finally *overriding*, for $d_1, d_2 \in {}^{V}\!A$

$$d_1 \triangledown d_2 = d_1\big|_{V \setminus \operatorname{dom} d_2} \mathbin{\dot\cup} d_2.$$

Construction ${}^{\mathcal{V}}\!\mathcal{A}$ demonstrates bifunctorial behavior in the following sense. Let $\sigma \colon \mathcal{V} \nrightarrow \mathcal{V}'$ be a partial S-sorted map, and $h \colon \mathcal{A} \to \mathcal{A}'$ be an S-sorted map. They induce several total maps between nominative set domains: function ${}^{\sigma}\!\mathcal{A} \colon {}^{\mathcal{V}'}\!\mathcal{A} \to {}^{\mathcal{V}}\!\mathcal{A}$ that maps nominative set $d \in {}^{\mathcal{V}'}\!\mathcal{A}$ to nominative set $d \circ \sigma$, function ${}^{\mathcal{V}}h \colon {}^{\mathcal{V}}\!\mathcal{A} \to {}^{\mathcal{V}}\!\mathcal{A}'$ that maps $d \in {}^{\mathcal{V}}\!\mathcal{A}$ to $h \circ d$, and function ${}^{\sigma}h \colon {}^{\mathcal{V}'}\!\mathcal{A} \to {}^{\mathcal{V}}\!\mathcal{A}'$ defined as $d \mapsto h \circ d \circ \sigma$. Notice that functions induced by change of set of values and set of names commute under composition:

$$ {}^{\mathcal{V}}h \circ {}^{\sigma}\!\mathcal{A} = {}^{\sigma}h = {}^{\sigma}\!\mathcal{A}' \circ {}^{\mathcal{V}'}h. \tag{3}$$

In these terms we have

$$r^{v_1\dots v_n}_{x_1\dots x_n} d = d \circ \sigma^{v_1\dots v_n}_{x_1\dots x_n} = {}^{\sigma^{\bar{v}}_{\bar{x}}}\mathcal{A}(d).$$

Ignoring the sorts does not change the functorial behavior of ${}^{\mathcal{V}}\!\mathcal{A}$.

4.2 Quasiary predicates and operations

Let $Bool = \{\top, \bot\}$ be a Boolean set. The *quasiary predicate* over S-sorted set of names $\mathcal{V}$ and S-sorted set of values $\mathcal{A}$ is a partial Boolean-valued function: ${}^{\mathcal{V}}\!\mathcal{A} \nrightarrow Bool$. The quasiary predicates over set of names $\mathcal{V}$ and set of values $\mathcal{A}$ are called $(\mathcal{V}, \mathcal{A})$-quasiary predicates for short. Let $Pr^{\mathcal{V}}_{\mathcal{A}} = \{p \mid p\colon {}^{\mathcal{V}}\!\mathcal{A} \nrightarrow Bool\}$.

The truth and falsity domains of $p \in Pr^{\mathcal{V}}_{\mathcal{A}}$ are respectively $\top(p) = \{d \mid p(d){\downarrow} = \top\} = p^{-1}(\{\top\})$, $\bot(p) = p^{-1}(\{\bot\})$.

Definition 6. *The* extension *of a partial predicate p is a pair of its truth and falsity domains: $\|p\| = (\top(p), \bot(p))$.*

Notice that sets in the extension of a predicate are necessarily disjoint. There is a 1-1 correspondence between extensions and partial predicates. Also there is a natural ordering of extensions:

$$\|p\| \subseteq \|p'\| \text{ if } \top(p) \subseteq \top(p') \text{ and } \bot(p') \subseteq \bot(p).$$

Definition 7. *A predicate p is* irrefutable *if $\bot(p) = \emptyset$.*

Like the domain of nominative data ${}^{\mathcal{V}}\!\mathcal{A}$, the construction $Pr^{\mathcal{V}}_{\mathcal{A}}$ also has bifunctorial behavior. Given partial S-sorted map $\sigma\colon \mathcal{V} \nrightarrow \mathcal{V}'$ and total S-sorted map $h\colon \mathcal{A} \to \mathcal{A}'$, there are total maps $Pr^{\mathcal{V}}_{h}\colon Pr^{\mathcal{V}}_{\mathcal{A}'} \to Pr^{\mathcal{V}}_{\mathcal{A}}$, $Pr^{\sigma}_{\mathcal{A}}\colon Pr^{\mathcal{V}}_{\mathcal{A}} \to Pr^{\mathcal{V}'}_{\mathcal{A}}$, $Pr^{\sigma}_{h}\colon Pr^{\mathcal{V}}_{\mathcal{A}'} \to Pr^{\mathcal{V}'}_{\mathcal{A}}$ realized as follows. Let $p \in Pr^{\mathcal{V}}_{\mathcal{A}'}$, $q \in Pr^{\mathcal{V}}_{\mathcal{A}}$, then

$$\begin{aligned} Pr^{\mathcal{V}}_{h}(p) &= p \circ {}^{\mathcal{V}}h \\ Pr^{\sigma}_{\mathcal{A}}(q) &= q \circ {}^{\sigma}\!\mathcal{A} \\ Pr^{\sigma}_{h}(p) &= p \circ {}^{\sigma}h. \end{aligned}$$

Once again, notice that maps induced by change of set of values and set of names commute under composition

$$Pr_h^{\mathcal{V}'} \circ Pr_{\mathcal{A}'}^{\sigma} = Pr_h^{\sigma} = Pr_{\mathcal{A}}^{\sigma} \circ Pr_h^{\mathcal{V}}. \tag{4}$$

Analogously to the quasiary predicates, we consider *quasiary operations* over S-sorted set of names $\mathcal{V}$ and S-sorted set of values $\mathcal{A}$ as partial functions: ${}^{\mathcal{V}}\!\mathcal{A} \nrightarrow A_s$ for some $s \in S$. The quasiary operations over set of names $\mathcal{V}$ and set of values $\mathcal{A}$ are called $(\mathcal{V}, \mathcal{A})$-quasiary operations for short. Let $Fn_{\mathcal{A},s}^{\mathcal{V}} = \{f \mid f\colon {}^{\mathcal{V}}\!\mathcal{A} \nrightarrow A_s\}$, $\mathcal{F}n_{\mathcal{A}}^{\mathcal{V}} = \left(Fn_{\mathcal{A},s}^{\mathcal{V}}\right)_{s\in S}$.

Similarly to the domain of nominative data ${}^{\mathcal{V}}\!\mathcal{A}$ and class of quasiary predicates $Pr_{\mathcal{A}}^{\mathcal{V}}$ construction $\mathcal{F}n_{\mathcal{A}}^{\mathcal{V}}$ also demonstrates functorial behavior, but this time only by parameter $\mathcal{V}$. Given a partial S-sorted map $\sigma\colon \mathcal{V} \nrightarrow \mathcal{V}'$, there is a total S-sorted map $Fn_{\mathcal{A}}^{\sigma}\colon \mathcal{F}n_{\mathcal{A}}^{\mathcal{V}} \to \mathcal{F}n_{\mathcal{A}}^{\mathcal{V}'}$ realized as follows: $Fn_{\mathcal{A},s}^{\sigma}(f) = f \circ {}^{\sigma}\!\mathcal{A}$, where $f \in Fn_{\mathcal{A},s}^{\mathcal{V}}$. Elements of $Fn_{\mathcal{A},s}^{\mathcal{V}}$ can also be thought as functorial algebras in S-$\mathbb{S}et_{part}$ (for the functor $(H_s)_{s\in S}$ such that $H_s(\mathcal{A}) = {}^{\mathcal{V}}\!\mathcal{A}$, and $H_{s'}(\mathcal{A}) = \emptyset$ for $s' \in S \setminus \{s\}$) [15, p. 142-143].

Given $d \in {}^{\mathcal{V}}\!\mathcal{A}$, distinct $v_i \in V$, $f_i \in Fn_{\mathcal{A},s_i}^{\mathcal{V}}$, such that $T_V(v_i) = s_i$, $i = \overline{1,n}$, there is a nominative data $[v_1 \mapsto f_1(d), \dots, v_n \mapsto f_n(d)] \in {}^{\mathcal{V}}\!\mathcal{A}$ defined as follows

$$v \mapsto a \in_n [v_1 \mapsto f_1(d), \dots, v_n \mapsto f_n(d)] \text{ if } \exists i \in \overline{1,n}. v = v_i, f_i(d){\downarrow} = a.$$

For short $[v_1 \mapsto f_1(d), \dots, v_n \mapsto f_n(d)]$ is written as $[\bar{v} \mapsto \bar{f}(d)]$. Let us introduce *substitution* operation $[v_1 \mapsto f_1, \dots, v_n \mapsto f_n]$ for nominative sets:

$$[v_1 \mapsto f_1, \dots, v_n \mapsto f_n]\, d = d\big|_{V\setminus\{v_i\}} \nabla [\bar{v} \mapsto \bar{f}(d)].$$

For short $[v_1 \mapsto f_1, \dots, v_n \mapsto f_n]\, d$ is written as $[\,\bar{v} \mapsto \bar{f}\,]\, d$.

4.3 Models

The sets $\mathcal{F}n_{\mathcal{A}}^{\mathcal{V}}$, $Pr_{\mathcal{A}}^{\mathcal{V}}$ are used as a carrier sets for most composition-nominative logics. The terms are interpreted as quasiary operations and formulas as quasiary predicates. Compositions have fixed interpretation for CNLs and are defined as follows

$$\begin{aligned}
{}^{\text{`}}x(d) &\cong d_{T_V(x)}(x).\\
\|p \vee q\| &= (\top(p) \cup \top(q), \bot(p) \cap \bot(q))\\
\|\neg p\| &= (\bot(p), \top(p))\\
\|\exists x p\| &= \big(\{d \mid d \nabla x \mapsto a \in \top(p) \text{ for some } a \in A_{T_V(x)}\},\\
&\qquad \{d \mid d \nabla x \mapsto a \in \bot(p) \text{ for all } a \in A_{T_V(x)}\}\big)\\
[\,\bar{v} \mapsto \bar{f}\,]\, g(d) &\cong g([\,\bar{v} \mapsto \bar{f}\,]\, d)\\
[\,\bar{v} \mapsto \bar{f}\,]\, p(d) &\cong p([\,\bar{v} \mapsto \bar{f}\,]\, d).
\end{aligned} \tag{5}$$

Here $d \in {}^{\mathcal{V}}\!\mathcal{A}$, $x, v_i \in V$, $g \in Fn^{\mathcal{V}}_{\mathcal{A},T_V(x)}$, $f_i \in Fn^{\mathcal{V}}_{\mathcal{A},T_V(v_i)}$, $i = \overline{1,n}$, $p, q \in Pr^{\mathcal{V}}_{\mathcal{A}}$; $[\,\bar{v} \mapsto \bar{f}\,]\, g(d)$ denotes $[v_1 \mapsto f_1, \ldots, v_n \mapsto f_n]\, g(d)$, likewise $[\,\bar{v} \mapsto \bar{f}\,]\, p(d)$ denotes $[v_1 \mapsto f_1, \ldots, v_n \mapsto f_n]\, p(d)$. In these terms

$$R^{\bar{v}}_{\bar{x}} p(d) \cong [\,\bar{v} \mapsto {}^{\text{`}}\bar{x}\,]\, p(d) \cong p([\,\bar{v} \mapsto {}^{\text{`}}\bar{x}\,]\, d) \cong p \circ {}^{\sigma^{\bar{v}}_{\bar{x}}}\!\mathcal{A}(d).$$

That is

$$R^{\bar{v}}_{\bar{x}} p = p \circ {}^{\sigma^{\bar{v}}_{\bar{x}}}\!\mathcal{A} = Pr^{\sigma^{\bar{v}}_{\bar{x}}}_{\mathcal{A}}(p).$$

Definition 8. *A* first-order algebra of $(\mathcal{V}, \mathcal{A})$-quasiary predicates *is a tuple* $(Pr, \mathcal{F}n, \mathcal{A}; Comp)$*, where* $Comp$ *are compositions defined in* (5) *and sets* $Pr \subseteq Pr^{\mathcal{V}}_{\mathcal{A}}$, $\mathcal{F}n \subseteq \mathcal{F}n^{\mathcal{V}}_{\mathcal{A}}$ *are closed under compositions.*

Definition 9. *Given a signature* $\Sigma = (S, \mathcal{V}, P, \mathcal{F})$*, a* Σ-model *of many-sorted first-order composition-nominative logic is a quadruple* $(Pr, \mathcal{F}n, \mathcal{A}, I)$ *such that* $(Pr, \mathcal{F}n, \mathcal{A}; Comp)$ *forms a first-order* $(\mathcal{V}, \mathcal{A})$*-quasiary predicates algebra and* $I = (I_P, I_F)$*, where* $I_P \colon P \to Pr$ *and* $I_F \colon \mathcal{F} \to \mathcal{F}n$ *are total and* S*-sorted total maps respectively.*

Interpretation of formulas and terms in a model is straightforward. The details are presented in section 6.

4.4 Model homomorphisms

Consider the conventional case of first-order logic. Model homomorphisms are functions $h \colon A \to B$ with operation and predicate preservation property. For each arity n due to contravariant powerset functor there is an induced map $\mathcal{P}_n(h)\colon \mathcal{P}(B^n) \to \mathcal{P}(A^n)$ between n-ary predicates. Preservation of n-ary predicate symbol $\pi \in P_n$ means $M_\pi \subseteq \mathcal{P}_n(h)(M'_\pi)$.

An analogous construction for quasiary case is presented in subsection 4.2. Let $h\colon \mathcal{A} \to \mathcal{A}'$ be a total S-sorted map. Consider total map $Pr^{\mathcal{V}}_h\colon Pr^{\mathcal{V}}_{\mathcal{A}'} \to Pr^{\mathcal{V}}_{\mathcal{A}}$ induced by h. Let us check its properties in regards to algebraic structure.

Proposition 3. *Function $Pr^{\mathcal{V}}_h$ preserves disjunction, negation and renomination compositions. If h is surjective, it also preserves existential quantifier composition.*

Proof. Let $p \in Pr^{\mathcal{V}}_{\mathcal{A}'}$, then

$$\top(Pr^{\mathcal{V}}_h(p)) = \{d \mid {}^{\mathcal{V}}h(d) \in \top(p)\} = \left({}^{\mathcal{V}}h\right)^{-1}(\top(p)).$$

Therefore

$$\|Pr^{\mathcal{V}}_h(p)\| = \Big(\left({}^{\mathcal{V}}h\right)^{-1}(\top(p)), \left({}^{\mathcal{V}}h\right)^{-1}(\bot(p))\Big) = \left({}^{\mathcal{V}}h\right)^{-1}\|p\|.$$

Let $p, q \in Pr^{\mathcal{V}}_{\mathcal{A}'}$, then

$$\begin{aligned} \|Pr^{\mathcal{V}}_h(\neg p)\| &= \left({}^{\mathcal{V}}h\right)^{-1}(\bot(p), \top(p)) = \|\neg Pr^{\mathcal{V}}_h(p)\|, \\ \|Pr^{\mathcal{V}}_h(p \vee q)\| &= \left({}^{\mathcal{V}}h\right)^{-1}(\top(p) \cup \top(q), \bot(p) \cap \bot(q)) \\ &= \|Pr^{\mathcal{V}}_h(p) \vee Pr^{\mathcal{V}}_h(q)\|, \end{aligned}$$

where preservation of unions and intersections by preimage is used.

For the renomination composition we use commutativity (4):

$$Pr^{\mathcal{V}}_h(R^{\bar{v}}_{\bar{x}}p) = Pr^{\mathcal{V}}_h \circ Pr^{\sigma^{\bar{v}}_{\bar{x}}}_{\mathcal{A}'}(p) = Pr^{\sigma^{\bar{v}}_{\bar{x}}}_{\mathcal{A}} \circ Pr^{\mathcal{V}}_h(p) = R^{\bar{v}}_{\bar{x}} Pr^{\mathcal{V}}_h(p).$$

Finally, if $h\colon \mathcal{A} \to \mathcal{A}'$ is surjective, then

$$
\begin{aligned}
\top(Pr_h^{\mathcal{V}}(\exists xp)) &= \Big\{d \in {}^{\mathcal{V}}\!\mathcal{A} \mid (h \circ d)\nabla x \mapsto a' \in \top(p) \text{ for some } a' \in A'_{T(x)}\Big\}\\
&= \big\{d \mid (h \circ d)\nabla x \mapsto h_s(a) \in \top(p) \text{ for some } a \in A_{T(x)}\big\}\\
&= \big\{d \mid h \circ (d\nabla x \mapsto a) \in \top(p) \text{ for some } a \in A_{T(x)}\big\}\\
&= \top(\exists x Pr_h^{\mathcal{V}}(p)).\\
\bot(Pr_h^{\mathcal{V}}(\exists xp)) &= \Big\{d \mid (h \circ d)\nabla x \mapsto a' \in \bot(p) \text{ for all } a' \in A'_{T(x)}\Big\}\\
&= \big\{d \in {}^{\mathcal{V}}\!\mathcal{A} \mid (h \circ d)\nabla x \mapsto h_s(a) \in \bot(p) \text{ for all } a \in A_{T(x)}\big\}\\
&= \bot(\exists x Pr_h^{\mathcal{V}}(p)).
\end{aligned}
$$

That is $Pr_h^{\mathcal{V}}(\exists xp) = \exists x Pr_h^{\mathcal{V}}(p)$. $\square$

Since there is no direct transformation between $\mathcal{F}n_{\mathcal{A}}^{\mathcal{V}}$ and $\mathcal{F}n_{\mathcal{A}'}^{\mathcal{V}}$, we cannot establish similar property for arbitrary substitution but we can do it for some subset of operations.

Definition 10. *Given a total function $h\colon \mathcal{A} \to \mathcal{A}'$, an operation $f \in Fn_{\mathcal{A},s}^{\mathcal{V}}$ is* h-related *to operation $f' \in Fn_{\mathcal{A}',s}^{\mathcal{V}}$ if the following diagram commutes.*

$$
\begin{tikzcd}
{}^{\mathcal{V}}\!\mathcal{A} \arrow[r, "f"] \arrow[d, "{}^{\mathcal{V}}h"'] & A_s \arrow[d, "h_s"] \\
{}^{\mathcal{V}}\!\mathcal{A}' \arrow[r, "f'"'] & A'_s
\end{tikzcd}
$$

The next proposition summarizes interaction between h and operation compositions.

Proposition 4. *For arbitrary map $h\colon \mathcal{A} \to \mathcal{A}'$ and name $x \in V$ composition ‘$x \in Fn_{\mathcal{A},T_V(x)}^{\mathcal{V}}$ is h-related to ‘$x \in Fn_{\mathcal{A}',T_V(x)}^{\mathcal{V}}$. If $g \in Fn_{\mathcal{A},s}^{\mathcal{V}}$, $f_i \in Fn_{\mathcal{A},s_i}^{\mathcal{V}}$ are h-related to $g' \in Fn_{\mathcal{A},s}^{\mathcal{V}}$, $f'_i \in Fn_{\mathcal{A}',s_i}^{\mathcal{V}}$, then substitution $[\,\bar{v} \mapsto \bar{f}\,]\,g$ is h-related to $[\,\bar{v} \mapsto \bar{f}'\,]\,g'$. If $f_i \in Fn_{\mathcal{A},s_i}^{\mathcal{V}}$ are h-related to $f'_i \in Fn_{\mathcal{A}',s_i}^{\mathcal{V}}$, $p \in Pr_{\mathcal{A}'}^{\mathcal{V}}$, then*

$$
Pr_h^{\mathcal{V}}([\,\bar{v} \mapsto \bar{f}'\,]\,p) = [\,\bar{v} \mapsto \bar{f}\,]\,Pr_h^{\mathcal{V}}(p). \tag{6}
$$

Proof. If $d \in {}^{\mathcal{V}}\!\mathcal{A}$, then $h_s(`x(d)) \cong h_s(d_s(x)) \cong {}^{\mathcal{V}}\!h(d)_s(x) \cong `x({}^{\mathcal{V}}\!h(d))$, where $s = T_V(x)$. For the second property we prove the commutativity of the diagram

$$\begin{tikzcd}
{}^{\mathcal{V}}\!\mathcal{A} \arrow[r, "{[\,\bar{v}\mapsto\bar{f}\,]}"] \arrow[d, "{}^{\mathcal{V}}\!h"'] & {}^{\mathcal{V}}\!\mathcal{A} \arrow[r, "g"] \arrow[d, "{}^{\mathcal{V}}\!h"] & A_s \arrow[d, "h_s"] \\
{}^{\mathcal{V}}\!\mathcal{A}' \arrow[r, "{[\,\bar{v}\mapsto\bar{f}'\,]}"'] & {}^{\mathcal{V}}\!\mathcal{A}' \arrow[r, "g'"'] & A'_s
\end{tikzcd}$$

The right rectangle is commutative by condition. Let $d \in {}^{\mathcal{V}}\!\mathcal{A}$, $v \in V$. Suppose $v \neq v_i$ for all $i = \overline{1,n}$. Then

$${}^{\mathcal{V}}\!h([\,\bar{v} \mapsto \bar{f}\,]\, d)(v) \cong h([\,\bar{v} \mapsto \bar{f}\,]\, d(v)) \cong h(d(v)) \cong [\,\bar{v} \mapsto \bar{f}'\,]\, ({}^{\mathcal{V}}\!h(d))(v).$$

Otherwise, if $v = v_i$ for some $i \in \overline{1,n}$, then

$$\begin{aligned}{}^{\mathcal{V}}\!h([\,\bar{v} \mapsto \bar{f}\,]\, d)(v) &\cong h([\,\bar{v} \mapsto \bar{f}\,]\, d(v_i)) \cong h(i_s(f_i(d))) \cong i'_s(f'_i({}^{\mathcal{V}}\!h(d))) \\ &\cong [\,\bar{v} \mapsto \bar{f}'\,]\, {}^{\mathcal{V}}\!h(d)(v),\end{aligned}$$

where $s = T_V(v_i)$ and $i_s\colon A_s \to A$, $i'_s\colon A'_s \to A'$ are canonical injections. This gives us commutativity of left rectangle. As a result outer rectangle is also commutative, i.e. the second property holds.

Let $p \in Pr^{\mathcal{V}}_{\mathcal{A}'}$, then

$$\begin{aligned}Pr^{\mathcal{V}}_h([\,\bar{v} \mapsto \bar{f}'\,]\, p) &= p \circ [\,\bar{v} \mapsto \bar{f}'\,] \circ {}^{\mathcal{V}}\!h \\ &= p \circ {}^{\mathcal{V}}\!h \circ [\,\bar{v} \mapsto \bar{f}\,] = [\,\bar{v} \mapsto \bar{f}\,]\, Pr^{\mathcal{V}}_h(p).\end{aligned}$$

Here we used the commutativity of left rectangle once again. □

Thus we only need to formalize preservation of predicates by map h. There are several ways to accomplish this. Here we do it similarly to the conventional case using the extensions of quasiary predicates.

Definition 11. *A* $(S, \mathcal{V}, P, (F_s)_{s\in S})$-model homomorphism $h\colon (Pr, \mathcal{F}n, \mathcal{A}, I) \to (Pr', \mathcal{F}n', \mathcal{A}', I')$ *is a total* S*-sorted map* $h\colon \mathcal{A} \to \mathcal{A}'$ *such that* $Pr^{\mathcal{V}}_h(Pr') \subseteq Pr$, $I_{F,s}(\alpha)$ *is* h*-related to* $I'_{F,s}(\alpha)$ *for all* $\alpha \in F_s$, $s \in S$, *and* $\|I_P(\pi)\| \subseteq \|Pr^{\mathcal{V}}_h(I'_P(\pi))\|$ *for all* $\pi \in P$.

Proposition 5. *$(S, \mathcal{V}, P, \mathcal{F})$-models and $(S, \mathcal{V}, P, \mathcal{F})$-model homomorphisms form a category* $\mathrm{Mod}(S, \mathcal{V}, P, \mathcal{F})$.

If this notion of homomorphism is too strict, other options include different relations between predicate extensions [8].

5 Model transformation

Now we need to figure out the change of model under signature morphism. Signature morphism has several components. Each of them cause some change of the model. We consider them one-by-one starting with predicate and operation symbols component, then following with change of names and ending with the change of sorts.

The simplest is the change of operation and predicate symbols. It only affects the interpretation functions for operation and predicate symbols. In the new model they become $(I'_P \circ \varphi_P, I'_F \circ \varphi_F)$. Due to properties of φ they are correct interpretations for the set of sorts S.

Consider the following commutative diagram

$$\begin{tikzcd}
V \arrow[r, "\varphi_V"] \arrow[d, "T_V"'] \arrow[dr, "T''_V"] & V' \arrow[d, "T'_V"] \\
S \arrow[r, "\varphi_S"'] & S'
\end{tikzcd}$$

It shows that model transformation is performed sequentially: first, according to the right triangle and then, according to the left triangle.

5.1 Change of names

Recall that name component of signature morphism is a 1-1 S-sorted map $\varphi_V : \varphi_S(\mathcal{V}) \to \mathcal{V}'$. Due to injectivity of φ_V there is a partial map $\psi_V : \mathcal{V}' \nrightarrow \varphi_S(\mathcal{V})$ such that $\varphi_V \circ \psi_V = \mathrm{id}_{\varphi_V(\varphi_S(\mathcal{V}))}$, $\psi_V \circ \varphi_V = \mathrm{id}_{\varphi_S(\mathcal{V})}$. It induces a total function ${}^{\psi_V}\!\mathcal{A} \colon {}^{\varphi_S(\mathcal{V})}\!\mathcal{A} \to {}^{\mathcal{V}'}\!\mathcal{A}$. Notice that ${}^{\psi_V}\!\mathcal{A}(d)(v')\uparrow$ for all $v' \in V' \setminus \varphi_V(V)$. There are also total functions $Pr^{\psi_V}_{\mathcal{A}} : Pr^{\mathcal{V}'}_{\mathcal{A}} \to Pr^{\varphi_S(\mathcal{V})}_{\mathcal{A}}$, $Fn^{\psi_{\mathcal{V}}}_{\mathcal{A},s'} \colon Fn^{\mathcal{V}'}_{\mathcal{A},s'} \to Fn^{\varphi_S(\mathcal{V})}_{\mathcal{A},s'}$. They are required to be able to jump from $\mathcal{V}'$-quasiary predicate model to $\mathcal{V}$-quasiary predicate model as Mod-functor implies. Before working out change of the model let us see how ψ_V affects the extension of quasiary predicate and how it interacts with compositions.

Lemma 6. *The following diagram commutes in the category of sets and partial mappings*

$$\begin{tikzcd}
V \arrow[r, "\varphi_V"] \arrow[d, "\sigma^{\bar{v}}_{\bar{x}}"'] & V' \arrow[r, "\psi_V"] \arrow[d, "\sigma^{\varphi_V(\bar{v})}_{\varphi_V(\bar{x})}"'] & V \arrow[d, "\sigma^{\bar{v}}_{\bar{x}}"] \\
V \arrow[r, "\varphi_V"'] & V' \arrow[r, "\psi_V"'] & V
\end{tikzcd}$$

Proof. Outer rectangle commutes because $\psi_V \circ \varphi_V = \mathrm{id}_V$. Notice that if $v \in V' \setminus \varphi_V(V)$, then value for both paths of right rectangle are undefined since $\psi(v)\uparrow$, $\sigma^{\varphi_V(\bar{v})}_{\varphi_V(\bar{x})}(v) = v$. Therefore right rectangle commutes. Left rectangle commutes because $\operatorname{dom}\psi_V = \varphi_V(V)$ and ψ_V is injective. $\square$

Proposition 7. *Let $\varphi_V\colon \varphi_S(\mathcal{V}) \to \mathcal{V}'$ be a name component of signature morphism. Then $Pr^{\psi_V}_{\mathcal{A}}\colon Pr^{\mathcal{V}'}_{\mathcal{A}} \to Pr^{\varphi_S(\mathcal{V})}_{\mathcal{A}}$, $Fn^{\psi_{\mathcal{V}}}_{\mathcal{A},s'}\colon Fn^{\mathcal{V}'}_{\mathcal{A},s'} \to Fn^{\varphi_S(\mathcal{V})}_{\mathcal{A},s'}$ preserve compositions in the following sense. Let $p', q' \in Pr^{\mathcal{V}'}_{\mathcal{A}}$, $x, v_j, x_j, u_i \in V$, $g' \in Fn^{\mathcal{V}'}_{\mathcal{A},s'}$, $f'_i \in Fn^{\mathcal{V}'}_{\mathcal{A},T''_V(u_i)}$, then*

$$\begin{aligned}
Pr^{\psi_V}_{\mathcal{A}}(\neg p') &= \neg Pr^{\psi_V}_{\mathcal{A}}(p')\\
Pr^{\psi_V}_{\mathcal{A}}(p' \vee q') &= Pr^{\psi_V}_{\mathcal{A}}(p') \vee Pr^{\psi_V}_{\mathcal{A}}(q')\\
Pr^{\psi_V}_{\mathcal{A}}(R^{\varphi_V(\bar{v})}_{\varphi_V(\bar{x})} p') &= R^{\bar{v}}_{\bar{x}} Pr^{\psi_V}_{\mathcal{A}}(p')\\
Pr^{\psi_V}_{\mathcal{A}}(\exists \varphi_V(x) p') &= \exists x Pr^{\psi_V}_{\mathcal{A}}(p')\\
Fn^{\psi_V}_{\mathcal{A},T''_V(x)}(`\varphi_V(x)) &= `x\\
Fn^{\psi_V}_{\mathcal{A},s'}([\,\varphi_V(\bar{u}) \mapsto \bar{f}'\,]\, g') &= [\bar{u} \mapsto Fn^{\psi_V}_{\mathcal{A},T''_V(\bar{u})}(\bar{f}')]\, Fn^{\psi_V}_{\mathcal{A},s'}(g')\\
Pr^{\psi_V}_{\mathcal{A}}([\,\varphi_V(\bar{u}) \mapsto \bar{f}'\,]\, p') &= [\bar{u} \mapsto Fn^{\psi_V}_{\mathcal{A},T''_V(\bar{u})}(\bar{f}')]\, Pr^{\psi_V}_{\mathcal{A}}(p').
\end{aligned}$$

Proof. The proof is similar to the proof of proposition 3. We directly check the properties and switch to extensions of predicates where needed. Suppose that $p', q' \in Pr_{\mathcal{A}}^{\mathcal{V}'}$. Then

$$\top(Pr_{\mathcal{A}}^{\psi_V}(p')) = \{d \in {}^{\varphi_S(\mathcal{V})}\mathcal{A} \mid {}^{\psi_V}\mathcal{A}(d) \in \top(p')\} = {}^{\psi_V}\mathcal{A}^{-1}(\top(p')),$$

where ${}^{\psi_V}\mathcal{A}^{-1}(D)$ is a preimage of D under map ${}^{\psi_V}\mathcal{A}$. The same goes for the falsity domain. Therefore

$$\|Pr_{\mathcal{A}}^{\psi_V}(p')\| = \left({}^{\psi_V}\mathcal{A}^{-1}(\top(p')), {}^{\psi_V}\mathcal{A}^{-1}(\bot(p'))\right) = {}^{\psi_V}\mathcal{A}^{-1}(\|p'\|).$$

Respectively

$$\begin{aligned}
\|Pr_{\mathcal{A}}^{\psi_V}(\neg p')\| &= {}^{\psi_V}\mathcal{A}^{-1}(\bot(p'), \top(p')) = \|\neg Pr_{\mathcal{A}}^{\psi_V}(p')\|, \\
\|Pr_{\mathcal{A}}^{\psi_V}(p' \vee q')\| &= {}^{\psi_V}\mathcal{A}^{-1}(\top(p') \cup \top(q'), \bot(p') \cap \bot(q')) \\
&= \|Pr_{\mathcal{A}}^{\psi_V}(p') \vee Pr_{\mathcal{A}}^{\psi_V}(q')\|.
\end{aligned}$$

Here we used the properties of the preimage of a function.

For the existential quantifier let $p' \in Pr_{\mathcal{A}}^{\mathcal{V}'}$, then

$$\begin{aligned}
\top(\exists x Pr_{\mathcal{A}}^{\psi_V}(p')) &= \left\{d \mid (d \nabla x \mapsto a) \circ \psi_V \in \top(p') \text{ for some } a \in A_{T''_V(x)}\right\} \\
&= \left\{d \mid d \circ \psi_V \nabla \varphi_V(x) \mapsto a \in \top(p') \text{ for some } a \in A_s\right\} \\
&= \top(Pr_{\mathcal{A}}^{\psi_V}(\exists \varphi_V(x) p')).
\end{aligned}$$

Here we used the definition of ψ_V and the following property of nominative sets. For $d \in {}^{\mathcal{V}}\mathcal{A}$, partial function $\sigma \colon \mathcal{V}' \nrightarrow \mathcal{V}$ we have

$$\begin{aligned}
(d \nabla x \mapsto a) \circ \sigma &= \left(d\big|_{V \setminus \{x\}} \cup [x \mapsto a]\right) \circ \sigma \\
&= \left(d \circ \sigma\big|_{\sigma^{-1}(V) \setminus \sigma^{-1}(\{x\})} \cup \left[x' \mapsto a \mid \sigma(x')\downarrow = x)\right]\right) \\
&= d \circ \sigma \nabla \left[x' \mapsto a \mid \sigma(x')\downarrow = x)\right].
\end{aligned}$$

Repeating for falsity domain and combining we derive

$$\exists x Pr_{\mathcal{A}}^{\psi_V}(p') = Pr_{\mathcal{A}}^{\psi_V}(\exists \varphi_V(x) p').$$

For renomination by lemma 6 we immediately have

$$R^{\bar{v}}_{\bar{x}} Pr^{\psi_V}_{\mathcal{A}}(p') = Pr^{\sigma^{\bar{v}}_{\bar{x}}} \circ Pr^{\psi_V}_{\mathcal{A}}(p') = Pr^{\psi_V}_{\mathcal{A}}(R^{\varphi_V(\bar{v})}_{\varphi_V(\bar{x})} p').$$

Now let us consider denomination. For $s' = T''_V(x)$ we have

$$Fn^{\psi_V}_{\mathcal{A},s'}(`\varphi_V(x))(d) \cong {}^{\psi_V}\!A(d)_{s'}(\varphi_V(x)) \cong d_{s'}(\psi_V(\varphi_V(x)) \cong `x(d).$$

Let $g' \in Fn^{\mathcal{V}'}_{\mathcal{A},s'}$, $f'_i \in Fn^{\mathcal{V}'}_{\mathcal{A},T''_V(u_i)}$, $u_i \in V$, $i = \overline{1,n}$. Notice that

$$\begin{aligned}([\bar{u} \mapsto Fn^{\psi_V}_{\mathcal{A},T''_V(\bar{u})}(\bar{f}')]\, d) \circ \psi_V &= \Big(d\Big|_{V \setminus \{u_i\}} \nabla [\bar{u} \mapsto \bar{f}'(d \circ \psi_V)]\Big) \circ \psi_V \\ &= [\, \varphi_V(\bar{u}) \mapsto \bar{f}'\,]\,(d \circ \psi_V).\end{aligned}$$

Here we used equality

$$[\bar{u} \mapsto \bar{f}'(d')] \circ \psi_V = [v \mapsto f'_i(d') \mid \psi_V(v){\downarrow} = u_i] = [\varphi_V(\bar{u}) \mapsto \bar{f}'(d')].$$

Then

$$\begin{aligned}Fn^{\psi_V}_{\mathcal{A},s'}([\, \varphi_V(\bar{u}) \mapsto \bar{f}'\,]\, g')(d) &\cong [\, \varphi_V(\bar{u}) \mapsto \bar{f}'\,]\, g'(d \circ \psi_V) \\ &\cong g'([\, \varphi_V(\bar{u}) \mapsto \bar{f}'\,]\, d \circ \psi_V) \\ &\cong g'(([\bar{u} \mapsto Fn^{\psi_V}_{\mathcal{A},T''_V(\bar{u})}(\bar{f}')]\, d) \circ \psi_V) \\ &\cong [\bar{u} \mapsto Fn^{\psi_V}_{\mathcal{A},T''_V(\bar{u})}(\bar{f}')]\, Fn^{\psi_V}_{\mathcal{A},s'}(g')(d).\end{aligned}$$

And similarly

$$Pr^{\psi_V}_{\mathcal{A}}([\, \varphi_V(\bar{u}) \mapsto \bar{f}'\,]\, p')(d) \cong [\bar{u} \mapsto Fn^{\psi_V}_{\mathcal{A},T''_V(\bar{u})}(\bar{f}')]\, Pr^{\psi_V}_{\mathcal{A}}(p')(d). \quad \square$$

5.2 Change of base

We start with commutative triangle

$$\begin{tikzcd} & V \arrow[dl, "T_V"'] \arrow[dr, "T''_V"] & \\ S \arrow[rr, "\varphi_S"'] & & S' \end{tikzcd}$$

According to it $(V, T''_V) = \varphi(\mathcal{V})$. A new S-sorted set of values is obtained from S'-sorted $\mathcal{A}$ using pullback functor $\varphi^*_S(\mathcal{A}) = (A_{\varphi_S(s)})_{s \in S}$. Fundamental for the construction of quasiary predicates and operations of new model is to understand the connection between nominative data of the two models. Here we use 1-1 correspondence

$$\frac{d\colon \mathcal{V} \nrightarrow \varphi^*_S(\mathcal{A})}{d^{\#}\colon \varphi_S(\mathcal{V}) \nrightarrow \mathcal{A}}$$

realized as follows:

$$\begin{aligned} d^{\#}_{s'} &= \dot{\bigcup_{\varphi_S(s)=s'}} d_s \\ d_s &= d^{\#}_{\varphi_S(s)}\Big|_{V_s}. \end{aligned}$$

It is actually a conventional adjunction $\varphi_S \dashv \varphi^*_S$ but extended to partial S-sorted maps [13, sec. 9.7].

Let $p \in Pr^{\varphi_S(\mathcal{V})}_{\mathcal{A}}$, $s \in S$, $f \in Fn^{\varphi_S(\mathcal{V})}_{\mathcal{A},\varphi_S(s)}$. Then there are $p^{\#} \in Pr^{\mathcal{V}}_{\varphi^*_S(\mathcal{A})}$, $f^{\#} \in Fn^{\mathcal{V}}_{\varphi^*_S(\mathcal{A}),s}$ defined as

$$\begin{aligned} p^{\#}(d) &\cong p(d^{\#}) \\ f^{\#}(d) &\cong f(d^{\#}). \end{aligned}$$

Notice there is an instance of $f^{\#}$ for each s' such that $\varphi_S(s') = \varphi_S(s)$.

By construction, the transition $d \mapsto d^{\#}$ preserves the operation of domain restriction and disjoint union of nominative sets. Respectively

$$\begin{aligned} (d_1 \nabla d_2)^{\#} &= d_1^{\#} \nabla d_2^{\#} \\ (d \nabla x \mapsto a)^{\#} &= d^{\#} \nabla x \mapsto a \\ (d \nabla [\bar{v} \mapsto \bar{a}])^{\#} &= d^{\#} \nabla [\bar{v} \mapsto \bar{a}] \\ [\,\bar{u} \mapsto \bar{f}^{\#}(d)\,]^{\#} &= [\,\bar{u} \mapsto \bar{f}(d^{\#})\,]. \end{aligned}$$

Proposition 8. *The correspondence $p \mapsto p^{\#}$, $(f,s) \mapsto f^{\#}$ preserves compositions in the following sense:*

$$
\begin{aligned}
(p \vee q)^{\#} &= p^{\#} \vee q^{\#} \\
(\neg p)^{\#} &= \neg p^{\#} \\
(R^{\bar{v}}_{\bar{x}} p)^{\#} &= R^{\bar{v}}_{\bar{x}} p^{\#} \\
(\exists x p)^{\#} &= \exists x p^{\#} \\
(\text{`}x)^{\#} &= \text{`}x \\
([\,\bar{u} \mapsto \bar{f}\,]\, g)^{\#} &= [\bar{u} \mapsto \bar{f}^{\#}]\, g^{\#} \\
([\,\bar{u} \mapsto \bar{f}\,]\, p)^{\#} &= [\bar{u} \mapsto \bar{f}^{\#}]\, p^{\#}.
\end{aligned}
$$

Proof. We immediately check

$$
\begin{aligned}
\|(p \vee q)^{\#}\| &= (\{d \mid d^{\#} \in \top(p \vee q)\}, \{d \mid d^{\#} \in \bot(p \vee q)\}) \\
&= (\top(p^{\#}) \cup \top(q^{\#}), \bot(p^{\#}) \cap \bot(q^{\#})) = \|p^{\#} \vee q^{\#}\|. \\
\|(\neg q)^{\#}\| &= (\{d \mid d^{\#} \in \bot(q)\}, \{d \mid d^{\#} \in \top(q)\}) = \|\neg q^{\#}\|. \\
\top((\exists x p)^{\#}) &= \{d \mid d^{\#} \triangledown x \mapsto a \in \top(p) \text{ for some } a \in A_{T''_V(x)}\} \\
&= \{d \mid (d \triangledown x \mapsto a)^{\#} \in \top(p) \text{ for some } a \in A_{\varphi_P(T_V(x))}\} \\
&= \top(\exists x p^{\#}). \\
\bot((\exists x p)^{\#}) &= \{d \mid (d \triangledown x \mapsto a)^{\#} \in \bot(p) \text{ for all } a \in A_{\varphi_P(T_V(x))}\} \\
&= \bot(\exists x p^{\#}). \\
[\,\bar{u} \mapsto \bar{f}\,]\, d^{\#} &= d^{\#} \triangledown [\,\bar{u} \mapsto \bar{f}(d^{\#})\,] \\
&= (d \triangledown [\,\bar{u} \mapsto \bar{f}^{\#}(d)\,])^{\#} = ([\,\bar{u} \mapsto \bar{f}^{\#}\,]\, d)^{\#}. \\
([\,\bar{u} \mapsto \bar{f}\,]\, p)^{\#}(d) &\cong p([\,\bar{u} \mapsto \bar{f}\,]\, d^{\#}) \\
&\cong p(([\,\bar{u} \mapsto \bar{f}^{\#}\,]\, d)^{\#}) \cong [\bar{u} \mapsto \bar{f}^{\#}]\, p^{\#}(d). \\
([\,\bar{u} \mapsto \bar{f}\,]\, g)^{\#}(d) &\cong g([\,\bar{u} \mapsto \bar{f}\,]\, d^{\#}) \\
&\cong g(([\,\bar{u} \mapsto \bar{f}^{\#}\,]\, d)^{\#}) \cong [\bar{u} \mapsto \bar{f}^{\#}]\, g^{\#}(d). \\
(\text{`}x)^{\#}(d) &\cong d^{\#}_{T''_V(x)}(x) \cong \dot{\bigcup_{\varphi_S(s)=\varphi_S(T_V(x))}} d_s(x) \cong \text{`}x(d).
\end{aligned}
$$

$$(R_{\bar{x}}^{\bar{v}}p)^{\#}(d) \cong p(d^{\#} \circ \sigma_{\bar{x}}^{\bar{v}}) \cong p((d \circ \sigma_{\bar{x}}^{\bar{v}})^{\#}) \cong R_{\bar{x}}^{\bar{v}}p^{\#}(d). \qquad \square$$

Suppose $Pr \subseteq Pr_{\mathcal{A}}^{\varphi_S(\mathcal{V})}$, then we denote $Pr^{\#} = \{p^{\#} \mid p \in Pr\}$.

5.3 Reduct functor

Next we provide combined model transformation.

Proposition 9. *Let* $\varphi\colon (S, \mathcal{V}, P, (F_s)_{s\in S}) \to (S', \mathcal{V}', P', \mathcal{F}')$ *be a signature morphism and* $M' = (Pr', (Fn'_{s'})_{s'\in S'}, \mathcal{A}, I')$ *be a* $(S', \mathcal{V}', P', \mathcal{F}')$*-model. Then there is a* $(S, \mathcal{V}, P, (F_s)_{s\in S})$*-model*

$$Mod(\varphi)(M') = (Pr, (Fn_s)_{s\in S}, \varphi_S^*(\mathcal{A}), (I_P, I_F)), \tag{7}$$

where

$$\begin{aligned} Pr &= Pr_{\mathcal{A}}^{\psi_V}(Pr')^{\#}, \quad Fn_s = Fn_{\mathcal{A},\varphi_S(s)}^{\psi_V}(Fn'_{\varphi_S(s)})^{\#}, \\ I_P(\pi) &= Pr_{\mathcal{A}}^{\psi_V}(I'_P(\varphi_P(\pi)))^{\#} \text{ for } \pi \in P, \\ I_{F,s}(\alpha) &= Fn_{\mathcal{A},\varphi_S(s)}^{\psi_V}(I'_{F,\varphi_S(s)}(\varphi_F(\alpha)))^{\#} \text{ for } \alpha \in F_s. \end{aligned}$$

Proof. Considerations above show that Pr is closed under quasiary predicates compositions. For instance, assume that $p_{1,2} = Pr_{\mathcal{A}}^{\psi_V}(p'_{1,2})^{\#} \in Pr$. Then $p_1 \vee p_2 = Pr_{\mathcal{A}}^{\psi_V}(p'_1 \vee p'_2)^{\#} \in Pr$. Class $(Fn_s)_{s\in S}$ is also closed under operation composition. For instance, suppose that $v_i \in V$, $f_i = Fn_{\mathcal{A},T''_V(v_i)}^{\psi_V}(f'_i)^{\#} \in Fn_{T_V(v_i)}$, $i = \overline{1,n}$, $g = Fn_{\mathcal{A},\varphi_S(s)}^{\psi_V}(g')^{\#} \in Fn_s$. Then

$$[\,\bar{v} \mapsto \bar{f}\,]\, g = Fn_{\mathcal{A},\varphi_S(s)}^{\psi_V}([\,\varphi_V(\bar{v}) \mapsto \bar{f}'\,]\, f')^{\#} \in Fn_s.$$

Thus $(Pr, (Fn_s)_{s\in S}, \varphi_S^*(\mathcal{A}); Comp(\mathcal{V}, \varphi_S^*(\mathcal{A})))$ is indeed a $(\mathcal{V}, \mathcal{A})$-quasiary predicate algebra. The following diagram

$$P \xrightarrow{\varphi_P} P' \xrightarrow{I'_P} Pr' \xrightarrow{Pr_{\mathcal{A}}^{\psi_V}} Pr_{\mathcal{A}}^{\psi_V}(Pr') \xrightarrow{\#} Pr$$

shows that I_P has proper type $P \to Pr$. The same way $I_{F,s}\colon F_s \to Fn_s$. $\square$

Corollary 10. *Given signatures* $\Sigma = (S, \mathcal{V}, P, \mathcal{F})$, $\Sigma' = (S', \mathcal{V}', P', \mathcal{F}')$, *morphism* $\varphi\colon \Sigma \to \Sigma'$ *and* Σ'*-models* $M' = (Pr', \mathcal{F}n', \mathcal{A}, I')$, $M'_1 = (Pr'_1, \mathcal{F}n'_1, \mathcal{A}_1, I'_1)$, *any* Σ'*-model homomorphism* $h\colon M' \to M'_1$ *induces* Σ*-model homomorphism* $\varphi^*_S(h)\colon \mathrm{Mod}(\varphi)(M') \to \mathrm{Mod}(\varphi)(M'_1)$.

Proof. There are three conditions to check here: that map $Pr^{\mathcal{V}}_{\varphi^*_S(h)}$ satisfies image condition, that it preserves predicates from P and that map $\varphi^*_S(h)$ preserves operations from $\mathcal{F}$.

By definition of $(S', \mathcal{V}', P', \mathcal{F}')$-model homomorphism, commutativity (4) and properties of images we have

$$Pr^{\psi_V}_{\mathcal{A}}(Pr')^{\#} \subseteq Pr^{\psi_V}_{\mathcal{A}}(Pr^{\mathcal{V}'}_h(Pr'_1))^{\#} = Pr^{\mathcal{V}}_{\varphi^*_S(h)}(Pr^{\psi_V}_{\mathcal{A}_1}(Pr'_1)^{\#}).$$

Here we also used properties of adjunction:

$$Pr^{\mathcal{V}}_h(p)^{\#}(d) \cong p(h \circ d^{\#}) \cong p((\varphi^*_S(h) \circ d)^{\#}) \cong Pr^{\mathcal{V}}_{\varphi^*_S(h)}(p^{\#})(d).$$

Similarly, for any $\pi \in P$ we have

$$\|Pr^{\psi_V}_{\mathcal{A}}(I'_P(\pi'))^{\#}\| \subseteq \|Pr^{\mathcal{V}}_{\varphi^*_S(h)}(Pr^{\psi_V}_{\mathcal{A}_1}(I'_{1P}(\pi'))^{\#})\|,$$

where $\pi' = \varphi_P(\pi)$.

Finally, for any $\alpha \in F_s$ we have

$$\begin{aligned}
\varphi^*_S(h)_s(I_{F,s}(\alpha)(d)) &\cong h_{\varphi_S(s)} \circ I'_{F,\varphi_S(s)}(\alpha') \circ {}^{\psi_V}\mathcal{A}(d^{\#}) \\
&\cong I'_{1F,\varphi_S(s)}(\alpha') \circ {}^{\mathcal{V}'}h \circ {}^{\psi_V}\mathcal{A}(d^{\#}) \\
&\cong I'_{1F,\varphi_S(s)}(\alpha') \circ {}^{\psi_V}\mathcal{A}_1(h \circ d^{\#}) \\
&\cong I'_{1F,\varphi_S(s)}(\alpha') \circ {}^{\psi_V}\mathcal{A}_1((\varphi^*_S(h) \circ d)^{\#}) \\
&\cong I_{1F,s}(\alpha) \circ {}^{\mathcal{V}}\varphi^*_S(h)(d).
\end{aligned}$$

where $\alpha' = \varphi_F(\alpha) \in F'_{\varphi_S(s)}$. Here we used preservation of α' by h, adjunction and commutativity for induced maps (3). ☐

Corollary 11. *The construction given by* (7) *and*

$$\mathrm{Mod}(\varphi)(h) = \varphi^*_S(h)$$

extends Mod *to a functor* $\mathbb{S}ig \to \mathbb{C}at$.

When Mod is known, the model resulting from the application of the reduct functor $\mathrm{Mod}(\varphi)$ to M, i.e. $\mathrm{Mod}(\varphi)(M)$, is often abbreviated as $M\big|_{\varphi}$.

6 Satisfaction relation

First, we extend the interpretation to all terms and formulas. Due to the definition of quasiary predicate algebras it is quite easy. Let $\Sigma = (S, \mathcal{V}, P, \mathcal{F})$, $t \in \mathrm{Ter}(\Sigma)$, $\Phi \in \mathrm{Sen}(\Sigma)$, $M = (Pr, (Fn_s)_{s \in S}, \mathcal{A}, I) \in |\mathrm{Mod}(\Sigma)|$. We define $M(\Phi) \in Pr$, $M_{T(t)}(t) \in Fn_{T(t)}$ inductively:

$$\begin{aligned}
M_s(\alpha) &= I_{F,s}(\alpha), \text{ where } \alpha \in F_s \\
M_{T_V(x)}(`x) &= `x \\
M_{T(t')}([\,\bar{v} \mapsto \bar{t}\,]\, t') &= [\bar{v} \mapsto M_{T(\bar{t})}(\bar{t})]\, M_{T(t')}(t') \\
M(\pi) &= I_P(\pi) \\
M(\Phi \vee \Psi) &= M(\Phi) \vee M(\Psi) \\
M(\neg\Phi) &= \neg M(\Phi) \\
M(\exists x \Phi) &= \exists x M(\Phi) \\
M([\,\bar{v} \mapsto \bar{t}\,]\, \Phi) &= [\bar{v} \mapsto M_{T(\bar{t})}(\bar{t})]\, M(\Phi).
\end{aligned}$$

In the right-hand side we use the interpretation of composition symbols given in subsection 4.3. The interpretation is respected by homomorphisms in the following sense.

Proposition 12. *Let $h\colon M \to M_1$ be a Σ-model homomorphism, where $M = (Pr, \mathcal{F}n, \mathcal{A}, I)$, $M_1 = (Pr_1, \mathcal{F}n_1, \mathcal{A}_1, I_1)$ are Σ-models. Then $M(t)$ is h-related to $M'(t)$ for any Σ-term t.*

Proof. By induction on term structure and proposition 4. ☐

Corollary 13. *Let $h\colon M \to M_1$ be a Σ-model homomorphism, where $M = (Pr, \mathcal{F}n, \mathcal{A}, I)$, $M_1 = (Pr_1, \mathcal{F}n_1, \mathcal{A}_1, I_1)$. Then for arbitrary Σ-terms t_i, $i = \overline{1, n}$ and predicate $p_1 \in Pr_1$, the following holds*

$$Pr_h^{\mathcal{V}}([\bar{v} \mapsto M_1(\bar{t})]\, p_1) = [\bar{v} \mapsto M(\bar{t})]\, Pr_h^{\mathcal{V}}(p_1).$$

Definition 12. *A formula* $\Phi \in \mathrm{Sen}(\Sigma)$ *is* satisfied *by* Σ*-model* $M = (Pr, \mathcal{F}n, \mathcal{A}, I)$*, if the predicate* $M(\Phi)$ *is irrefutable, i.e.* $\bot(M(\Phi)) = \emptyset$*. This is denoted by* $M \models \Phi$*.*

Let us see how change of notation affects interpretation of a formula.

Proposition 14. *Given a* Σ*-term* t*, formula* $\Phi \in \mathrm{Sen}(\Sigma)$*, signature* $\Sigma' = (S', \mathcal{V}', P', (F'_s)_{s' \in S'})$*,* Σ'*-model* $M' = (Pr', \mathcal{F}n', \mathcal{A}, I')$ *and signature morphism* $\varphi\colon \Sigma \to \Sigma'$*, the following holds:*

$$Fn^{\psi_V}_{\mathcal{A},\varphi_S(s)}(M'_{\varphi_S(s)}(\varphi(t)))^{\#} = M'\big|_{\varphi,s}(t), \text{ where } s = T(t)$$
$$Pr^{\psi_V}_{\mathcal{A}}(M'(\varphi(\Phi)))^{\#} = M'\big|_{\varphi}(\Phi).$$

Proof. By induction on structure of term t and formula Φ respectively. Let $\alpha \in F_s$ be an operation symbol. Then

$$\begin{aligned}
Fn^{\psi_V}_{\mathcal{A},\varphi_S(s)}(M'_{\varphi_S(s)}(\varphi_F(\alpha)))^{\#} &= Fn^{\psi_V}_{\mathcal{A},\varphi_S(s)}(I'_{F,\varphi_S(s)}(\varphi_F(\alpha)))^{\#} \\
&= M'\big|_{\varphi,s}\varphi_F(\alpha). \\
Fn^{\psi_V}_{\mathcal{A},\varphi_S(T_V(x))}(M'_{\varphi_S(T_V(x))}(`\varphi_V(x)))^{\#} &= Fn^{\psi_V}_{\mathcal{A},\varphi_S(T_V(x))}(`\varphi_V(x))^{\#} \\
&= `x^{\#} = M'\big|_{\varphi,T_V(x)}(`x).
\end{aligned}$$

In the latter we used propositions 7 and 8. For the next case in addition to them we use induction hypothesis. Let t, t_i be Σ-terms, $v_i \in V$, $T_V(v_i) = T(t_i) = s_i$, $T(t) = s$, $\varphi_S(s) = s'$, $i = \overline{1,n}$. Then

$$\begin{aligned}
Fn^{\psi_V}_{\mathcal{A},s'}(M'(\varphi([\,\bar{v} \mapsto \bar{t}\,]\,t)))^{\#} &= Fn^{\psi_V}_{\mathcal{A},s'}\left([\varphi_V(\bar{v}) \mapsto M'_{\bar{s}}(\varphi(\bar{t}))]\, M'(\varphi(t))\right)^{\#} \\
&= [\bar{v} \mapsto M'\big|_{\varphi,\bar{s}}(\bar{t})]\, M'\big|_{\varphi,s}(t') \\
&= M'\big|_{\varphi,s}([\,\bar{v} \mapsto \bar{t}\,]\,t').
\end{aligned}$$

The same can be done for formulas. Let $\pi \in P$ be a predicate symbol, then

$$Pr^{\psi_V}_{\mathcal{A}}(M'(\varphi_P(\pi)))^{\#} = Pr^{\psi_V}_{\mathcal{A}}(I'_P(\varphi_P(\pi)))^{\#} = M'\big|_{\varphi}(\pi).$$

For the rest of cases we use propositions 7, 8, induction hypothesis and just proven property for terms:

$$\begin{aligned}
Pr_{\mathcal{A}}^{\psi_V}(M'(\varphi(\Phi \vee \Psi)))^{\#} &= Pr_{\mathcal{A}}^{\psi_V}\left(M'(\varphi(\Phi))^{\#} \vee M'(\varphi(\Psi))\right)^{\#} \\
&= Pr_{\mathcal{A}}^{\psi_V}\left(M'(\varphi(\Phi))\right)^{\#} \vee Pr_{\mathcal{A}}^{\psi_V}\left(M'(\varphi(\Psi))\right)^{\#} \\
&= M'\big|_{\varphi}(\Phi) \vee M'\big|_{\varphi}(\Psi) = M'\big|_{\varphi}(\Phi \vee \Psi). \\
Pr_{\mathcal{A}}^{\psi_V}(M'(\varphi(\neg\Phi)))^{\#} &= \neg\left(Pr_{\mathcal{A}}^{\psi_V}(M'(\varphi(\Phi)))^{\#}\right) \\
&= \neg M'\big|_{\varphi}(\Phi) = M'\big|_{\varphi}(\neg\Phi). \\
Pr_{\mathcal{A}}^{\psi_V}(M'(\varphi(\exists x\Phi)))^{\#} &= Pr_{\mathcal{A}}^{\psi_V}(\exists\varphi_V(x) M'(\varphi(\Phi)))^{\#} \\
&= \exists x Pr_{\mathcal{A}}^{\psi_V}(M'(\varphi(\Phi)))^{\#} = M'\big|_{\varphi}(\exists x\Phi).
\end{aligned}$$

The case for substitution in formula basically repeats case for substitution in term. □

Corollary 15. *Given formula* $\Phi \in \mathrm{Sen}(\Sigma)$, Σ'*-model* M' *and signature morphism* $\varphi\colon \Sigma \to \Sigma'$, *then*

$$M' \models \varphi(\Phi) \text{ if and only if } M'\big|_{\varphi} \models \Phi.$$

Proof. By previous proposition $\perp(M'\big|_{\varphi}(\Phi))^{\#} = {}^{\psi_V}\!\mathcal{A}^{-1}(\perp(M'(\varphi(\Phi)))$, where $\mathcal{A}$ is the carrier of M'. Due to properties of images the satisfaction condition holds. □

By proposition 2, and corollaries 11, 15 we have

Theorem 1. *Constructed* $(\mathbb{S}ig, \mathrm{Sen}, \mathrm{Mod}, \models)$ *form an institution.*

This result finishes construction of institution **FOCNL** for many-sorted first-order composition-nominative logic.

7 Conclusion

This paper proves that many-sorted first-order composition-nominative logic forms an institution. For this all necessary constituents of institution are provided. Homomorphisms between models of many-sorted first-order CNL are introduced. This construction can be considered as an extension of the institution for (pure) first-order CNL [8], [9]. It can be developed further to accommodate programming logics like [4]. Other line of research suggests studying the distinctive features of obtained institutions compared to more conventional ones.

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