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Extensionality, Proper Classes, and Quantum Non-Individuality

William J. Greenberg

Abstract

In this paper I address two questions: (1) What distinguishes proper classes from sets? (2) Are proper classes and quantum particles individuals?

Against the familiar response to (1) that proper classes are too big to be sets, I propose that it is not a difference in size that distinguishes such collections but a difference in individuation. The linchpin of my proposal and centerpiece of an NBG-like fragment of class and set theory ("NBG": von Neumann-Bernays-Gödel), is an Axiom of Restricted Extensionality according to which sets are individuated by their members but proper classes are not. This setting (I call it NBG$^-$) I show to be equi-consistent with its NBG counterpart.

I answer (2) by exhibiting a parallelism in NBG$^-$ between proper classes and quantum particles, the former unindividuated by their members and the latter unindividuated by their relational properties. Since both violate the (weak) principle of the identity of indiscernibles as well as the principle of reflexive identity, in NBG$^-$ neither proper classes nor quantum particles are individuals.

1 Three Principles

Modulo the deductive apparatus of First-Order Logic with Weak Identity[1], Russell's Paradox follows from three principles: *Unrestricted*

[1]In FOL=W : $x = y \to y = x$ and $(x = y \,\&\, y = z) \to x = z$ are theses but $x = x$ is not. Every proof in FOL=W is a proof in FOL=. So FOL=W is a sub-theory of FOL=.

Extensionality, *Restricted Comprehension*, and *Unrestricted Pairing*. Concerning the first of these Michael Potter writes:

> Various theories of [classes] have been proposed since the 1900s. What they all share is the axiom of extensionality, which asserts that if x and y are [classes] then
>
> $$\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y.$$
>
> The fact that they share this is just a matter of definition: objects which do not satisfy extensionality are not [classes]. ([12])

Restricted Comprehension says that for every condition $P(x)$, some y contains just the sets satisfying $P(x)$.

$$\exists y \forall x(x \in y \leftrightarrow (\text{set } x \;\&\; Px)).$$

And *Unrestricted Pairing* says that for every w, u and some y: identity-with-w or identity-with-u is necessary and sufficient for membership in y:

$$\forall x(x \in y \leftrightarrow (x = w \vee x = u)).$$

Individually, each of these is plausible. But no consistent theory features all three. For (A,B,C) prove (D),[2] engendering Russell's Paradox.

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[2]

1. $\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y$	*Unrestricted Extensionality*
2. $\exists y \forall x(x \in y \leftrightarrow (\text{set } x \;\&\; Px))$	*Restricted Comprehension*
3. $\forall t \forall w \exists y \forall x(x \in y \leftrightarrow (x = t \vee x = w))$	*Unrestricted Pairing*
4. $\forall t \forall w \forall x \exists y(x \in y \leftrightarrow x = t \vee x = w)$	3, Quantifier Shift
5. $\forall x \exists y(x \in y \leftrightarrow x = x)$	4,UI
6. $\forall x \exists y(x = x \rightarrow x \in y)$	5
7. $\forall x[x = x \rightarrow \exists y(x \in y)]$	6
8. $\forall x(x = x) \rightarrow \forall x \exists y(x \in y)$	7
9. $\forall x(x = x)$	Corollary of 1
10. $\forall x \exists y(x \in y)$	8,9
11. $\forall x(\text{set } x)$	10, definition of *set*
12. $\exists y \forall x(x \in y \leftrightarrow Px)$	2,11

(A)	$\forall x \forall y(\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y)$	(*Unrestricted Extensionality*)
(B)	$\exists y \forall x(x \in y \leftrightarrow \text{set } \& \ Px)$	(*Restricted Comprehension*)
(C)	$\forall w \forall u \exists y \forall x(x \in y \leftrightarrow x = w \vee x = u)$	(*Unrestricted Pairing*)
(D)	$\exists y \forall x(x \in y \leftrightarrow Px)$	(*Unrestricted Comprehension*)

(D) can be avoided by replacing (B) with (B′), as in Zermelo Set Theory;

$$(\mathrm{B}') \ \forall z \exists y \forall x(x \in y \leftrightarrow (x \in z \ \& \ Px)) \quad (\textit{Separation})$$

or by replacing (C) with (C') as in NBG*, a sub-theory of NBG;

$$(\mathrm{C}') \ \forall w \forall u((\text{set } w \ \& \ \text{set } u) \rightarrow \exists y \forall x(x \in y \leftrightarrow (x = w \vee x = u))) \quad (\textit{Restricted Pairing})$$

or by replacing (A) with (A′), as in NBG$^-$: an NBG-like theory with *Restricted Extensionality* and *Unrestricted Pairing*:

(A′)	$\forall x \forall y(\forall z(z \in x \leftrightarrow z \in y) \rightarrow ((\text{set } x \ \& \ \text{set } y) \leftrightarrow x = y))$	(*Restricted Extensionality*)
(B)	$\exists y \forall x(x \in y \leftrightarrow (\text{set } x \ \& \ Px))$	(*Restricted Comprehension*)
(C)	$\forall w \forall u \exists y \forall x(x \in y \leftrightarrow x = w \vee x = u)$	(*Unrestricted Pairing*)

2 Proper Classes in NBG$^-$

From (A′) it follows that identity is reflexive for sets (T1) but irreflexive for proper classes (T2).

T1: $\forall x(x = x \leftrightarrow \text{set } x)$

T2: $\forall x(\neg(x = x) \leftrightarrow \text{prop } x)$[3]

From (B) it follows that there is a class of non-self-membered sets (T3),

T3: $\exists y \forall x(x \in y \leftrightarrow (\text{set } x \ \& \ \neg(x \in x)))$,

which is not a set but a proper class (T4) – and thus not self-identical (T5).

[3] Prop $x \overset{\text{def}}{=} \neg(\text{set } x)$

T4: $\forall y(\forall x(x \in y \leftrightarrow (\text{set } x \ \& \ \neg(x \in x))) \rightarrow \text{prop } y)$

T5: $\forall y(\forall x(x \in y \leftrightarrow (\text{set } x \ \& \ \neg(x \in x))) \rightarrow \neg(y = y))$

Hence there is no universe class (T6).

T6: $\neg\exists y \forall x(x \in y)$

(A′, B) secure an empty class (T7); a pair class (T8: *aka* C); a sum class (T9); a power class (T10); a class of self-identicals (T11); and a class of sets (T12).

T7: $\exists y \forall x \neg(x \in y)$	(*Empty Class*)[4]
T8: $\forall w \forall u \exists y \forall x(x \in y \leftrightarrow x = w \vee x = u)$	(*Pair Class*)[5]
T9: $\forall z \exists y \forall x(x \in y \leftrightarrow \exists w(w \in z \ \& \ x \in w))$	(*Sum Class*)[6]
T10: $\forall z \exists y \forall x(x \in y \leftrightarrow (\text{set } x \ \&$ $\forall w(w \in x \rightarrow w \in z)))$	(*Power Class*)
T11: $\exists y \forall x(x \in y \leftrightarrow x = x)$	(*Class of Self-Identicals*)[7]
T12: $\exists y \forall x(x \in y \leftrightarrow \text{set } x)$	(*Class of Sets*)

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[4]1. $\forall x(x \in y \leftrightarrow (\text{set } x \ \& \ \neg(x = x)))$	B, EI
2. $\exists x(x \in y) \leftrightarrow \exists x(\text{set } x \ \& \ \neg(x = x))$	1
3. $\forall x(\text{set } x \leftrightarrow x = x)$	A′
4. $\neg\exists x(x \in y)$.	2, 3
[5]1. Show $\forall a \forall b \exists y \forall x(x \in y \leftrightarrow x = a \vee x = b)$	
2. Show $\exists y \forall x(x \in y \leftrightarrow x = a \vee x = b)$	
3. $\forall a \forall b \exists y \forall x(x \in y \leftrightarrow (\text{set } x \ \& \ (x = a \vee x = b)))$	B
4. $x \in y \leftrightarrow (\text{set } x \ \& \ (x = a \vee x = b))$	3, UI, EI
5. $(x = a \vee x = b) \rightarrow x = x$	"=" is weakly reflexive
6. $x = x \rightarrow \text{set } x$	A′
7. $(x = a \vee x = b) \rightarrow \text{set } x$	5, 6
8. $(x = a \vee x = b) \rightarrow x \in y$	4,6,7
9. $x \in y \rightarrow (x = a \vee x = b)$	4
10. $x \in y \leftrightarrow (x = a \vee x = b)$	8, 9
11. $\forall x(x \in y \leftrightarrow x = a \vee x = b)$	10, UG
12. $\exists y \forall x(x \in y \leftrightarrow x = a \vee x = b)$	11, EG: Cancel *Show* line 2
13. $\forall a \forall b \exists y \forall x(x \in y \leftrightarrow x = a \vee x = b)$	2, UG: Cancel *Show* line 1
[6]1. Show $\forall z \exists y \forall x(x \in y \leftrightarrow \exists w(w \in z \ \& \ x \in w))$	
2. $\forall z \exists y \forall x(x \in y \leftrightarrow (\text{set } x \ \& \ \exists w(w \in z \ \& \ x \in w)))$	
3. $\exists w(w \in z \ \& \ x \in w) \rightarrow \text{set } x$	
4. $\forall z \exists y \forall x(x \in y \leftrightarrow \exists w(w \in z \ \& \ x \in w))$	Cancel *Show* line 1
[7]1. $\exists y \forall x(x \in y \leftrightarrow (\text{set } x \ \& \ x = x))$	B
2. $\text{set } x \leftrightarrow x = x$	A′
3. $\exists y \forall x(x \in y \leftrightarrow x = x)$	1, 2
4. $\exists y \forall x(x \in y \leftrightarrow \text{set } x)$	2,3

Remark 1: "Set x" doesn't appear on the right-hand side of T8 or T9 because it is redundant.

Remark 2: From T8 it follows that the "singleton" of a non-self-identical is empty.[8]

3 Some Classes Are Not Sets

Conventional wisdom decrees that some classes are not sets, either because they are infinite totalities "too large" to be sets ([8], 44 *ff*; [11], 264 *ff*), or because their members "are not all present at any rank of the iterative hierarchy". ([7], 104) According to John Bell, however, infinite totalities are not problematic per se. He writes:

> ... set theory... as originally formulated, does contain contradictions, which result not from admitting infinite totalities per se, but rather from countenancing totalities consisting of all entities of a certain abstract kind, "manys" which, on pain of contradiction, cannot be regarded as "ones". So it was in truth not the finite/infinite opposition, but rather the one/many opposition, which led set theory to inconsistency. This is well illustrated by the infamous Russell paradox, discovered in 1901. ([1], 173)

My treatment of Russell's paradox squares with Bell's observation. *Restricted Comprehension* provides for classes of non-self-membered sets, but on pain of contradiction these "manys" cannot be treated as "one", as would be the case if they were subject to *Unrestricted Extensionality.* Indeed, from *Restricted Comprehension* it follows that every predicate is associated with (perhaps empty) classes of sets which satisfy it. In NBG^- (and its extensions), whether such "manys" can be "ones" – that is, sets – depends not on their size or rank, but on whether their "oneness" would spawn contradiction ([1], op. cit.).

[8] "Consider a thing, a say, and its unit set $\{a\}$. . . If anything x is not a member of the unit set $\{a\}$ then that thing x is not a. And conversely, if anything x is not a then that thing x is not a member of the unit set $\{a\}$." ([3], 82)

4 Proper Classes, Sets, and Models

Suppose $\forall z(z \in x \leftrightarrow z \in y)$. Are x and y identical? Are x and y sets? Unlike NBG*, in NBG$^-$ identity and set-hood go hand-in-hand: equi-membered x and y are identical *iff* these are sets.[9] But from (A$'$,B) it does not follow that equi-membered classes are sets. Therefore, although (A$'$,B) prove T7-T12, they do not make equi-membered classes identical: *unlike sets, classes are not individuated by their members.*

(A$'$,B) are satisfied by a non-self-identical, non-element. But such an entity violates model-theoretic restrictions enunciated by Ruth Marcus, who in "Dispensing With Possibilia" writes:

> The notion of an individual object or thing is an indispensable primitive for theories of meaning grounded in standard model theoretic semantics. One begins with a domain of individuals, and there are no prima facie constraints as to what counts as an individual except those of a most general and seemingly redundant kind. *Each individual must be distinct from every other and identical to itself* (emphasis added). ([9], 39)

5 NBG$^-$ and NBG*

NBG$^-$ and NBG* are *deviations*[10] of one another, for $\neg\forall x(x = x)$ is a theorem of NBG$^-$ and $\forall x(x = x)$ a theorem of NBG*. I will now show that NBG* and NBG$^-$ are *definitional extensions* of one another as well. To show this I will define "=" in terms of "I" and "$\in$" in NBG*, and "I" in terms of "$\in$" in NBG$^-$; and then show that NBG* $\vdash$ NBG$^-$, and NBG$^-$ $\vdash$ NBG*.

[9] In *Elementary Logic*, Mates writes, "... we have explicated the term 'relation' in such a way that whatever cannot be a member of a set cannot be related by any relation. Thus insofar as identity is a relation in this sense, such a thing cannot even stand in this relation to itself. This would hold not only of the set of all objects that are not members of themselves, but also of sets described by phrases that give no hint of impending difficulties. The problem is closely related to *Russell's Antinomy*, and once again every way out seems unintuitive." ([10], 157-8)

[10] "One system is a deviation of another if it shares the vocabulary of the first, but has a different system of theorems/valid inferences." ([5], 3)

NBG*:

($\in$)	1*:	$\forall x\forall y(\forall z(z \in x \leftrightarrow z \in y) \rightarrow \forall z(x \in z \leftrightarrow y \in z))$
(I)	2*:	$\forall x\forall y(\forall z(z \in x \leftrightarrow z \in y) \rightarrow x\mathrm{I}y)$
	3*:	$\forall x\forall y(x\mathrm{I}y \rightarrow \forall z(z \in x \leftrightarrow z \in y))$
(Set)	4*:	$\exists y\forall x(x \in y \leftrightarrow (\text{set } x \text{ \& } Px))$
	5*:	$\forall w\forall u((\text{set } w \text{ \& set } u) \rightarrow \exists y\forall x(x \in y \leftrightarrow (x\mathrm{I}w \vee x\mathrm{I}u)))$
(Def)	D1*:	$\text{set } x \overset{\text{def}}{=} \exists y(x \in y)$
	D2*:	$x = y \overset{\text{def}}{=} (x\mathrm{I}y \text{ \& set } x \text{ \& set } y)$

NBG$^-$:

($\in$)	1^-:	$\forall x\forall y(\forall z(z \in x \leftrightarrow z \in y) \rightarrow \forall z(x \in z \leftrightarrow y \in z))$
(=)	2^-:	$\forall x\forall y(\forall z(z \in x \leftrightarrow z \in y) \rightarrow ((\text{set } x \text{ \& set } y) \leftrightarrow x = y))$
	3^-:	$\forall x\forall y(x = y \rightarrow \forall z(z \in x \leftrightarrow z \in y))$
(Set)	4^-:	$\exists y\forall x(x \in y \leftrightarrow (\text{set } x \text{ \& } Px))$
(Def)	$\text{D}1^-$:	$\text{set } x \overset{\text{def}}{=} \exists y(x \in y)$
	$\text{D}2^-$:	$x\mathrm{I}y \overset{\text{def}}{=} \forall z(z \in x \leftrightarrow z \in y)$

NBG* $\vdash$ NBG$^-$: Since $1^- = 1^*$ and $4^- = 4^*$, to show that NBG* $\vdash$ NBG$^-$ I will show that NBG* $\vdash 2^-, 3^-$

Proof of 2^-:

1. Show $\forall x\forall y(\forall z(z \in x \leftrightarrow z \in y) \rightarrow ((\text{set } x \text{ \& set } y) \leftrightarrow x = y))$	
2. $\forall z(z \in x \leftrightarrow z \in y)$	Assume
3. Show $(\text{set } x \text{ \& set } y) \leftrightarrow x = y$	
4. $x = y \overset{\text{def}}{=} (x\mathrm{I}y \text{ \& set } x \text{ \& set } y)$	D2*
5. $x = y \rightarrow (\text{set \& set } y)$	4
6. Show $(\text{set } x \text{ \& set } y) \rightarrow x = y$	
7. $\text{set } x \text{ \& set } y$	Assume
8. Show $x = y$	
9. $x\mathrm{I}y \text{ \& set } x \text{ \& set } y$	2*, 2, 7
10. $x = y$	9, D2*: Cancel *Show* line 8
11. $(\text{set } x \text{ \& set } y) \rightarrow x = y$	7, 8: Cancel *Show* line 6
12. $(\text{set } x \text{ \& set } y) \leftrightarrow x = y$	5, 6: Cancel *Show* line 3
13. $\forall x\forall y(\forall z(z \in x \leftrightarrow z \in y) \rightarrow ((\text{set } x \text{ \& set } y) \leftrightarrow x = y))$	2, 3: Cancel *Show* line 1

Proof of 3^- :

1. Show $\forall x\forall y(x = y \to \forall z(z \in x \leftrightarrow z \in y))$	
2. $x = y$	Assume
3. Show $\forall z(z \in x \leftrightarrow z \in y))$	
4. $x = y \overset{\text{def}}{=} (x\mathrm{I}y \ \&\ \text{set } x \ \&\ \text{set } y)$	D2*
5. $x\mathrm{I}y$	2, 4
6. $\forall x\forall y(x\mathrm{I}y \to \forall z(z \in x \leftrightarrow z \in y))$	3*
7. $\forall z(z \in x \leftrightarrow z \in y)$	5, 6: Cancel *Show* line 3
8. $\forall x\forall y(x\mathrm{I}y \to \forall z(z \in x \leftrightarrow z \in y))$	2, 3: Cancel *Show* line 1

$\mathrm{NBG}^- \vdash$ NBG^*: Since $1^- = 1^*$ and $4^- = 4^*$, to show that $\mathrm{NBG}^- \vdash \mathrm{NBG}^*$ I will show that $\mathrm{NBG}^- \vdash 2^*, 3^*, 5^*$.

Proof of 2*:

1. Show $\forall x\forall y(\forall z(z \in x \leftrightarrow z \in y) \to x\mathrm{Iy})$	
2. $\forall z(z \in x \leftrightarrow z \in y)$	Assume
3. Show $x\mathrm{Iy}$	
4. $x\mathrm{Iy} \overset{\text{def}}{=} \forall \mathrm{z}(\mathrm{z} \in \mathrm{x} \leftrightarrow \mathrm{z} \in \mathrm{y})$	$\mathrm{D2}^-$
5. $x\mathrm{Iy}$	2, 4: Cancel *Show* line 3
6. $\forall x\forall y(\forall z(z \in x \leftrightarrow z \in y) \to x\mathrm{Iy})$	2, 3: Cancel *Show* line 1

Proof of 3*:

1. Show $\forall x\forall y(x\mathrm{Iy} \to \forall \mathrm{z}(\mathrm{z} \in \mathrm{x} \leftrightarrow \mathrm{z} \in \mathrm{y}))$	
2. $x\mathrm{Iy} \overset{\text{def}}{=} \forall \mathrm{z}(\mathrm{z} \in \mathrm{x} \leftrightarrow \mathrm{z} \in \mathrm{y})$	$\mathrm{D2}^-$
3. $\forall x\forall y(x\mathrm{Iy} \to \forall \mathrm{z}(\mathrm{z} \in \mathrm{x} \leftrightarrow \mathrm{z} \in \mathrm{y}))$	2: Cancel *Show* line 1

Proof of 5*:

1. Show $\forall w\forall u((\text{set } w \ \&\ \text{set } u) \to \exists y\forall x(x \in y \leftrightarrow (x\mathrm{Iw} \vee \mathrm{xIu})))$	
2. set w & set u	Assume
3. Show $\exists y\forall x(x \in y \leftrightarrow (x\mathrm{Iw} \vee \mathrm{xIu}))$	
4. $\exists y\forall x(x \in y \leftrightarrow (\text{set } x \ \&\ Px))$	4^-
5. $\exists y\forall x(x \in y \leftrightarrow (\text{set } x \ \&\ x = w \vee x = u))$	Instance of 4
6. $(x = w \vee x = u) \to \text{set } x$	1^-
7. $x \in y \to \text{set } x$	$D2^-$
8. $\exists y\forall x(x \in y \leftrightarrow (x = w \vee x = u))$	5, 6, 7
9. $x = w \leftrightarrow \forall z(z \in x \leftrightarrow z \in w)$	3^-
10. $x = u \leftrightarrow \forall z(z \in x \leftrightarrow z \in u)$	3^-
11. $\exists y\forall x(x \in y \leftrightarrow (\forall z(z \in x \leftrightarrow z \in w) \vee \forall z(z \in x \leftrightarrow z \in)))$	8, 9, 10
12. $\exists y\forall x(x \in y \leftrightarrow x\mathrm{Iw} \vee \mathrm{xIu})$	11, $\mathrm{D2}^-$: Cancel *Show* line 3
13. $\forall w\forall u((\text{set } w \ \&\ \text{set } u) \to \exists y\forall x(x \in y \leftrightarrow (x\mathrm{Iw} \vee \mathrm{xIu})))$	2, 3: Cancel *Show* line 1

Each a definitional extension of the other, NBG* and NBG$^-$are accordingly equi-consistent. The question thus arises, Which of these two systems – NBG* (in which identity is reflexive and proper classes are individuated by their members), or NBG$^-$ (in which identity is non-reflexive and proper classes are not individuated by their members) – should be employed as a setting for theories in which all sets are classes, but some classes are not sets?

6 Classes Into Sets

(5^-, 6^-, 7^-, 8^-) constitute – as sets: pair classes, sum classes, power classes, and sub-classes of sets. For it follows from (5^-, 6^-, 7^-, 8^-) that these are individuated by their members.

5^-: $\forall y(\forall x(x \in y \leftrightarrow (x = a \vee x = b)) \rightarrow \text{set } y)$ (*Pair Set*)
6^-: $\forall z\forall y(\forall x(x \in y \leftrightarrow \exists w(w \in z \;\&\; x \in w)) \rightarrow \text{set } y)$ (*Sum Set*)
7^-: $\forall z\forall y(\forall x(x \in y \leftrightarrow (\text{set } x \;\&\; \forall w(w \in x \rightarrow w \in z))) \rightarrow \text{set } y)$ (*Power Set*)
8^-: $\forall z\forall y(\forall x(x \in z \rightarrow x \in y) \rightarrow (\text{set } y \rightarrow \text{set } z))$ (*Subsets*)

To guarantee an empty set, Z and its extensions require an axiom of infinity or an axiom of set existence; and Lemmon's NBG requires an axiom, "set $\emptyset$". ([6], 46) In NBG$^-$ no such apparatus is required, for (1^-, 2^-, 5^-) guarantee an empty set.

Thus (1^-, 2^-, 5^-) prove T13,

T13: $\forall y(\forall x\neg(x \in y) \rightarrow \text{set } y)$ (*Empty Set*)

which together with T7: $\exists y\forall x\neg(x \in y)$ establish a unique empty set.

Proof of T13:
Suppose y empty. From (T7, T8, 5^-) we have

$$\exists z(\text{set } z \;\&\; \forall x(x \in z \leftrightarrow x = y)),$$

and so by EI: set z & $\forall x(x \in z \leftrightarrow x = y)$. Hence $\exists x(x \in z) \leftrightarrow \exists x(x = y)$. Now suppose, contrary to T13, that y is a proper class. Then $\neg(y = y)$, $\neg\exists x(x = y)$, and $\neg\exists x(x \in z)$, so that z and y have the same members. So because z is a set, from T14 it follows that y is a set:

T14: $\forall x\forall y(\forall z(z \in x \leftrightarrow z \in y) \to (\text{set } x \leftrightarrow \text{set } y))$ (*Equi-Equi*)[11]

So if y is a proper class, y is a set. Hence y is a set.

7 NBG$^-$ and Foundation

4^- provides for a class of self-membered sets:

T15: $\exists y\forall x(x \in y \leftrightarrow x \in x)$ (*Class of self-membered sets*)

The *Anti-Foundation* axiom 9^- would constitute this class as a set.

9^-: $\forall y(\forall x(x \in y \leftrightarrow x \in x) \to \text{set } y)$ (*Anti-Foundation*)

But a *Foundation* axiom such as $9^{-\prime}$ would constitute such a class as a proper class.

$9^{-\prime}$: $\forall y(\forall x(x \in y \leftrightarrow x \in x) \to \neg(\text{set } y))$ (*Foundation*)

8 NBG$^-$ and the Identity of Indiscernibles

Here is a set-theoretic gloss on the weak version of Leibniz's principle of the Identity of Indiscernibles (*PII*):

$\forall x\forall y(\forall z(x \in z \leftrightarrow y \in z) \to x = y)$ (*Unrestricted PII*)

Unrestricted *PII* is refuted in NBG$^-$. For by satisfying $\neg(x = x)$, proper classes refute $\forall x(x = x)$, a corollary of *PII. PII* must thus be restricted, by excluding proper classes from its range of application, *thus*:

$\forall x\forall y(\forall z(x \in z \leftrightarrow y \in z) \to ((\text{set } x \;\&\; \text{set } y) \leftrightarrow x = y))$ (*Restricted PII*)

And to save pairing and extensionality, which together prove unrestricted *PII*, either pairing or extensionality must be restricted, as in NBG* or NBG$^-$.

[11] If x,y are equi-membered, from 1^- it follows that x is an element iff y is an element. So set x iff set y.

9 Quasi-Set Theories and $\neg(\mathbf{x} = \mathbf{x})$

Quasi-Set theories deal with collections of indistinguishable objects such as quantum particles. Such theories recognize two kinds of entities: *M-Atoms*, which "have the properties of standard *Ur-elemente* of ZFU"; and *m-atoms*, which "represent the elementary basic entities of quantum physics". To m-atoms "the concept of identity does not apply." In Quasi-Set theories "this exclusion is achieved by restricting the concept of formula: expressions like $x = y$ are not well formed if x and y denote m-atoms. The equality symbol is not a primitive logical symbol" ([4], 276]

Whereof they cannot speak, thereof must Quasi-Set theories remain silent. But by remaining silent about the distinctness of indiscernible elementary particles, Quasi-Set theories dissimulate a relation whose trivial proof does nothing to diminish the bearing of the distinctness of indiscernibles on the Principle of Reflexive Identity (*PRI*).

For equi-propertied x and y, suppose $\neg(x = y)$. Because x lacks identity-with-y and x and y share their properties, y lacks identity-with-y and x identity-with-x. Hence for equi-propertied x and y: $\neg(x = y) \rightarrow (\neg(x = x) \ \&\ \neg(y = y))$. So distinct quantum particles with identical relational properties contravene *PRI* as well as unrestricted *PII*.

Except in the land of quasi-sets – where to defend ZFU from *Logic*, m-particles representing the elementary basic entities of quantum physics are not allowed to co-occur with the sign for identity.[12]

10 Summary and Conclusion

A: $\forall x \forall y(\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y)$	(*Unrestricted Extensionality*)
B: $\exists y \forall x(x \in y \leftrightarrow \text{set } x \ \&\ Px)$	(*Restricted Comprehension*)
C: $\forall w \forall u \exists y \forall x(x \in y \leftrightarrow x = w \vee x = u)$	(*Unrestricted Pairing*)

[12]Dean Rickles writes, "An immediate problem with the denial of primitive identities is, then, that it is unclear how one is able to support set theory. . . (I owe this point to Steven French). There are ways of accommodating the denial of primitive identities through the use of 'quasi-set theory' in which the identity relation is not a wellformed formula for indistinguishable objects (see French & Krause [1999] and Krause [1992])". ([13], 106)

Modulo a background logic in which identity is a partial equivalence relation, the inconsistency of (A,B,C) can be resolved by replacing B with B′, as in Z*;

Z*	
A: $\forall x \forall y(\forall z(z \in x \leftrightarrow z \in y) \to x = y)$	*Unrestricted Extensionality*
B′: $\forall z \exists y \forall x(x \in y \leftrightarrow (x \in z \,\& Px))$	*Separation*
C: $\forall w \forall u \exists y \forall x(x \in y \leftrightarrow x = w \vee x = u)$	*Unrestricted Pairing*

or by replacing C with C’, as in NBG*;

NBG*	
A: $\forall x \forall y(\forall z(z \in x \leftrightarrow z \in y) \to x = y)$	*Unrestricted Extensionality*
B: $\exists y \forall x(x \in y \leftrightarrow (\text{set } x \,\&\, Px))$	*Restricted Comprehension*
C′: $\forall w \forall u((\text{set } w \,\&\, \text{set } u) \to \exists y \forall x(x \in y \leftrightarrow (x = w \vee x = u)))$	*Restricted Pairing*

or by replacing A with A’, as in NBG$^-$.

NBG$^-$	
A′: $\forall x \forall y(\forall z(z \in x \leftrightarrow z \in y) \to ((\text{set } x \,\&\, \text{set } y) \leftrightarrow x = y))$	*Restricted Extensionality*
B: $\exists y \forall x(x \in y \leftrightarrow (\exists z(x \in z) \,\&\, Px))$	*Restricted Comprehension*
C: $\forall w \forall u \exists y \forall x(x \in y \leftrightarrow x = w \vee x = u)$	*Unrestricted Pairing*

Z* and NBG* are sub-theories of Z and NBG. NBG$^-$ and NBG* are deviations *and* definitional extensions of one another.

Highlighting the rivalry of NBG* and NBG$^-$, I have proposed NBG$^-$ – in which identity is reflexive for sets and classical particles, but irreflexive for proper classes and quantum particles – as a setting for class and set theory and framework for quantum non-individuality.[13]

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[13] As things now stand, non-self-identicals in NBG are non-elements, making their identification with quantum particles problematic. To surmount this obstacle, a more restrictive definition of "set" is required:

set $x \stackrel{\text{def}}{=} \exists y(x \in y \ \& \ \forall z(z \in y \to z = x))$ $\quad (\stackrel{\text{def}}{=} \mathit{Set}\ )$

From $(\stackrel{\text{def}}{=} \mathit{Set})$ and "set $x \leftrightarrow x = x$" (a corollary of 2^{-} – 2^{-} holding under both definitions of "set"), it follows that possession of an individuating property is necessary and sufficient for self-identity:

$x = x \leftrightarrow \exists y(x \in y \ \& \ \forall z(z \in y \to z = x))$ $\quad (\mathit{Ind})$

Hence $\neg(x = x)$ if, and only if, x is not a member of any unit class.

$\neg(x = x) \leftrightarrow \forall y(\neg(x \in y) \vee \exists z(z \in y \ \& \ \neg(z = x)))$ $\quad (\mathit{Ind})$

This will be the case if x is not a member of *any* class (think proper classes); or if every class that x belongs to is such that it contains an element that is not identical with x (think quantum particles). In both cases, x can be said to *lack individuality.*

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William J. Greenberg
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USA
E–mail: wgreenb@gmail.com