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Admissibility, compatibility, and deducibility in first-order sequent logics

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Abstract

The paper is about the notions of admissibility and compatibility and their significance for deducibility in different sequent logics including first-order classical and intuitionistic ones both without and with equality and, possibly, with modal rules. Results on the coextensivity of the proposed sequent calculi with usual Gentzen and Kanger sequent calculi as well as with their equality and modal extensions are given.

Keywords: First-order classical logic, first-order intuitionistic logic, first-order modal logic, sequent calculus, deducibility, admissibility, compatibility, coextensivity, validity.

1 Introduction

There is a great impact of methods originally developed for deduction in different logics on some branches of computer science. From the beginning it was realized that logical inference tools have strong influence on the development of such fields as automated theorem proving, knowledge management, data mining, etc. As a result, investigations in computer-made reasoning gave rise to the appearance of various methods for proof search in the classical first-order logic. Thus, Gentzen's sequent calculi [1] modified for their software implementation have found many applications. But in the case of the classical logic their practical usage as a logical engine of the intelligent systems has not received wide use: preference is usually given to the resolution-type methods. This is explained by higher efficiency of these methods as compared to sequent

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calculi, which is mainly connected with different possible orders of the quantifier rule applications in sequent calculi while the resolution-type methods, due to skolemization, are free from this deficiency.

In its turn, the deduction process in sequent calculi reflects sufficiently well natural theorem-proving methods which, as a rule, do not include preliminary skolemization so that inferences are performed within the scope of the signature of an initial theory. This feature of sequent calculi becomes important when some interactive mode of proof is developed since it is preferable to present the output information concerning the proof search in the form comprehensive for a man. Besides, preliminary skolemization is not a valid operation for many non-classical logics including the intuitionistic one while many of such logics are widely used in solving reasoning problems. That is the problem of the efficient quantifier manipulation makes its appearance.

When quantifier rules are applied, some substitution of selected terms for variables is made. For this step of deduction to be sound, certain restrictions are put on the substitution. A substitution, satisfying these restrictions, is said to be admissible. Here we show how Gentzen's notion of an admissible substitution can be modified so that computer-oriented sequent calculi can be finally obtained for both classical and non-classical logics. For simplicity, we give a complete description of our approach for the classical logic without equality and briefly discuss a way (utilizing additionally a so-called compatibility) to use it for the intuitionistic logic as well as for their equality and modal extensions.

We use modifications of the calculi LK and LJ without equality from [1] denoting them by mLK and mLJ respectively. Moreover, we convert mLK and mLJ in a certain way to logics with equality and/or modal rules. At that, we don't touch upon any procedure of selection of propositional rules and terms substituted, focusing our attention on quantifier handling only. Note that in contrary to [1], the antecedents and succedents of all the sequents under consideration are assumed to be multisets. As usual, the inference search in any calculus is of the form of the so-called inference tree "growing" from bottom to top in accordance with the order of counter-applying inference rules. An inference tree all leaves of which are axioms is called a proof tree.

2 Genzen's notion of admissibility

Classical quantifier rules, substituting arbitrary structure terms when applied from bottom to top, are usually of the following form slightly distinguished from that given in [1]:

$$\frac{\Gamma, A[t/x] \to \Delta}{\Gamma, \forall x A \to \Delta}\ (\forall : left) \qquad \frac{\Gamma \to A[t/x], \Delta}{\Gamma \to \exists x A, \Delta}\ (\exists : right)$$

where the term t is required to be free for the variable x in the formula A and $A[t/x]$ denotes the result of the simultaneous replacement in A of x by t. This restriction of the substitution of t for x gives Gentzen's (classical) notion of an admissible substitution, which proves to be sufficient for the needs of the proof theory. But it becomes useless from the point of view of efficiency of computer-oriented theorem proving. It is clear from the following example.

Consider a sequent $A_1, A_2 \to B$, where A_1 is $\forall x_1 \exists y_1 (R_1(x_1) \vee R_2(y_1))$, A_2 is $\forall x_2 \exists y_2 (R_1(y_2) \supset R_3(x_2))$, and B is $\exists x_3 \forall y_3 (R_2(x_3) \vee R_3(y_3))$. The provability of this sequent in LK will be established below, while here we notice that the quantifier rules should be applied to all the quantifiers occurring in A_1, A_2, and B. Therefore, the classical notion of an admissible substitution yields 90 ($= 6!/(2!{\cdot}2!{\cdot}2!)$) different orders of quantifier rule applications to $A_1, A_2 \to B$. It is clear that the resolution-type methods allow avoiding this redundant work.

3 Kanger's notion of admissibility

To optimize the procedure of applying the quantifier rules, in [3] S.Kanger suggested his Gentzen-type calculus, denoted here by K. In calculus K a "pattern" of an inference tree is first constructed with the help of special variables, the so called parameters and dummies. At some instants of time, an attempt is made to convert a "pattern" into a proof tree to complete the deduction process. In case of failure, the process is continued. The main difference between K and LK consists in a special modification of the above-given quantifier rules and in a certain splitting (in K) of the process of the "pattern" construction into stages. The rules $(\forall : left)$ and $(\exists : right)$ of K are as follows:

$$\frac{\Gamma, A[d/x] \to \Delta}{\Gamma, \forall x A \to \Delta}\ d/t_1, ..., t_n \qquad \frac{\Gamma \to A[d/x]\Delta}{\Gamma \to \exists x A, \Delta}\ d/t_1, ..., t_n$$

where $t_1, \ldots, t_n$ are terms occurring in the conclusion of the rules, d is a dummy, and $d/t_1, \ldots, t_n$ denotes that when an attempt is made to convert "pattern" into a proof tree, the dummy d must be replaced by one of the terms $t_1, \ldots, t_n$. The replacement of dummies by terms is made in the end of every stage, and at every stage the rules are applied in a certain order.

This scheme of the deduction construction in calculus K leads to the notion of a Kanger-admissible substitution, which is more efficient than of the Gentzen one. For example, in the above-given example it yields only 6 (=3!) variants of different possible orders of the quantifier rule applications (but none of these variants is preferable). Despite this, the Kanger-admissible substitutions still do not allow achieving the efficiency comparable with that when the skolemization is made. It is due to the fact that, as in case of the Gentzen notion of admissibility, it is required to select a certain order of the quantifier rule applications when an initial sequent is deduced, and, if it proves to be unsuccessful, the other order of applications is tried, and so on.

4 New notion of admissibility

For constructing the modification mLK of calculus LK from [1], let us introduce a new notion of an admissible substitution in order to get rid of the dependence of the deduction efficiency in sequent calculi on different possible orders of quantifier rule applications. The main idea is to determine, proceeding from the quantifier structures of formulas of an initial sequent and a substitution under consideration, would there exists a desired sequence of quantifier rules applications. (This notion was used in [4] in slightly modified form for another purpose.)

We assume that besides usual variables there are two countable sets of special variables, namely of parameters and dummies.

A *substitution* s is defined as a finite (maybe, empty) set of ordered pairs [5], every of which consists of a variable, say, x, and a term, say, t, and is written as t/x, where x is called a *variable* and t a *term* of

s. For a sequent tree D, by $D \cdot s$ denote the result of the simultaneous replacement of all the variables of s by the corresponding terms of s.

Let P be a set of sequences of parameters and dummies and s a substitution. Put $T(P, s) = \{\langle z, t, p \rangle \colon z$ is a variable of s, t a term of s, $p \in P$, and z lies in p to the left of some parameter of $t\}$. The substitution s is said to be *admissible* for P if and only if (1) the variables of s are dummies and (2) there are no triples $\langle z_1, t_1, p_1 \rangle, \ldots, \langle z_n, t_n, p_n \rangle \in T(P, s)$ such that $t_2/z_1 \in s, \ldots, t_n/z_{n-1} \in s, t_1/z_n \in s$ $(n > 0)$.

5 Admissibility and classical deducibility

As in the case of calculus LK, its modification mLK deals with sequents consisting of formulas, except that in mLK, to every formula, a (possibly, empty) sequence of parameters and dummies is additionally assigned. Thus, the sequents of mLK consist of pairs $\langle p, A \rangle$, where A is a formula and p a sequence (word) of parameters and dummies. Also, it will be assumed that the empty sequence is always added to each formula of an initial sequent (that is, a sequent to be proved).

The calculus mLK has the following rules:

Axioms:

$$\frac{}{\Gamma, \langle p, A \rangle \to \langle p, A \rangle, \Delta}$$

Propositional rules:

$$\frac{\Gamma, \langle p, A \rangle, \langle p, B \rangle \to \Delta}{\Gamma, \langle p, A \wedge B \rangle \to \Delta} \quad \frac{\Gamma \to \langle p, A \rangle, \Delta \quad \Gamma \to \langle p, B \rangle, \Delta}{\Gamma \to \langle p, A \wedge B \rangle, \Delta}$$

$$\frac{\Gamma, \langle p, A \rangle \to \Delta \quad \Gamma, \langle p, B \rangle \to \Delta}{\Gamma, \langle p, A \vee B \rangle \to \Delta} \quad \frac{\Gamma \to \langle p, A \rangle, \Delta}{\Gamma \to \langle p, A \vee B \rangle, \Delta} \quad \frac{\Gamma \to \langle p, B \rangle, \Delta}{\Gamma \to \langle p, A \vee B \rangle, \Delta}$$

$$\frac{\Gamma, \langle p, A \rangle \to \langle p, B \rangle, \Delta \quad \Gamma, \langle p, B \rangle \to \Delta}{\Gamma, \langle p, A \supset B \rangle \to \Delta} \quad \frac{\Gamma, \langle p, A \rangle \to \langle p, B \rangle, \Delta}{\Gamma \to \langle p, A \supset B \rangle, \Delta}$$

$$\frac{\Gamma \to \langle p, A \rangle, \Delta}{\Gamma, \langle p, \neg A \rangle \to \Delta} \quad \frac{\Gamma, \langle p, A \rangle \to \Delta}{\Gamma \to \langle p, \neg A \rangle, \Delta}$$

Contraction rules:

$$\frac{\Gamma, \langle p, A\rangle, \langle p, A\rangle \to \Delta}{\Gamma, \langle p, A\rangle \to \Delta}\ (Con \to) \quad \frac{\Gamma \to \langle p, A\rangle, \langle p, A\rangle \Delta}{\Gamma \to \langle p, A\rangle, \Delta}\ (\to Con)$$

Quantifier rules:

$$\frac{\Gamma, \langle pd, A[d/x]\rangle \to \Delta}{\Gamma, \langle p, \forall x A\rangle \to \Delta}\ (\forall : left') \qquad \frac{\Gamma \to \langle pz, A[z/x]\rangle, \Delta}{\Gamma \to \langle p, \forall x A\rangle, \Delta}\ (\forall : right')$$

$$\frac{\Gamma, \langle pz, A[z/x]\rangle \to \Delta}{\Gamma, \langle p, \exists x A\rangle \to \Delta}\ (\exists : left') \qquad \frac{\Gamma \to \langle pd, A[d/x]\rangle, \Delta}{\Gamma \to \langle p, \exists x A\rangle, \Delta}\ (\exists : right')$$

Here d is a new dummy, z a new parameter, p a sequence of parameters and dummies, Γ and Δ are arbitrary multisets of ordered pairs consisting of sequences (of dummies and parameters) and formulas, A and B are arbitrary formulas.

In what follows, the establishing of the deducibilty of a sequent $A_1, \ldots, A_m \to B_1, \ldots, B_n$ in LK is replaced by the establishing of the deducibilty of the so-called *initial sequent* $\langle, A_1\rangle, \ldots, \langle, A_m\rangle \to \langle, B_1\rangle, \ldots, \langle, B_n\rangle$ in mLK (or its modifications).

Applying first a rule from bottom to top to a sequent under consideration and afterwards to its "heirs", and so on, we finally obtain a so-called *inference tree* for this sequent.

Let D be an inference tree in mLK and s a substitution. If all the leaves of $D \cdot s$ are axioms, then D is called a *latent proof tree* in mLK w.r.t. s.

The main result concerning the calculus mLK is as follows.

Theorem 1. *Let $A_1, \ldots, A_m, B_1, \ldots, B_n$ be formulas of the first-order language. The sequent $A_1, \ldots, A_m \to B_1, \ldots, B_n$ is deducible in LK if and only if there exist an inference tree D in mLK for the initial sequent $\langle, A_1\rangle, \ldots, \langle, A_m\rangle \to \langle, B_1\rangle, \ldots, \langle, B_n\rangle$ and a substitution s of terms without dummies for all the dummies of D such that: (1) D is a latent proof tree in mLK w.r.t. s and (2) s is an admissible substitution for the set of all the sequences of parameters and dummies from D.*

Proof. (=>) Let D be a proof tree for an initial sequent $\langle, A_1\rangle, \ldots, \langle, A_m\rangle \rightarrow \langle, B_1\rangle, \ldots, \langle, B_n\rangle$ in the calculus mLK and s be a substitution converting all the leaves of D into axioms and being admissible for the set P of all sequences of parameters and dummies from D. Without loss of generality, it can be assumed that terms of s do not contain dummies for otherwise these dummies could be replaced by a special constant, say, c_0. Since s is admissible for P, it is possible to construct the following sequence p consisting of parameters and dummies occurring in the sequences of P:

(i) every $p' \in P$ is a subsequence of p and

(ii) s is admissible for $\{p\}$ (i.e. there is no an element $\langle z, t, p\rangle \in T(p, s)$ such that $t/z \in s$.

Such a sequence p can be generated, for example, by using the convolution algorithm from [4], applied to a list of all the sequences from P (in the convolution algorithm, parameters are treated as existence quantifiers and dummies as universal quantifiers). The property (i) of the sequence p and the definitions of the propositional and quantifier rules lead to the following observation:

When D is constructed, propositional, contraction, and quantifier rules are applied (from bottom to top) in the order that corresponds to looking through p from the left to right: i.e. when the first quantifier rule is applied, the first variable (a parameter or dummy) of p is generated, when the second quantifier rule is applied, the second variable (a parameter or dummy) of p is generated, and so on.

Now it is possible to convert the tree D into a proof tree D' for the initial sequent $A_1, \ldots, A_m \rightarrow B_1, \ldots, B_n$ in calculus LK. To do this, it is enough to "repeat" the process of the construction of D in the above-given order p and execute the following transformations:

1) Suppose that in a processed node of D one of the following rules was applied:

$$\frac{\Gamma, \langle pd, A[d/x]\rangle \rightarrow \Delta}{\Gamma, \langle p, \forall x A\rangle \rightarrow \Delta}\ (\forall : left') \text{ or } \frac{\Gamma \rightarrow \langle pd, A[d/x]\rangle, \Delta}{\Gamma \rightarrow \langle p, \exists x A\rangle, \Delta}\ (\exists : right')\ ,$$

where $t/d \in s$ for some term t. The term t is free for d in A, because the order of applications of quantifier rules is reflected by p, and the property (ii) is satisfied. Hence, the conditions of admissibility in Gentzen's (classical) sense are satisfied when the above-given rules $(\forall : left')$ and $(\exists : right')$ are replaced in D by the corresponding rules $(\forall : left)$ and $(\exists : right)$ of the calculus LK:

$$\frac{\Gamma, A[t/x] \to \Delta}{\Gamma, \forall x A \to \Delta}\ (\forall : left) \ \text{ or } \ \frac{\Gamma \to A[t/x]\Delta}{\Gamma \to \exists x A, \Delta}\ (\exists : right)$$

and all occurrences of d in D are replaced by t.

2) In all the other cases, the rules of the calculus mLK are replaced in D by their analogues from LK by a simple deleting of sequences of parameters and dummies from these rules.

It is evident that D' is an inference tree in the calculus LK. Furthermore, by the construction of D' it follows that all its leaves are axioms of the calculus LK. Thus, D' is a proof tree for the initial sequent A_1, $\ldots$, $A_m \to B_1, \ldots, B_n$ in LK.

(<=) Let D' be a proof tree for an initial sequent A_1, $\ldots$, A_m $\to B_1, \ldots, B_n$ in the calculus LK. Convert D' into a proof tree D for the initial sequent $\langle , A_1\rangle$, $\ldots$, $\langle , A_m\rangle \to \langle , B_1\rangle, \ldots, \langle , B_n\rangle$ in mLK. For this purpose, "repeat" (from bottom to top) the process of construction of D', replacing in D' every rule application by its analogue in mLK and subsequently generating a substitution s. (Initially s is the empty substitution.)

1) If an applied rule is one of the following:

$$\frac{\Gamma, A[t/x] \to \Delta}{\Gamma, \forall x A \to \Delta}\ (\forall : left) \ \text{ or } \ \frac{\Gamma \to A[t/x]\Delta}{\Gamma \to \exists x A, \Delta}\ (\exists : right)\ ,$$

then it is replaced by

$$\frac{\Gamma, \langle pd, A[d/x]\rangle \to \Delta}{\Gamma, \langle p, \forall x A\rangle \to \Delta}\ (\forall : left') \text{ or } \frac{\Gamma \to \langle pd, A[d/x]\rangle, \Delta}{\Gamma \to \langle p, \exists x A\rangle, \Delta}\ (\exists : right')$$

accordingly with adding t/d to the existing substitution s, where d is a new dummy, and substituting d for those occurrences of the term t into "heirs" of the formula $A[t/x]$ being the result of a replaced rule application inserting t.

2) In all the other cases, the replacement of the rules of LK by the rules of mLK is evident. (The rules ($\forall : left$) and ($\exists : right$) may be considered as those inserting new parameters).

Since D' is a proof tree in the calculus utilizing the classical notion of an admissible substitution, then it is clear that the finally generated substitution s is admissible (in the new sense) for a set of all sequences of parameters and dummies from D. Therefore, D is a proof tree for the initial sequent $\langle , A_1\rangle, \ldots, \langle , A_m\rangle \to \langle , B_1\rangle, \ldots, \langle , B_n\rangle$ in mLK. $\square$

To demonstrate the deduction technique, consider the sequent $A_1, A_2 \to B$ from the above-given example and establish its deducibility in the calculus LK. To do this, construct a proof tree for the initial sequent $\langle , A_1\rangle,\langle , A_2\rangle \to \langle , B\rangle$ in mLK and use Theorem 1.

Applying first the rule ($\to Con$) to the initial sequent and then only quantifier rules to the result in any order, the following sequent is deduced: $\langle d_1 z_1, R_1(x_1) \vee R_2(y_1)\rangle$, $\langle d_2 z_2, R_1(y_2) \supset R_3(x_2)\rangle \to \langle d_3 z_3, R_2(d_3) \vee R_3(x_3)\rangle$, $\langle d_4 z_4, R_2(d_4) \vee R_3(x_4)\rangle$, where $d_1, \ldots, d_4$ are dummies, $z_1, \ldots, z_4$ parameters.

Now let us apply propositional rules to the latter sequent as long as they are applicable. As a result, we construct an inference tree, say, D. If we take the substitution $s = \{z_2/d_1, z_3/d_2, c_0/d_3, z_1/d_4\}$ (c_0 is a special constant), then the following conclusions concerning s and D are valid: (1) every leaf from D is transformed into an axiom by applying of s to it and (2) s is admissible for the set of all sequences of dummies and parameters from D.

So, in accordance with Theorem 1 the sequent $A_1, A_2 \to B$ is deducible in the calculus LK.

Draw your attention to the fact that the selection of an order of the quantifier rules applications in mLK is immaterial; it can be any.

6 Admissibility, compatibility, and intuitionistic deducibility

The intuitionistic calculus LJ is distinguished from LK by that the succedent of any sequent in LJ should contain no more than one formula [1]. In this connection, it may seem that this restriction putting on mLK leads to a correct intuitionistic modification of the classical calculus mLK, say, mLJ. Unfortunately, it is not so, and the following example demonstrates this fact.

Consider the sequent $\neg\forall x P(x) \to \exists y \neg P(y)$. Obviously, it is deducible in LK while it *is not deducible* in LJ.

We can construct the following proof tree D in mLK for it:

$$\cfrac{\cfrac{\cfrac{\cfrac{\langle d, P(d)\rangle \to \langle z, P(z)\rangle}{\langle d, P(d)\rangle \to \langle , \forall x P(x)\rangle}}{\langle , \neg\forall x P(x)\rangle, \langle d, P(d)\rangle \to}}{\langle , \neg\forall x P(x)\rangle \to \langle d, \neg P(d)\rangle}}{\langle , \neg\forall x P(x)\rangle \to \langle , \exists y \neg P(y)\rangle}\ ,$$

where d is a dummy and z a parameter.

Consider the substitution $s = \{z/d\}$. It converts the upper sequent of D into an axiom and is admissible for D. By Theorem 1, the sequent $\neg\forall x P(x) \to \exists y \neg P(y)$ is deducible in LK.

The succedent of any sequent in D contains only one formula, i.e. D satisfies the intuitionistic requirement to inference rules. Therefore, the usage of only the new admissibility is not enough for providing the "sound" deducibility in mLJ for the intuitionistic case.

This situation can be corrected with the help of the notion of the so-called *compatibility* of a constructed proof tree with a selected substitution [6]. Because of the paper size limit, this notion will not be formally defined below. We note simply that after introducing both the notions of admissibility and compatibility in mLJ, they correlate with each other in such a way that provides the soundness of inference search. For example, the above-given tree D for the sequent $\neg\forall x P(x) \to \exists y \neg P(y)$ *is not compatible* with the unique "reasonable" substitution $s = \{z/d\}$, which implies that $\neg\forall x P(x) \to \exists y \neg P(y)$ is not deducible in the calculus LJ.

The following result takes place for intuitionistic logic.

Theorem 2. *Let $A_1, \ldots, A_m, B_1, \ldots, B_n$ be formulas of the first-order language. The sequent $A_1, \ldots, A_m \to B_1, \ldots, B_n$ is deducible in LJ if and only if there exist an inference tree D in mLJ for the initial* *sequent $\langle, A_1\rangle, \ldots, \langle, A_m\rangle \to \langle, B_1\rangle, \ldots, \langle, B_n\rangle$ and a substitution s of terms without dummies for all the dummies of D such that: (1) D is a latent proof tree in mLJ w.r.t. s, (2) s is an admissible substitution for the set of all sequences of parameters and dummies from D, and (3) D is compatible with s.*

Pay your attention to the fact that Theorems 1 and 2 are distinguished by only the presence of the item (3) in Theorem 2.

7 Admissibility, compatibility, and deducibility in equality and modal extensions

Let LK$^{\approx}$ and LJ$^{\approx}$ be, respectively, the calculi LK and LJ, in which the Kanger equality rules from [3] are incorporated, where $\approx$ denotes the equality symbol.

Let us introduce for mLK and mLJ the following modifications of the Kanger equality rules (denoting the corresponding equality extensions by mLK$^{\approx}$ and mLJ$^{\approx}$):

$$\frac{\Gamma|^{t'}_{t''},\ \langle p, t' \approx t''\rangle \to \Delta|^{t'}_{t''}}{\Gamma,\ \langle p, t' \approx t''\rangle \to \Delta} \qquad \frac{\Gamma|^{t'}_{t''},\ \langle p, t'' \approx t'\rangle \to \Delta|^{t'}_{t''}}{\Gamma,\ \langle p, t'' \approx t'\rangle \to \Delta},$$

where the terms t' and t'' do not contain dummies and $\Gamma|^{t'}_{t''}$ and $\Delta|^{t'}_{t''}$ denote the results of the simultaneous replacement of t' by t'' in Γ and Δ respectively.

As in [3], the introduced equality rules are applied in inference search in mLK$^{\approx}$ and mLJ$^{\approx}$ last of all, i.e. when it seems impossible to construct such a tree D without applying equality rules and select such a substitution s that the conditions (1), (2), and (3) from Theorems 1 and 2 are satisfied.

Let D be an inference tree constructed in mLK$^{\approx}$ (mLJ$^{\approx}$) without applying equality rules and s be a substitution. Suppose that after subsequent applying only the equality rules to all the leaves of $D \cdot s$ not being axioms, then to their "heirs", and so on, an inference tree is produced, each leaf of which is only an axiom. Then D is called a *latent proof tree* in mLK$^{\approx}$ (mLJ$^{\approx}$) w.r.t. s.

Theorem 3. *Let $A_1, \dots, A_m, B_1, \dots, B_n$ be formulas of the first-order language. The sequent $A_1, \dots, A_m \to B_1, \dots, B_n$ is deducible in $LK^{\approx}$ ($LJ^{\approx}$) if and only if there exist an inference tree D in $mLK^{\approx}$ ($mLJ^{\approx}$) for the initial sequent $\langle , A_1\rangle, \dots, \langle , A_m\rangle \to \langle , B_1\rangle, \dots, \langle , B_n\rangle$ and a substitution s of terms without dummies for all the dummies of D such that: (1) D is a latent proof tree in $mLK^{\approx}$ ($mLJ^{\approx}$) w.r.t. s, (2) s is an admissible substitution for the set of all the sequences of parameters and dummies from D, and, in the case of $mLJ^{\approx}$, (3) the tree D is compatible with s.*

Our way of the construction of modal calculi has a certain correlation with the papers [7] and [8], where necessary modal rules in a sequent form are simply added to Gentsen's calculi LK and LJ.

Doing the same for LK and LJ and $\mathrm{LK}^{\approx}$ and $\mathrm{LJ}^{\approx}$, we obtain their modal extensions LK+Mod_m, LJ+Mod_m, $\mathrm{LK}^{\approx}$+Mod_m, and $\mathrm{LJ}^{\approx}$+Mod_m, where Mod_m is a set of modal rules.

As to modal rules that can be added to LK, LJ, $\mathrm{LK}^{\approx}$, and $\mathrm{LJ}^{\approx}$, any such modal rule is considered to be of the following general form:

$$\frac{\Gamma, \Phi_1, \dots, \Phi_k \to \Psi_1, \dots, \Psi_r, \Delta}{\Gamma, \bigcirc_1(\Phi_1), \dots, \bigcirc_k(\Phi_k) \to \bigcirc'_1(\Psi_1), \dots, \bigcirc'_r(\Psi_r), \Delta},$$

where $\bigcirc_1, \dots, \bigcirc_r, \bigcirc'_1, \dots, \bigcirc'_r$ are modal operators and $\Phi_1, \dots, \Phi_k$, $\Psi_1, \dots, \Psi_r$ multisets of formulas (containing, possibly, modal operators). In particular, such approach makes possible to determine the calculus GK or GS4 from [8] based on using certain sequent rules for the standard modal operators $\Box$ and $\Diamond$.

Any modal rule of this form naturally determines the corresponding modal rule of the following form (that can be introduced in any of the calculi mLK, mLJ, $\mathrm{mLK}^{\approx}$, and $\mathrm{mLJ}^{\approx}$):

$$\frac{\Gamma', \langle p_1, \Phi_1\rangle, ..., \langle p_k, \Phi_k\rangle \to \langle q_1, \Psi_1\rangle, ..., \langle q_r, \Psi_r\rangle, \Delta'}{\Gamma', \langle p_1, \bigcirc_1(\Phi_1)\rangle, ..., \langle p_k, \bigcirc_k(\Phi_k)\rangle \to \langle q_1, \bigcirc'_1(\Psi_1)\rangle, ..., \langle q_r, \bigcirc'_r(\Psi_r)\rangle, \Delta'}$$

where $p_1, \dots, p_k, q_1, \dots, q_r$ are sequences of dummies and parameters.

Draw your attention to that any such rule satisfies the subformula property, which leads to the following result for modal extensions in virtue of Theorems 1, 2, and 3.

Theorem 4. *Let $A_1, \ldots, A_m, B_1, \ldots, B_n$ be formulas of the first-order language containing, possibly, modal operators. The sequent $A_1, \ldots, A_m \rightarrow B_1, \ldots, B_n$ is deducible in LK+Mod$_m$ (LJ+Mod$_m$, LK$^{\approx}$+Mod$_m$, LJ$^{\approx}$+Mod$_m$) if and only if there exist an inference tree D in LK+Mod$_m$ (LJ+Mod$_m$, LK$^{\approx}$+Mod$_m$, LJ$^{\approx}$+Mod$_m$) for the initial sequent $\langle, A_1\rangle, \ldots, \langle, A_m\rangle \rightarrow \langle, B_1\rangle, \ldots, \langle, B_n\rangle$ and a substitution s of terms without dummies for all the dummies of D such that: (1) D is a latent proof tree in LK+Mod$_m$ (LJ+Mod$_m$, LK$^{\approx}$+Mod$_m$, LJ$^{\approx}$+Mod$_m$) w.r.t. s, (2) s is an admissible substitution for the set of all the sequences of parameters and dummies from D, and, in the cases of LJ+Mod$_m$ and LJ$^{\approx}$+Mod$_m$, (3) the tree D is compatible with s.*

The Kanger calculus K without equality is coextensive with the Gentzen calculus LK [3]. It is easy to see that all the above-described constructions made for LK can be transferred to the case of K producing an analogue of mLK for K and its intuitionistic modification as well as their equality and modal extensions *retaining the results on coextensivity for all such modifications and extensions of K.*

Taking into consideration this and all the above-given theorems, *we can obtain the soundness and completeness theorem* for any of the introduced calculi if and only if this theorem is true for its Gentzen or Kanger analogue. For example, we conclude that the validity of a formula F in the classical (intuitionistic) logic with equality is equivalent to the deductibility of the initial sequent $\rightarrow \langle, F\rangle$ in mLK$^{\approx}$ (in mLJ$^{\approx}$).

8 Conclusion

The research presented in this paper demonstrates that the introduced notions of admissibility and compatibility lead to a good enough decision of the problem of quantifier handling in first-order logics. They can be easily built-in into the Gentzen calculi LK and LJ, which gives a good basis for constructing computer-oriented sequent calculi for classical and intuitionisticl logics as well as for their equality and modal extensions. Despite the questions of the machine implementation of such sequent calculi were not considered in the paper, note that the construction of efficient calculi requires optimizing the order of the propositional rule applications and selecting a method for generating a substitution which can produce a latent proof tree. Bypassing details, make a point that the Robinson unification algorithm combined with the new notion of admissibility (and compatibility) is suitable for generating such substitutions.

The suggested approach to the construction of methods for inference search in first-order logics corresponds well to a modern vision of the so-called Evidence Algorithm, EA, advances by V. M. Glushkov as early as 1970. For the classical logic, it has found its reflection in the deductive engine of the system for automated deduction SAD designed in the accordance with the EA requirements to automated theorem proving (see the Web-site “nevigal.org” as well as papers [9–14]).

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