Computer Science Journal of Moldova, vol.**23**, no.3(69), 2015

Convex graph covers

Radu Buzatu, Sergiu Cataranciuc

Abstract

We study some properties of minimum convex covers and minimum convex partitions of simple graphs. We establish existence of graphs with fixed number of minimum convex covers and minimum convex partitions. It is known that convex p-cover problem is NP-complete for $p \geq 3$ [5]. We prove that this problem is NP-complete in the case $p = 2$. Also, we study covers and partitions of graphs when respective sets are nontrivial convex.

Keywords: Convexity, graphs, convex covers, convex partitions.

1 Introduction

We denote by $G = (X; U)$ a simple graph with vertex set X and edge set U. The set of all vertices adjacent to $x \in X$ in G is denoted by $\Gamma(x)$.

Now we remind some notions defined in [1]: a) *metric segment* $\langle x, y\rangle$ is the set of all vertices lying on a shortest path between vertices $x, y \in X$; b) a set $S \subseteq X$ is called *convex* if $\langle x, y\rangle \subseteq S$ for all $x, y \in S$; c) *convex hull* of $S \subseteq X$, denoted $d - conv(S)$, is the smallest convex set containing S.

A family of sets is called *convex cover* of $G = (X; U)$ and is denoted by $\boldsymbol{\mathcal{P}}(G)$ if the following conditions hold:

1) every set of $\boldsymbol{\mathcal{P}}(G)$ is convex in G;
2) $X = \bigcup_{Y \in \boldsymbol{\mathcal{P}}(G)} Y$;
3) $Y \not\subseteq \bigcup_{\substack{Z \in \boldsymbol{\mathcal{P}}(G) \\ Z \neq Y}} Z$ for every $Y \in \boldsymbol{\mathcal{P}}(G)$.

If $|\mathcal{P}(G)| = p$, then this family is called *convex p-cover* of G and is denoted by $\mathcal{P}_p(G)$. The concept of *convex p-cover* of a graph was defined in [5]. In particular, a family $\mathcal{P}(G)$ is called *convex partition* of graph G if it is a convex cover of G and any two sets of $\mathcal{P}(G)$ are disjoint. A convex p-cover is called *convex p-partition* if it is a convex partition of a graph. Clearly, every graph G has convex 1-cover and convex n-cover. By Claude Berge [3], a set $S \subseteq X$ is a *clique* if every pair of vertices of S is adjacent in G. If $\mathcal{P}_p(G)$ is a convex p-partition and all the sets of $\mathcal{P}_p(G)$ are cliques, then $\mathcal{P}_p(G)$ is called *clique p-partition* of graph G.

Definition 1. *A convex cover $\mathcal{P}(G)$ of graph $G = (X; U)$ is called nontrivial convex cover if every set $Y \in \mathcal{P}(G)$ satisfies the inequalities: $3 \leq |Y| \leq |X| - 1$. Consequently the elements of $\mathcal{P}(G)$ are called nontrivial convex sets.*

Likewise, if a nontrivial convex cover $\mathcal{P}(G)$ is a convex partition, we say that $\mathcal{P}(G)$ is a *nontrivial convex partition.*

Definition 2. **[5]** *Convex cover number $\varphi_c(G)$ of a graph G is the least integer $p \geq 2$ for which G has a convex p-cover. Similarly, convex partition number $\theta_c(G)$ of a graph G is the least integer $p \geq 2$ for which G has a convex p-partition.*

Further, the least integer $p \geq 2$ for which graph G has a nontrivial convex p-cover is said to be *nontrivial convex cover number* $\varphi_{cn}(G)$. In the same way, the least integer $p \geq 2$ for which graph G has a nontrivial convex p-partition is said to be *nontrivial convex partition number* $\theta_{cn}(G)$.

Indeed, there are graphs for which there are no nontrivial convex p-covers or nontrivial convex p-partitions or both. For example, every convex simple graph has no nontrivial convex covers. A graph G is called *convex simple* if it does not contain nontrivial convex set [2].

Let us introduce the following notions.

Minimum convex cover $\mathcal{P}_{\varphi_c}(G)$ is the convex p-cover of graph G such that $p = \varphi_c(G)$;

Minimum convex partition $\mathcal{P}_{\theta_c}(G)$ is the convex p-partition of graph G such that $p = \theta_c(G)$;

Minimum nontrivial convex cover $\mathcal{P}_{\varphi_{cn}}(G)$ is the nontrivial convex p-cover of graph G such that $p = \varphi_{cn}(G)$;

Minimum nontrivial convex partition $\mathcal{P}_{\theta_{cn}}(G)$ is the nontrivial convex p-partition of graph G such that $p = \theta_{cn}(G)$.

It is obvious that for any graph G we have $\varphi_c(G) \leq \theta_c(X)$. As above, if $\mathcal{P}_{\varphi_{cn}}(G)$ and $\mathcal{P}_{\theta_{cn}}(G)$ exist, then $\varphi_{cn}(G) \leq \theta_{cn}(G)$. If $\mathcal{P}_{\varphi_{cn}}(G)$ exists, then $\varphi_c(G) \leq \varphi_{cn}(G)$. If $\mathcal{P}_{\theta_{cn}}(G)$ exists, then $\theta_c(G) \leq \theta_{cn}(G)$.

Also, we introduce the following concept.

Definition 3. *A vertex $x \in X$ is called resident in $\mathcal{P}(G)$ if x belongs to only one set of $\mathcal{P}(G)$.*

By definition, every set of $\mathcal{P}(G)$ contains at least one resident vertex in $\mathcal{P}(G)$. If $\mathcal{P}(G)$ is a convex partition of G, then all vertices of every set of $\mathcal{P}(G)$ are resident in $\mathcal{P}(G)$.

This paper is organized as follows. In Section 2 we describe some properties of minimum convex graph covers. In Section 3 we establish conditions for existence of graph G with given numbers $\varphi_c(G)$, $\theta_c(G)$ and $\varphi_{cn}(G)$, $\theta_{cn}(G)$. In Section 4 we prove that it is NP-complete to decide if a graph has a convex 2-cover. Deciding if a graph has convex 2-cover was declared an open problem in [5]. In addition, we prove that it is NP-complete to decide if a graph has nontrivial convex p-cover or nontrivial convex p-partition for $p \geq 2$.

2 Properties of minimum convex graph covers

Let $\mathcal{P}_{\varphi_c}(G)$ be the minimum convex cover of a simple connected graph G.

Theorem 1. *If $\varphi_c(G) \geq 3$, then for every two sets $A, B \in \mathcal{P}_{\varphi_c}(G)$, $A \neq B$, there exists $C \in \mathcal{P}_{\varphi_c}(G) \backslash \{A, B\}$ such that there exist $a \in A$, $b \in B$, $c \in C \backslash (A \cup B)$, where $c \in \langle a, b \rangle$.*

Proof. Assume the converse. Suppose there exist sets $A, B \in \mathcal{P}_{\varphi_c}(G)$, $A \neq B$, such that for all vertices $a \in A$, $b \in B$, we have $\langle a, b\rangle \subseteq A \cup B$. Thus, since $d-conv(A \cup B) = A \cup B$, we get the reduced convex cover number $\varphi_c(G)$. As $\varphi_c(G)$ is the least integer for which graph G has a convex p-cover, a contradiction follows. ■

Theorem 2. *If $\varphi_c(G) \geq 3$, then for each set $A \in \mathcal{P}_{\varphi_c}(G)$, there exist $B, C \in \mathcal{P}_{\varphi_c}(G)\backslash\{A\}$, $B \neq C$, such that there exist $a \in A\backslash(B \cup C)$, $b \in B$, $c \in C$, where $a \in \langle b, c\rangle$.*

Proof. Assume the converse. Suppose there exists a set $A \in \mathcal{P}_{\varphi_c}(G)$ such that for every two sets $B, C \in \mathcal{P}_{\varphi_c}(G)\backslash\{A\}$, $B \neq C$, we have $A \cap (\langle b, c\rangle\backslash(B \cup C)) = \emptyset$ for all vertices $b \in B$, $c \in C$. This yields that

$$d-conv(\bigcup_{S \in \mathcal{P}_{\varphi_c}(G)\backslash\{A\}} S) = \bigcup_{S \in \mathcal{P}_{\varphi_c}(G)\backslash\{A\}} S.$$

We obtain the convex 2-cover

$$\mathcal{P}_2(G) = \mathcal{P}_{\varphi_c}(G) = \{\bigcup_{S \in \mathcal{P}_{\varphi_c}(G)\backslash\{A\}} S, A\}.$$

Finally, $\varphi_c(G) = 2$. This contradicts the condition of the theorem that $\varphi_c(G) \geq 3$. ■

Considering nontrivial convex cover as a particular case of convex cover, Theorems 1 and 2 have two consequences.

Corollary 1. *If $\varphi_{cn}(G) \geq 3$, then for every two sets $A, B \in \mathcal{P}_{\varphi_{cn}(G)}$, $A \neq B$, there exists $C \in \mathcal{P}_{\varphi_{cn}}(G)\backslash\{A, B\}$ such that there exist $a \in A$, $b \in B$, $c \in C\backslash(A \cup B)$, where $c \in \langle a, b\rangle$.*

Corollary 2. *If $\varphi_{cn}(G) \geq 3$, then for each set $A \in \mathcal{P}_{\varphi_{cn}}(G)$, there exist $B, C \in \mathcal{P}_{\varphi_{cn}}(G)\backslash\{A\}$, $B \neq C$, such that there exist $a \in A\backslash(B \cup C)$, $b \in B$, $c \in C$, where $a \in \langle b, c\rangle$.*

Let $\alpha(G)$ be the vertex independence number of a graph G [3]. Next theorem is true.

Theorem 3. *Let $G = (X;\, U)$ be a simple connected graph and let S be a family of subsets of X with properties:*

a) $|S| \geq 2$;

b) each $Y \in S$ *is a clique;*

c) $X \setminus \bigcup_{Y \in S} Y$ *is not a clique;*

d) $Y \cap Z = \emptyset$ *for all* $Y, Z \in S$;

e) for each set $Y \in S$*, the equality* $\Gamma(y) = (Y \setminus \{y\}) \cup (X \setminus \bigcup_{Z \in S} Z)$ *is satisfied for every vertex* $y \in Y$.

Then the following conditions hold:

a) $\varphi_c(G) \geq \alpha(G)$, $\theta_c(G) \geq \alpha(G)$;

b) if $\boldsymbol{\mathcal{P}}_{\varphi_{cn}}(G)$ *exists, then* $\varphi_{cn}(G) \geq \alpha(G)$;

c) if $\boldsymbol{\mathcal{P}}_{\theta_{cn}}(G)$ *exists, then* $\theta_{cn}(G) \geq \alpha(G)$;

d) every convex set of G *is a clique.*

Proof. Consider two nonadjacent vertices a, b of $X \setminus \bigcup_{Y \in S} Y$ and two vertices y, z such that $y \in Y$, $z \in Z$, where $Y, Z \in S$, $Y \neq Z$. Note that y, z are by definition nonadjacent.

From property $e)$, it follows that $\bigcup_{Y \in S} Y \subseteq \langle a, b \rangle$ and $X \setminus \bigcup_{Y \in S} Y \subseteq \langle y, z \rangle$. Further, $X \setminus \bigcup_{Y \in S} Y \subseteq d - conv(\bigcup_{Y \in S} Y)$ and $\bigcup_{Y \in S} Y \subseteq d - conv(X \setminus \bigcup_{Y \in S} Y)$. Furthermore, we have $d - conv(\bigcup_{Y \in S} Y) = d - conv(X \setminus \bigcup_{Y \in S} Y) = X$. Thus, there is no convex set containing vertices a, b or y, z. This means that every convex set is a clique.

Let $M \subseteq X$ be the maximum independent set of G. In addition, from property $e)$ it follows that $M \subseteq X \setminus \bigcup_{Y \in S} Y$, or $M \subseteq \bigcup_{Y \in S} Y$ such that every element of M belongs to exactly one set of S. By the above, every convex cover of graph G has at least $|M| = \alpha(G)$ sets. This implies the inequalities:

$$\varphi_c(G) \geq \alpha(G),\ \theta_c(G) \geq \alpha(G).$$

Moreover, if $\boldsymbol{\mathcal{P}}_{\varphi_{cn}}(G)$ exists, then $\varphi_{cn}(G) \geq \alpha(G)$. Also, if $\boldsymbol{\mathcal{P}}_{\theta_{cn}}(G)$ exists, then $\theta_{cn}(G) \geq \alpha(G)$. ■

3 Existence of graphs with minimum convex covers

In this section several theorems regarding existence of simple connected graphs with fixed number of minimum convex covers and minimum convex partitions are proved.

For all $n \in N$, $n \geq 2$, there exists a graph on n vertices, which has a convex 2-cover or a convex 2-partition or both. For example, chain graph on $n \geq 2$ vertices has a convex 2-cover and a convex 2-partition. In addition, for every graph that has a nontrivial convex 2-cover the inequality $n \geq 4$ holds, because every set belonging to a convex 2-cover is nontrivial and has at least one resident vertex. On the other hand, for every graph that has a nontrivial convex 2-partition the inequality $n \geq 6$ is satisfied, because its sets are nontrivial and disjoint.

It is clear that for every graph G on n vertices, where $n = 2$ or $n = 3$, we have $\varphi_c(G) = \theta_c(G) = 2$.

First, we prove theorems regarding existence of graphs with fixed numbers $\varphi_c(G)$ and $\theta_c(G)$.

Theorem 4. *If G is a simple connected graph on $n \geq 4$ vertices, then $\varphi_c(G) \leq n - 2$.*

Proof. We distinguish two possible cases.

1) Let G be a graph that is not a convex simple graph. Suppose G has a nontrivial convex set S. Then, since $|S| \geq 3$, we obtain a convex p-cover of G such that $p = n - |S| + 1$. This convex p-cover consists of the set S and $n - |S|$ singletons (sets consisting of exactly one vertex). Substituting $|S| = n + 1 - p$ in $|S| \geq 3$, we get $p \leq n - 2$. So, $\varphi_c(G) \leq n - 2$.

2) Let G be a convex simple graph.

If $n = 4$, then G is a cycle. In this case, G has a convex 2-cover such that both convex sets consist of two adjacent vertices of G.

If $n \geq 5$, then graph G contains vertices x, y, such that $\Gamma(x) = \Gamma(y)$ and $|\Gamma(x)| \geq 3$ [2]. We choose two vertices u, v of $\Gamma(x)$. It is clear that u and v are nonadjacent, otherwise G is not a convex simple graph, because $\{x, u, v\}$ is a triangle, which is a nontrivial convex set. We get

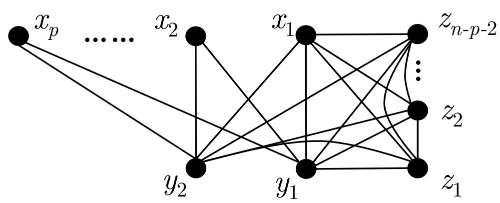

Figure 1. Graph G on n vertices such that $3 \leq p \leq n-2$, $\theta_c(G) = \varphi_c(G) = p$

a convex cover of G that consists of $p = n-2$ sets: $\{x, v\}$, $\{y, u\}$ and $p-4$ singletons.

By definition of $\varphi_c(G)$, we have $\varphi_c(G) \leq n-2$. ■

Corollary 3. *If G is a simple connected graph on $n \geq 4$ vertices, then $\theta_c(G) \leq n-2$.*

Theorem 5. *For any $p, n \in N$, $2 \leq p \leq n-2$, there exists a simple connected graph G on n vertices such that $\varphi_c(G) = p$.*

Proof. If $p = 2$, then take a chain graph G on n vertices for which $\varphi_c(G) = 2$.

If $p \geq 3$, we construct a graph $G = (X; U)$ as follows:

Step 1. let $X_1 = \{x_1, x_2, \ldots, x_p\}$, where any two vertices of X_1 are nonadjacent, i.e., X_1 is an independent set;

Step 2. if $p < n-2$, then define $X_2 = X_1 \cup Z$, where $Z = \{z_1, z_2, \ldots, z_{n-p-2}\}$ such that $Z \cup \{x_1\}$ is a clique. Otherwise, $X_2 = X_1$ and $Z = \emptyset$;

Step 3. $X = X_2 \cup \{y_1, y_2\}$, where $\Gamma(y_1) = \Gamma(y_2) = X_2$.

The resulted graph G is represented in Figure 1.

It is easy to verify that $|X| = n$.

Since X_1 is a maximum independent set in G, the independence number of this graph is $\alpha(G) = |X_1| = p$. The family $\{\{y_1\}, \{y_2\}\}$

satisfies the conditions of Theorem 3 in G. Thus, $\varphi_c(G) \geq p$ and every convex set of G is a clique.

It remains to show that there exists a convex p-cover of graph G.

Graph G has the convex p-cover $\mathcal{P}_p(G)$ that consists of cliques $\{x_1, y_1\} \cup Z$, $\{x_2, y_2\}$, $\{x_3\}$, $\{x_4\}$,..., $\{x_p\}$.

So, since $\mathcal{P}_p(G)$ is the convex p-cover of obtained graph G and $\varphi_c(G) \geq p$, it follows that $\mathcal{P}_p(G)$ is the minimum convex cover of G and $\varphi_c(G) = p$. ■

Corollary 4. *For any $p, n \in N$, $2 \leq p \leq n-2$, there exists a simple connected graph G on n vertices such that $\theta_c(G) = p$.*

Further, a few theorems regarding existence of graphs with fixed numbers $\varphi_{cn}(G)$ and $\theta_{cn}(G)$ are proposed.

Theorem 6. *For any $p, n \in N$, $2 \leq p \leq \lfloor \frac{n}{3} \rfloor$, there exists a simple connected graph G on n vertices such that $\theta_{cn}(G) = p$.*

Proof. We construct a graph $G = (X;\, U)$ as follows:

Step 1. let $X_1 = \{x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}, \ldots, x_{p,1}, x_{p,2}\}$, where $x_{i,1} \sim x_{i,2}$ for $1 \leq i \leq p$;

Step 2. if $3p < n$, then define $X_2 = X_1 \cup Z$, where $Z = \{z_1, z_2, \ldots, z_{n-3p}\}$ such that $Z \cup \{x_{1,1}, x_{1,2}\}$ is a clique. Otherwise, $X_2 = X_1$ and $Z = \emptyset$;

Step 3. $X = X_2 \cup Y$, where $Y = \{y_1, y_2, \ldots, y_p\}$ such that $\Gamma(y_i) = X_2$ for $1 \leq i \leq p$.

The obtained graph G is represented in Figure 2.

It is easy to verify that $|X| = n$.

Since Y is a maximum independent set in G, the independence number of this graph is $\alpha(G) = |Y| = p$. The family $\{\{y_1\}, \{y_2\}, \ldots, \{y_p\}\}$ satisfies the conditions of Theorem 3 in G. Thus, $\theta_c(G) \geq p$ and if there exists $\mathcal{P}_{\theta_{cn}}(G)$, then $\theta_{cn}(G) \geq p$. Also, every convex set of G is a clique.

It remains to show that there exists a nontrivial convex p-partition of graph G.

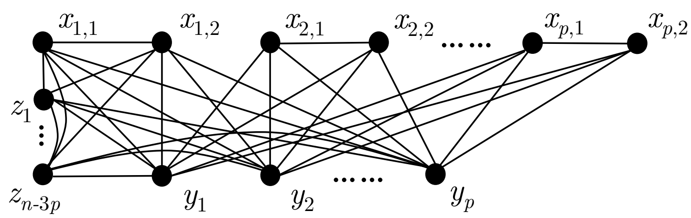

Figure 2. Graph G on n vertices such that $2 \leq p \leq \lfloor \frac{n}{3} \rfloor$, $\theta_{cn}(G) = p$

Graph G has a nontrivial convex p-partition $\boldsymbol{\mathcal{P}}_p(G)$ that consists of cliques $\{x_{1,1}, x_{1,2}, y_1\} \cup Z$, $\{x_{2,1}, x_{2,2}, y_2\}$, $\{x_{3,1}, x_{3,2}, y_3\}, \ldots$, $\{x_{p,1}, x_{p,2}, y_p\}$.

So, since $\boldsymbol{\mathcal{P}}_p(G)$ is a nontrivial convex p-partition of obtained graph G and $\theta_{cn}(G) \geq p$, it follows that $\boldsymbol{\mathcal{P}}_p(G)$ is the minimum nontrivial convex partition of G and $\theta_{cn}(G) = p$. ■

Let C_4 be a cycle on 4 vertices.

Theorem 7. *If G is a simple connected graph on 4 vertices, then $\varphi_{cn}(G) = 2$ if and only if $G \neq C_4$.*

Proof. By definition of nontrivial convex cover, G has a nontrivial convex p-cover if and only if $p = 2$. In Figure 3 simple connected graphs on 4 vertices are represented. It can be easily checked that every graph from Figure 3, except the cycle C_4, has a nontrivial convex 2-cover. Now, if we recall that nontrivial convex cover number is the least integer $p \geq 2$ for which graph G has a nontrivial convex p-cover, we get $\varphi_{cn}(G) = 2$ for every simple connected graph G on 4 vertices, where $G \neq C_4$. ■

Theorem 8. *If G is a simple connected graph on $n \geq 5$ vertices, then $\varphi_{cn}(G) < n - 2$.*

Proof. There is no a nontrivial convex n-cover or a nontrivial convex $(n-1)$-cover, because every convex set of nontrivial convex cover has

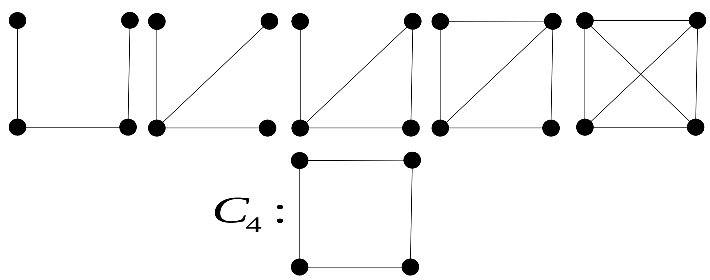

Figure 3. All simple connected graphs on 4 vertices

at least one resident vertex in this convex cover and contains at least three vertices.

Let us prove that for every graph G the inequality $\varphi_{cn}(G) < n-2$ is satisfied. The proof is by reductio ad absurdum. We can assume without loss of generality that there exists a graph $G = (X\ ,U)$ such that $\varphi_{cn}(G) = n-2$. It is required that $n \geq 5$, consequently $\varphi_{cn}(G) \geq 3$. Let $\boldsymbol{\mathcal{P}}_{\varphi_{cn}}(G)$ be the minimum nontrivial convex cover of G. In this case, every set $S \in \boldsymbol{\mathcal{P}}_{\varphi_{cn}}(G)$ satisfies equality $|S| = 3$ and contains exactly one resident vertex in $\boldsymbol{\mathcal{P}}_{\varphi_{cn}}(G)$. Further, there are two vertices $x, y \in X$, which are common for all sets of $\boldsymbol{\mathcal{P}}_{\varphi_{cn}}(G)$. Notice that $x \sim y$, otherwise connectivity of nontrivial convex sets of $\boldsymbol{\mathcal{P}}_{\varphi_{cn}}(G)$ implies $\Gamma(x) = \Gamma(y) = X \backslash \{x, y\}$ and furthermore $d - conv(\{x, y\}) = X$. According to Corollaries 1 and 2, all vertices of set $X \backslash \{x, y\}$ are nonadjacent in G. Finally, we obtain a nontrivial convex 2-cover $\boldsymbol{\mathcal{P}}_2(G) = \boldsymbol{\mathcal{P}}_{\varphi_{cn}}(G) = \{\{x, y, z\}, X \backslash \{z\}\}$, where $z \in X \backslash \{x, y\}$. Thus, $\varphi_{cn}(G) = 2$. This contradiction concludes the proof. ■

Theorem 9. *For any $p, n \in N$, $2 \leq p \leq n-3$, there exists a simple connected graph G on n vertices such that $\varphi_{cn}(G) = p$.*

Proof. We construct a graph $G = (X;\ U)$ as follows:

Step 1. let $X_1 = \{x_1, x_2, \ldots, x_p\}$, where all vertices of X_1 are nonadjacent, i.e., X_1 is an independent set;

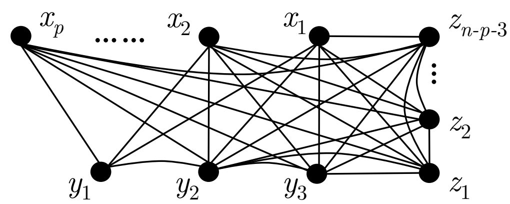

Figure 4. Graph G on n vertices such that $2 \leq p \leq n-3$, $\varphi_{cn}(G) = p$

Step 2. if $p < n-3$, then define $X_2 = X_1 \cup Z$, where $Z = \{z_1, z_2, \ldots, z_{n-p-3}\}$ such that $Z \cup \{x\}$ is a clique for all $x \in X_1$. Otherwise, $X_2 = X_1$ and $Z = \emptyset$;

Step 3. $X = X_2 \cup Y$, where $Y = \{y_1, y_2, y_3\}$ such that $\Gamma(y_1) = X_1 \cup \{y_2\}$, $\Gamma(y_2) = X_2 \cup \{y_1, y_3\}$ and $\Gamma(y_3) = X_2 \cup \{y_2\}$.

The obtained graph G is represented in Figure 4.

It is easy to verify that $|X| = n$.

Since X_1 is a maximum independent set in G, the independence number of this graph is $\alpha(G) = |X_1| = p$. The family $\{\{x_1\}, \{x_2\}, \ldots, \{x_p\}\}$ satisfies the conditions of Theorem 3 in G. Thus, $\varphi_c(G) \geq p$ and if there exists $\boldsymbol{\mathcal{P}}_{\varphi_{cn}}(G)$, then $\varphi_{cn}(G) \geq p$. Also, every convex set of G is a clique.

It remains to show that there exists a nontrivial convex p-cover of graph G.

Graph G has a nontrivial convex p-cover $\boldsymbol{\mathcal{P}}_p(G)$ that consist of cliques $\{x_1, y_2, y_3\} \cup Z$, $\{x_2, y_1, y_2\}$, $\{x_3, y_1, y_2\}$, …, $\{x_p, y_1, y_2\}$.

So, since $\boldsymbol{\mathcal{P}}_p(G)$ is a nontrivial convex p-cover of obtained graph G and $\varphi_{cn}(G) \geq p$, it follows that $\boldsymbol{\mathcal{P}}_p(G)$ is the minimum nontrivial convex cover of G and $\varphi_{cn}(G) = p$. ■

4 NP-completeness

Let us examine the complexity of convex cover problems.

Deciding whether a graph G has a convex p-cover, for $p \geq 2$, is said to be convex p-cover problem. Similarly, deciding whether a graph G has a convex p-partition, for $p \geq 2$, is said to be convex p-partition problem. In the same way, we introduce nontrivial convex p-cover and nontrivial convex p-partition problems, where nontrivial convex covers and nontrivial convex partitions are considered.

It was shown in [4], [6] that the convex p-partition problem is NP-complete for $p \geq 2$. Also, we know that the convex p-cover problem is NP-complete for $p \geq 3$ [5]. Deciding if a graph has a convex 2-cover was declared an open problem in the paper [5].

We prove that the convex 2-cover problem is NP-complete.

The complexity of this case is proved by reducing the NP-complete 1-IN-3 3 SAT problem [8] to a convex 2-cover problem.

1-*IN*-3 3 *SAT problem:*

Instance: Set $V = \{v_1, v_2, \ldots, v_n\}$ of variables, collection $\mathcal{C} = \{c_1, c_2, \ldots, c_m\}$ of clauses over V such that each clause $c \in \mathcal{C}$ has $|c| = 3$ and no negative literals.

Question: Is there a truth assignment for V such that each clause in $\mathcal{C}$ has exactly one true literal?

We say that $\mathcal{C}$ is *satisfiable* if there exists a truth assignment for V such that $\mathcal{C}$ is satisfiable and each clause in $\mathcal{C}$ has exactly one true variable.

Theorem 10. *The convex* 2*-cover problem is NP-complete.*

Proof. We mention that this problem is in NP, because verifying if a set is convex can be done in polynomial time [7]. Further, we reduce 1-IN-3 3 SAT to the convex 2-cover problem. First, we determine the structure of a particular graph $G = (X; U)$ for a convex 2-cover from a generic instance $(V, \mathcal{C})$ of 1-IN-3 3 SAT. Next, we prove that $\mathcal{C}$ is satisfiable if and only if G has a convex 2-cover. For this purpose, we prove that a convex 2-cover of G defines a truth assignment that satisfies $(V, \mathcal{C})$. At the same time, we prove that a truth assignment that satisfies $(V, \mathcal{C})$ defines a convex 2-cover of G.

Let graph G be given by vertex set X and edge set U.

The vertex set X consists of:

a) $\boldsymbol{\mathscr{V}} = \{\boldsymbol{\nu}_1, \boldsymbol{\nu}_2, \ldots, \boldsymbol{\nu}_n\}$, $Y = \{f, y_1, y_2, y_3, y_4, y_5\}$,
$Z = \{t, z_1, z_2, z_3, z_4, z_5\}$;
b) $F = \{f_j | 1 \leq j \leq m\}$, $T = \{t_j | 1 \leq j \leq m\}$;
c) $L = \{l_j^i | 1 \leq j \leq m, 1 \leq i \leq 3\}$, $\boldsymbol{\mathscr{L}} = \{\boldsymbol{\ell}_j^i | 1 \leq j \leq m, 1 \leq i \leq 3\}$,
$Q = \{q_j^i | 1 \leq j \leq m, 1 \leq i \leq 3\}$;

We get $X = \boldsymbol{\mathscr{V}} \cup Y \cup Z \cup F \cup T \cup L \cup Q \cup \boldsymbol{\mathscr{L}}$. Every variable $v_i \in V$ corresponds to vertex $\boldsymbol{\nu}_i \in \boldsymbol{\mathscr{V}}$. Every clause $\boldsymbol{c}_j \in \boldsymbol{\mathscr{C}}$ corresponds to eleven vertices: f_j, l_j^1, l_j^2, l_j^3, $\boldsymbol{\ell}_j^1$, $\boldsymbol{\ell}_j^2$, $\boldsymbol{\ell}_j^3$, q_j^1, q_j^2, q_j^3, t_j.

The edge set U satisfies conditions:

a) $\boldsymbol{\mathscr{V}} \cup Q$ is a clique in G;
b) $\Gamma(f) = \boldsymbol{\mathscr{V}} \cup Q \cup F \cup \{y_3, y_4\}$ and $\Gamma(t) = \boldsymbol{\mathscr{V}} \cup Q \cup T \cup \{z_3, z_4\}$;
c) $\Gamma(y_5) = F \cup \{y_3, y_4\}$ and $\Gamma(z_5) = T \cup \{z_3, z_4\}$;
d) there exist the following edges: $\{y_1, y_3\}$, $\{y_1, y_4\}$, $\{y_2, y_3\}$, $\{y_2, y_4\}$, $\{z_1, z_3\}$, $\{z_1, z_4\}$, $\{z_2, z_3\}$, $\{z_2, z_4\}$;
e) every clause $\boldsymbol{c}_j = \{v_a, v_b, v_c\}$, $1 \leq j \leq m$, corresponds to eighteen edges: $\{l_j^1, v_a\}$, $\{l_j^2, v_b\}$, $\{l_j^3, v_c\}$, $\{l_j^1, f_j\}$, $\{l_j^2, f_j\}$, $\{l_j^3, f_j\}$, $\{\boldsymbol{\ell}_j^1, t_j\}$, $\{\boldsymbol{\ell}_j^2, t_j\}$, $\{\boldsymbol{\ell}_j^3, t_j\}$, $\{q_j^1, \boldsymbol{\ell}_j^1\}$, $\{q_j^2, \boldsymbol{\ell}_j^2\}$, $\{q_j^3, \boldsymbol{\ell}_j^3\}$, $\{l_j^1, \boldsymbol{\ell}_j^2\}$, $\{l_j^1, \boldsymbol{\ell}_j^3\}$, $\{l_j^2, \boldsymbol{\ell}_j^1\}$, $\{l_j^2, \boldsymbol{\ell}_j^3\}$, $\{l_j^3, \boldsymbol{\ell}_j^1\}$, $\{l_j^3, \boldsymbol{\ell}_j^2\}$.

We skip the trivial case $|\boldsymbol{\mathscr{C}}| = 1$ of 1-IN-3 3 SAT problem. Let us consider $|\boldsymbol{\mathscr{C}}| \geq 2$.

If $G = (X; U)$ has a convex 2-cover, then $\boldsymbol{\mathscr{C}}$ is satisfiable.

Let $\boldsymbol{\mathscr{P}}_2(G) = \{S_f, S_t\}$ be a convex 2-cover of G. For every $i, j \in \{1, 2\}$ we have $d - conv(\{y_i, z_j\}) = X$.

Let $y_1, y_2 \in S_f$, $z_1, z_2 \in S_t$ and let $S_1 = \{y_3, y_4, y_5, f\} \cup F$, $S_2 = \{z_3, z_4, z_5, t\} \cup T$.

Let us enumerate some properties:

Property 1: $S_1 \cap S_t = \emptyset$ and $S_2 \cap S_f = \emptyset$.

We notice what $S_1 \subseteq d - conv(\{y_1, y_2\})$, $S_2 \subseteq d - conv(\{z_1, z_2\})$. Consequently we have $S_1 \subseteq S_f$, $S_2 \subseteq S_t$.

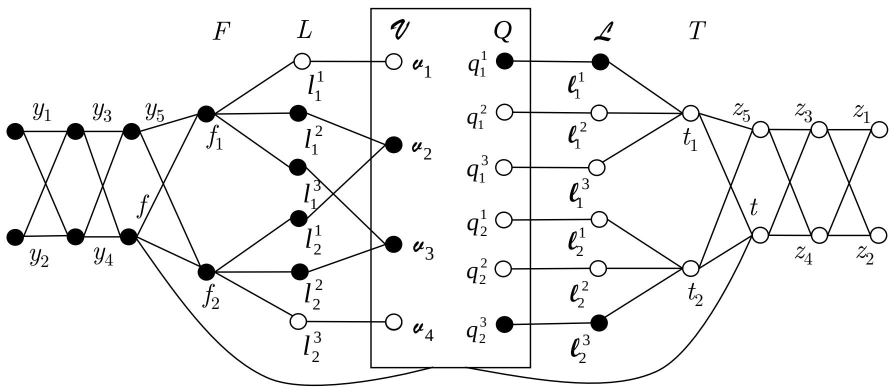

Figure 5. The convex 2-cover of the graph G for the instance $(V, \boldsymbol{\mathcal{C}}) = (\{v_1, v_2, v_3, v_4\}, \{\{v_1, v_2, v_3\}, \{v_2, v_3, v_4\}\})$

Moreover, for each $u \in F \cup \{f\}$, we get $d - conv(\{u, t\} \cup T) \subseteq d - conv(\{f\} \cup F \cup S_2) \subseteq d - conv(S_1 \cup S_2) \subseteq d - conv(S_f \cup S_t) = X$. This implies that $u \notin S_t$ for each $u \in F \cup \{f\}$. Similarly, for each $u \in T \cup \{t\}$, we get $d - conv(\{u, f\} \cup F) \subseteq d - conv(\{t\} \cup T \cup S_1) \subseteq d - conv(S_1 \cup S_2) \subseteq d - conv(S_f \cup S_t) = X$. This implies that $u \notin S_f$ for each $u \in T \cup \{t\}$. Thus, $S_1 \cap S_t = \emptyset$ and $S_2 \cap S_f = \emptyset$.

Property 2: Sets $L, \boldsymbol{\mathcal{V}}, Q, \boldsymbol{\mathcal{L}}$ are uniquely interdependent.

If vertex l_j^i belongs to S_t, then $\Gamma(l_j^i) \cap \boldsymbol{\mathcal{V}} \subseteq S_t$ and $\boldsymbol{\ell}_j^k$ belongs to S_t for $1 \leq k \leq 3$, $k \neq i$.

If vertex $\boldsymbol{v}_i$ belongs to S_t, then $\Gamma(\boldsymbol{v}_i) \cap L \subseteq S_t$ and for all $l_j^a \in \Gamma(\boldsymbol{v}_i) \cap L$ vertices $\boldsymbol{\ell}_j^k$ belong to S_t for $1 \leq k \leq 3$, $k \neq a$.

Vertex $\boldsymbol{\ell}_j^i$ belongs to S_f if and only if q_j^i belongs to S_f. If vertex $\boldsymbol{\ell}_j^i$ belongs to S_f, then $L' = \{l_j^k | 1 \leq k \leq 3, k \neq i\} \subseteq S_f$ and $\Gamma(l_j^k) \cap \boldsymbol{\mathcal{V}}$ is contained in S_f for all $l_j^k \in L'$.

Property 3: Exactly one vertex of $L_j = \{l_j^1, l_j^2, l_j^3\}$ belongs to S_t, for $1 \leq j \leq m$, and exactly one vertex of $\boldsymbol{\mathcal{L}}_j = \{\boldsymbol{\ell}_j^1, \boldsymbol{\ell}_j^2, \boldsymbol{\ell}_j^3\}$ belongs to S_f, for $1 \leq j \leq m$.

Exactly one vertex of every set $L_j = \{l_j^1, l_j^2, l_j^3\}$, $1 \le j \le m$, belongs to S_t. In the converse case, if two vertices $\{l_j^a, l_j^b\}$ of L_j belong to S_t, then f_j belongs to S_t. By Property 1, we get a contradiction. If no vertex of $L_j = \{l_j^1, l_j^2, l_j^3\}$ belongs to S_t, then $L_j \subseteq S_f$, $\boldsymbol{\mathcal{L}}_j = \{\boldsymbol{\ell}_j^1, \boldsymbol{\ell}_j^2, \boldsymbol{\ell}_j^3\} \subseteq S_f$ and t_j belongs to S_f. Now by Property 1, we have a contradiction.

In the same way, exactly one vertex of every set $\boldsymbol{\mathcal{L}}_j = \{\boldsymbol{\ell}_j^1, \boldsymbol{\ell}_j^2, \boldsymbol{\ell}_j^3\}$, $1 \le j \le m$, belongs to S_f.

We associate $\boldsymbol{\mathcal{V}}$ with V and L with $\boldsymbol{\mathcal{C}}$ such that convex 2-cover represents a truth assignment for $\boldsymbol{\mathcal{V}}$, where the variable v_i is true if and only if the vertex $\boldsymbol{u}_i \in S_t$.

It follows from Properties 1, 2 and 3 that if G has a convex 2-cover $\boldsymbol{\mathcal{P}}_2(G) = \{S_f, S_t\}$, then $\boldsymbol{\mathcal{C}}$ is satisfiable. Let us remark that sets S_f, S_t are nontrivial and disjoint.

If $\boldsymbol{\mathcal{C}}$ is satisfiable, then $G = (X;\, U)$ has a 2-convex cover.

Suppose that there exists a truth assignment which satisfies $(V, \boldsymbol{\mathcal{C}})$. We construct a convex 2-cover $\boldsymbol{\mathcal{P}}_2(G) = \{S_f, S_t\}$ as follows:

Step 1. Define $S_t = \{z_1, z_2, z_3, z_4, z_5, t\} \cup T$;
Step 2. For each true variable v_i of V we add vertex $\boldsymbol{u}_i$ and the set $L' = \Gamma(\boldsymbol{u}_i) \cap L$ to S_t and for each $l_j^a \in L'$ we add vertices q_j^b, $\boldsymbol{\ell}_j^b$ to S_t such that $\boldsymbol{\ell}_j^b \sim l_j^a$ and $q_j^b \sim \boldsymbol{\ell}_j^b$;
Step 3. Define $S_f = X \backslash S_t$.

Clearly, for the resulting convex 2-cover $\boldsymbol{\mathcal{P}}_2(G) = \{S_f, S_t\}$ Properties 1, 2 and 3 are satisfied. Hence, if $\boldsymbol{\mathcal{C}}$ is satisfiable, then G has convex 2-cover. Note also that the sets S_f and S_t are nontrivial and disjoint.

In Figure 5 the graph G, which corresponds to a particular instance $(V, \boldsymbol{\mathcal{C}}) = (\{v_1, v_2, v_3, v_4\}, \{\{v_1, v_2, v_3\}, \{v_2, v_3, v_4\}\})$ is represented. Sets $Q \cup \boldsymbol{\mathcal{V}} \cup \{f\}$ and $Q \cup \boldsymbol{\mathcal{V}} \cup \{t\}$ generate cliques in G. White vertices belong to S_t and black vertices belong to S_f. The white vertices of $\boldsymbol{\mathcal{V}}$ represent the variables of V set to true. All edges between L and $\boldsymbol{\mathcal{L}}$ are represented in Figure 6. ■

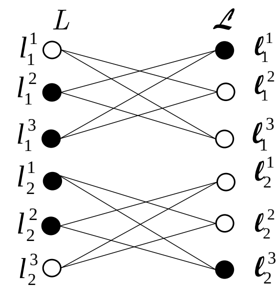

Figure 6. Edges between L and $\mathcal{L}$

Now if we recall that convex p-cover problem is NP-complete, for $p \geq 3$, we can affirm that convex p-cover problem is NP-complete for $p \geq 2$.

Corollary 5. *Convex* 2*-partition problem, nontrivial convex* 2*-cover problem and nontrivial convex* 2*-partition problem are NP-complete.*

Proof. By construction in the previous theorem of a particular graph $G = (X; U)$ for convex 2-cover problem from a generic instance $(V, \mathcal{C})$ of 1-IN-3 3 SAT problem, we conclude that G can be covered only by nontrivial disjoint convex sets. Hence, every convex 2-cover is a convex 2-partition in G. Moreover, every convex 2-cover is a nontrivial convex 2-cover in G. ■

Taking into account Theorem 10 and Corollary 5, we affirm that Theorem 10 is stronger than Theorem 4 in [6], which proves only NP-completeness of convex 2-partition problem.

Furthermore, we prove that nontrivial convex p-cover problem, for $p \geq 3$, is NP-complete. We reduce NP-complete clique p-partition problem, for $p \geq 3$ [9], to a nontrivial convex p-cover problem.

Clique p-partition problem:
Instance: Graph $G = (X; U)$ and $p \in N$, $p \geq 3$.
Question: Is there a partition of X into p disjoint cliques?

Theorem 11. *The nontrivial convex p-partition problem is NP-complete for* $p \geq 3$.

Proof. The problem is in NP, because determining if a set is convex can be done in polynomial time [7].

Let $G = (X; U)$ be a generic graph of clique p-partition problem. Without loss of generality, it can be assumed that X is not a clique. We obtain a particular graph $G' = (X'; U')$ of nontrivial convex p-partition problem from G by adding auxiliary sets $Y = \{y_1, y_2, \ldots, y_p\}$ and $Z = \{z_1, z_2, \ldots, z_p\}$ to X such that $X' = X \cup Y \cup Z$, where $\Gamma(y_i) = X \cup \{z_i\}$ and $\Gamma(z_i) = X \cup \{y_i\}$ for $1 \leq i \leq p$.

Graph G' satisfies the conditions of Theorem 3. Thus, every convex set of G' is a clique.

If $\mathcal{P}_p(G)$ is a clique p-partition of G, $p \geq 3$, then we obtain a nontrivial convex p-partition $\mathcal{P}_p(G')$ of G' by addition of set $\{y_i, z_i\}$ to X_i, where $X_i \in \mathcal{P}_p(G)$, for $1 \leq i \leq p$.

On the other hand, a nontrivial convex p-partition $\mathcal{P}_p(G')$ of G', $p \geq 3$, implies existence of a clique p-partition $\mathcal{P}_p(G)$ of G by subtraction of set $\{y_i, z_i\}$ from X'_i , where $X'_i \in \mathcal{P}_p(G')$, for $1 \leq i \leq p$. ■

Corollary 6. *The nontrivial convex p-cover problem is NP-complete for* $p \geq 3$.

Proof. The problem is also in NP, because determining if a set is convex can be done in polynomial time [7].

We know that any proper convex set of graph G' constructed in previous theorem, is a clique. Let $\mathcal{P}_p(G')$ be a nontrivial convex p-cover of G'. We get a family of sets $\mathcal{P} = \{X_1, X_2, \ldots, X_p\}$ such that $X_i = X'_i \backslash \{y_i, z_i\}$, where $X'_i \in \mathcal{P}_p(G')$ for $1 \leq i \leq p$. Removing from $\mathcal{P}$ all sets contained in the union of other sets of the family $\mathcal{P}$ we obtain a convex k-partition $\mathcal{P}_k(G)$ of G such that $k \leq p$, where G is a graph of clique p-partition problem. Note also that if any graph has a clique q-partition and there exists a set S of this partition that is not a singleton, then dividing S into two cliques, we get a clique $(q+1)$-partition. Thus, G has a clique p-partition.

On the other hand, we know from previous theorem that every clique p-partition of G, $p \geq 3$, implies existence of nontrivial convex

p-partition of G'. Now, if we recall that every nontrivial convex p-partition is a nontrivial convex p-cover, we deduce that every clique p-partition of G, $p \geq 3$, implies existence of nontrivial convex p-cover of G'. ■

We affirm that nontrivial convex p-cover problem and nontrivial convex p-partition problem are NP-complete for $p \geq 2$. Indeed, this follows from Theorems 11 and from Corollaries 5 and 6.

5 Conclusion

We prove that the problem of deciding if a graph has a convex 2-cover is NP-complete. Since Theorem 10 proves NP-completeness of convex 2-cover problem and Corollary 5, as consequence of Theorem 10, proves NP-completeness of convex 2-partition problem, we conclude that Theorem 10 is stronger than Theorem 4 in [6], which proves only NP-completeness of convex 2-partition problem. We affirm that convex p-cover problem and convex p-partition problem are NP-complete for $p \geq 2$.

Also, we prove that it is NP-complete to decide if a graph has a nontrivial convex p-cover or nontrivial convex p-partition for $p \geq 2$.

We discover some properties of minimum convex covers and minimum convex partitions of graphs. We establish conditions for existence of graph G with given numbers $\varphi_c(G)$, $\theta_c(G)$ and $\varphi_{cn}(G)$, $\theta_{cn}(G)$.

References

[1] V. Bolteansky, P.Soltan. *Combinatorial geometry of the various classes of convex sets.* Chişinău, 1978. (in Russian)

[2] S. Cataranciuc, N.Sur. *D-convex simple and quasi-simple graphs.* Chişinău, Republic of Moldova, 2009. (in Romanian)

[3] C. Berge. *Graphs and Hypergraphs.* New York: Elsevier, 1973.

[4] D. Artigas, M.C.Dourado, J.L.Szwarcfiter. *Convex Partitions of Graphs.* Electronic Notes in Discrete Mathematics, vol. 29 (2007), pp. 147–151.

[5] D. Artigas, S.Dantas, M.C.Dourado, J.L.Szwarcfiter. *Convex covers of graphs.* Matemática Contemporânea, Sociedade Brasileira de Matemática, vol. 39 (2010), pp. 31–38.

[6] D. Artigas, S.Dantas, M.C.Dourado, J.L.Szwarcfiter. *Partitioning a graph into convex sets.* Discrete Mathematics, vol. 311 (2011), pp. 1968–1977.

[7] M.C. Dourado, J.G.Gimbel, F.Protti, J.L.Szwarcfiter. *On the computation of the hull number of a graph.* Discrete Mathematics, vol. 309 (2009), pp. 5668–5674.

[8] T. J. Schaefer. *The complexity of satisfiability problems.* Proceeding STOC '78 Proceedings of the tenth annual ACM symposium on Theory of computing, ACM New York, NY, USA , 1978, pp. 216–226.

[9] R. Karp. *Reducibility among combinatorial problems.* In: R.E. Miller and J.W. Thatcher (Eds.), Complexity of Computer Computations, Plenum, New York, 1972, pp. 85–103.

Radu Buzatu, Sergiu Cataranciuc
Received July 27, 2015

Radu Buzatu
State University of Moldova
60 A. Mateevici, MD-2009, Chişinău, Republic of Moldova
E–mail: radubuzatu@gmail.com

Sergiu Cataranciuc
State University of Moldova
60 A. Mateevici, MD-2009, Chişinău, Republic of Moldova
E–mail: s.cataranciuc@gmail.com