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Fundamental theorems of extensional untyped λ-calculus revisited

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Abstract

This paper presents new proofs of three following fundamental theorems of the untyped extensional λ-calculus: the η-Postponement theorem, the $\beta\eta$-Normal form theorem, and the Normalization theorem for $\beta\eta$-reduction. These proofs do not involve any special extensions of the standard language of λ-terms but nevertheless are shorter and much more comprehensive than their known analogues.

Keywords: extensional untyped λ-calculus, $\beta\eta$-reduction, postponement of η-reduction, η-Postponement theorem, $\beta\eta$-Normal form theorem, Normalization theorem for $\beta\eta$-reduction.

1 Introduction

The untyped version of the λ-calculus is considered.

Its variables are denoted by symbols x, y and z, λ-terms by t, p, q, u and w, redeces by $\triangle$ (of cause, indices are sometimes used). All the other denotations used in the paper are completely standard or otherwise will be introduced separately.

Throughout the paper the variable convention is assumed to be satisfied; hence the conditions $x \notin t$ and $x \notin FV(t)$ say the same.

Recall some basic facts concerning η- and $\beta\eta$-reduction. By definition, the notion η of η-reduction is $\{ < \lambda x.wx, w > \mid x \notin w \}$, the notion $\beta\eta$ of $\beta\eta$-reduction is $\beta \cup \eta$. The notions of η- and $\beta\eta$-reduction induce, in the usual way, the dictionaries of their derivative notions (such as η- and $\beta\eta$-redeces, the relations of one-step and multi-step η-

and $\beta\eta$-reduction, η- and $\beta\eta$-reduction sequences, etc.). Remark that the notion of η-reduction is strongly normalizing since the contraction of an η-redex in any λ-term decreases its length.

The extensional untyped λ-calculus studies properties of the notion of $\beta\eta$-reduction as well as of its derivative relations, especially $\twoheadrightarrow_{\beta\eta}$ (multi-step $\beta\eta$-reduction) and $=_{\beta\eta}$ ($\beta\eta$-convertibility). Along with the Church-Rosser theorem for $\beta\eta$-reduction (and not taking the results on λ-representability into account), the most important results in the extensional λ-calculus are: 1) the η-Postponement theorem, 2) the $\beta\eta$-Normal form theorem, 3) the Normalization theorem for $\beta\eta$-reduction.

In the paper, new proofs of the theorems 1) – 3) are constructed. These proofs do not involve any special extensions of the standard language of λ-terms but nevertheless are shorter and much more comprehensive than their original or known analogues. (For example, the original proof of the theorem 3) by J.W.Klop takes over 20 pages and is technically very complicated.) The new proofs are arranged in the following logical order: 1) $\Rightarrow$ 2) $\Rightarrow$ 3).

2 Postponable binary relations

Our proof of the η-Postponement theorem exploits some general properties which are more convenient to be observed and studied in the general set-theoretic situation.

Definition 1. Given a set A and binary relations Q and R on A, then R is said to be *postponable* after Q if the following diagram holds:

$$\begin{tikzcd} & \cdot \arrow[dl, dashed, "Q"'] \arrow[dr, "R"] & \\ \cdot \arrow[dr, dashed, "R"'] & & \cdot \arrow[dl, "Q"] \\ & \cdot & \end{tikzcd}$$

(Here and in the sequel, the language of diagrams of binary relations is used. In the general case, such a diagram is a configuration on the plane consisting of points some of which may be labelled by elements of a fixed set A, and arrows between points, each obligatorily labelled by a binary relation on A. Each arrow can be of two sorts: usual or dotted. If a diagram contains a usual arrow from a to b and labelled by R, then the latter expresses that $a\,R\,b$; if a point has no label, then it is considered to be bounded by a universal quantifier (restricted by A). At that, precisely those arrows are dotted that lead to or start with the elements, the existence of which is being claimed (together with the conditions imposed by the labels). Thus, each diagram (containing at least one dotted arrow) determines a certain implicative statement. For example, the diagram from the last definition means the following:

$$\forall a,b,c\in A\left[\, a\,R\,b \;\;\&\;\; b\,Q\,c \;\;\Rightarrow\;\; \exists\, d\in A\,(a\,Q\,d \;\;\&\;\; d\,R\,c)\,\right],$$

i.e. that $R\circ Q \subseteq Q\circ R$, where $\circ$ denotes the usual composition of binary relations.)

Note that if binary relations $Q=f$ and $R=g$ are functions, then g is postponable after f if and only if $g\circ f = f\circ g$, that is f and g commute with each other. Therefore, in this case f is postponable after g as well. However in the general case the latter is not valid, i.e. the notion of postponability is not symmetric.

By R^+ and R^* denote, resp., the transitive and reflexive-transitive closures of a binary relation R.

It can easily be proved that if R is postponable after Q, then R^* is postponable after Q^* as well. Containing this statement as a particular case, the following result can be viewed as its natural generalization.

Postponement Lemma. *Let Q and R be binary relations on a set A such that for any triple of elements of A, at least one of the following diagrams holds:*

$$\begin{tikzcd} & \cdot \arrow[dl, dashed, "Q"'] \arrow[dr, "R"] & \\ \cdot \arrow[dr, dashed, "R^*"'] & & \cdot \arrow[dl, "Q"] \\ & \cdot & \end{tikzcd}$$

or

$$\begin{tikzcd} \cdot \arrow[dd, dashed, "Q^+"'] \arrow[dr, "R"] & \\ & \cdot \arrow[dl, "Q"] \\ \cdot & \end{tikzcd}$$

Then R^* is postponable after Q^* and $(Q\cup R)^* = Q^*\circ R^*$.

Proof. Since $R^* \circ Q^* \subseteq (Q \cup R)^*$, for proving the postponability, it is sufficient to show that $(Q \cup R)^* \subseteq Q^* \circ R^*$, which also substantiates the second statement (the inclusion opposite to the latter holds trivially). Let a and c be any elements of A with $a\,(Q \cup R)^* c$. Obviously, it can be assumed that $a \neq c$. Then there is, for some elements $b_1, \ldots, b_{n-1} \in A$, a valid sequence of the form

$$b_0\, S_0\, b_1\, S_1\, b_2\, S_2 \ldots S_{n-2}\, b_{n-1}\, S_{n-1}\, b_n\,, \tag{1}$$

where $b_0 = a$, $b_n = c$, and $S_i \in \{Q, R\}$ for every $i \in \{0, n-1\}$.

Consider its leftmost "two-step" segment of the form $b_k R\, b_{k+1} Q\, b_{k+2}$ (if there is no such a segment, there is nothing to prove). If, for simplicity, the "Q-prefix" of (1) is empty, then (1) has the following form:

$$b_0 R\, b_1 R\, b_2 R \ldots R\, b_{k-1} R\, \underline{b_k\, R\, b_{k+1} Q\, b_{k+2}}\, S_{k+2}\, b_{k+3}\, S_{k+3} \ldots S_{n-1} b_n\,. \tag{2}$$

Applying one of the diagrams from the conditions of the lemma to the underlined segment, one of the following sequences will be obtained:

$$b_0 R \ldots R\, b_{k-1} R\, \underline{b_k\, Q\, b'_{k+1} R\, b'_{k+1,\,1} R \ldots R\, b'_{k+1,\,m} R\, b_{k+2}}\, S_{k+2} \ldots S_{n-1}\, b_n$$

(in the case of the left diagram), where m is a natural number, or

$$b_0 R \ldots R\, b_{k-1} R\, \underline{b_k\, Q\, b'_{k+1,\,1} Q \ldots Q\, b'_{k+1,\,m} Q\, b_{k+2}}\, S_{k+2} \ldots S_{n-1}\, b_n$$

(in the case of the right diagram), where m is a positive natural number.

Comparing the obtained sequences with the previous one, notice that in the both cases, the position of the leftmost occurrence of Q is one item to the left than that was in (2). Therefore, the proof can be completed by induction, applied to the set of sequences of the form (1) that is considered to be lexicographically ordered in accordance with the positions of all occurrences of Q in a sequence under consideration when reading it from left to right. $\square$

Remark. The Postponement lemma states less than that was proved. Actually, the above given proof determines an algorithm of reconstructing each sequence (1) to a form $b_0\, Q \ldots Q\, d\, R \ldots R\, b_n$ which will be referred to as *postponing* R after Q. (Of cause, this algorithm makes sense provided the conditions of the lemma are satisfied.)

3 Postponement of η-reduction

Note that when considering diagrams over the set of λ-terms some arrows of which are labelled by one-step reductions, it is often convenient to introduce additional labels for (some of) these arrows for indicating the contracted redex occurrences. (Examples are given below.)

Given a one-step $\beta\eta$-reduction sequence $\sigma\colon t_1 \xrightarrow{\triangle}_{\beta\eta} t_2$. If $\triangle_1$ is such a $\beta\eta$-redex occurrence in t_1 that has exactly one residual in t_2 (w.r.t. σ), then the latter will be denoted by $\overrightarrow{\triangle}_1$. If $\triangle_2$ is a $\beta\eta$-redex occurrence in t_2, then it can be trivially verified that there can be at most one $\beta\eta$-redex occurrence in t_1 for which $\triangle_2$ is a residual of (or belongs to the set of residuals of); in this case, it is denoted by $\overleftarrow{\triangle}_2$, i.e. $\overleftarrow{\triangle}_2$ is a "coresidual" of $\triangle_2$ w.r.t. σ.

η-Postponement Theorem. [1; 2] *Every finite $\beta\eta$-reduction sequence $\sigma\colon t_1 \twoheadrightarrow_{\beta\eta} t_2$ can be reconstructed into a sequence of the form $\sigma'\colon t_1 \twoheadrightarrow_\beta u \twoheadrightarrow_\eta t_2$, for some λ-term u.*

Proof. Let us verify the diagrams from the conditions of the Postponement lemma (with $Q = \longrightarrow_\beta$ and $R = \longrightarrow_\eta$). For a given two-step $\beta\eta$-reduction sequence of the form $t_1 \xrightarrow{\triangle_\eta}_\eta t_2 \xrightarrow{\triangle_\beta}_\beta t_3$, note that the coresidual $\overleftarrow{\triangle}_\beta$ always exists and is always a β-redex occurrence in t_1. Let $\triangle_\eta \equiv \lambda x.wx$, $\triangle_\beta \equiv (\lambda y.p')q'$ and $\overleftarrow{\triangle}_\beta \equiv (\lambda z.p)q$. Consider all the possible cases of mutual locations of the redeces $\triangle_\eta$ and $\overleftarrow{\triangle}_\beta$ in t_1:

<table>
<tr><td colspan="3">1. $\triangle_\eta \cap \overleftarrow{\triangle}_\beta = \varnothing$</td></tr>
<tr><td colspan="3">2. $\triangle_\eta \supset \overleftarrow{\triangle}_\beta$</td></tr>
<tr><td rowspan="3">3. $\triangle_\eta \subset \overleftarrow{\triangle}_\beta$</td><td>3.1. $\triangle_\eta \equiv \lambda z.p \equiv \lambda x.(\lambda y.p')x$</td><td>(hence $z \equiv x$ and $q \equiv q'$)</td></tr>
<tr><td>3.2. $\triangle_\eta \subseteq p$</td><td>(hence $z \equiv y$ and $q \equiv q'$)</td></tr>
<tr><td>3.3. $\triangle_\eta \subseteq q$</td><td>(hence $z \equiv y$ and $p \equiv p'$)</td></tr>
</table>

The cases 1, 2 and 3.2 are trivial: one only needs to reverse the order of contractions (first, to contract $\overleftarrow{\triangle}_\beta$ in t_1 and then, to contract the residual $\overrightarrow{\triangle}_\eta$ of $\triangle_\eta$ in the resulting term), which all lead to a diagram of the form:

$$\begin{tikzcd}
 & t_1 \arrow[dl, dashed, "\overleftarrow{\triangle}_\beta", "\beta"' near end] \arrow[dr, "\triangle_\eta", "\eta"' near end] & \\
\cdot \arrow[dr, dashed, "\overrightarrow{\triangle}_\eta"', "\eta" near end] & & t_2 \arrow[dl, "\triangle_\beta", "\beta"' near end] \\
 & t_3 &
\end{tikzcd}$$

Case 3.1:

$$\begin{tikzcd}
 & \ldots(\lambda x.(\lambda y.p')x)q'\ldots \arrow[dl, dashed, "(\lambda x.(\lambda y.p')x)q'", "\beta"' near end] \arrow[dr, "\lambda x.(\lambda y.p')x", "\eta"' near end] & \\
\ldots(\lambda y.p')q'\ldots \arrow[dr, dashed, "(\lambda y.p')q'"', "\beta" near end] & & \ldots(\lambda y.p')q'\ldots \arrow[dl, "(\lambda y.p')q'", "\beta"' near end] \\
 & \ldots p'[y := q']\ldots &
\end{tikzcd}$$

Case 3.3. First, consider the subcase when $q \equiv \triangle_\eta \equiv \lambda x.wx$:

$$\begin{tikzcd}
 & \ldots(\lambda y.p')(\lambda x.wx)\ldots \arrow[dl, dashed, "(\lambda y.p')(\lambda x.wx)", "\beta"' near end] \arrow[dr, "\lambda x.wx", "\eta"' near end] & \\
\ldots p'[y := \lambda x.wx]\ldots \arrow[dr, dashed, "\eta" near end] & & \ldots(\lambda y.p')w\ldots \arrow[dl, "(\lambda y.p')w", "\beta"' near end] \\
 & \ldots p'[y := w]\ldots &
\end{tikzcd}$$

Finally, in the general conditions of the case 3.3, one has $q \equiv \ldots\lambda x.wx\ldots$, which leads to a diagram similar to the latter. $\square$

4 $\beta\eta$-Normal forms

Note that for a λ-term of the form $(\lambda x.wx)q$, where $x \notin w$, the both contractions $(\lambda x.wx)q \xrightarrow{(\lambda x.wx)q}_\beta wq$ and $(\lambda x.wx)q \xrightarrow{\lambda x.wx}_\eta wq$ lead to the same result wq.

Definition 2. An η-redex occurrence $\triangle_\eta \equiv \lambda x.wx$ in a λ-term t is called *β-replaceable*, if $\triangle_\eta$ is a re-part of some β-redex occurrence in t, i.e. there is a term q such that $(\lambda x.wx)q$ is a β-redex occurrence in t.

Lemma 1. *Given a finite η-reduction sequence $\sigma\colon t \twoheadrightarrow_\eta t'$ in which neither of the contracted η-redeces is β-replaceable. If t' is a β-normal form, then so is t.*

Proof. Obviously, it is sufficient to prove the lemma for the case of a one-step η-reduction sequence $\sigma\colon t \xrightarrow{\triangle_\eta}_\eta t'$. If t is not a β-normal form, then it contains some β-redex occurrence $(\lambda x.p)q$ and hence $t \equiv \ldots(\lambda x.p)q\ldots$. Since $\triangle_\eta$ is not β-replaceable, it follows that $\triangle_\eta \not\equiv \lambda x.p$. Then, evidently, $(\lambda x.p)q$ has a nonempty residual in t' which is a β-redex occurrence. Thus, t' is not a β-normal form. $\square$

$\beta\eta$-Normal Form Theorem. [1; 3] *An arbitrary λ-term t has a $\beta\eta$-normal form $\Leftrightarrow$ t has a β-normal form.*

Proof. The sufficiency is obvious, since if t has a β-normal form t', then any η-redex contraction in t' does not create new β-redeces and decreases the length of t'.

Let us prove the necessity. Suppose a λ-term t has a $\beta\eta$-normal form t'. By the Church-Rosser theorem for $\beta\eta$-reduction ([1; 2]), then there is a $\beta\eta$-reduction sequence $\sigma\colon t \twoheadrightarrow_{\beta\eta} t'$. Moreover, it can be assumed without loss of generality that neither of the η-redeces being contracted in σ is β-replaceable (since any such an η-redex can be replaced in σ with the corresponding β-redex). Furthermore, from the analysis of the diagrams from the proof of the η-Postponement theorem it can be concluded that when postponing η-reduction in σ, no contracted β-replaceable η-redeces can emerge. Therefore, there exists a $\beta\eta$-reduction sequence of the form $\sigma'\colon t \twoheadrightarrow_\beta u \twoheadrightarrow_\eta t'$, for some λ-term u, that does not contract neither of the β-replaceable η-redeces. Lemma 1 finally implies that u is the needed β-normal form of t. $\square$

5 Normalization theorem for $\beta\eta$-reduction

Recall that for a given notion R of reduction and λ-term t, an R-leftmost redex is any such an R-redex occurrence in t, the position (in t) of the first symbol of which cannot be strictly to the right of the position of the first symbol of any other R-redex occurrence. Therefore, this notion is not deterministic in the general case, i.e. for some notion R of reduction, a term t may have two or more distinct leftmost R-redex occurrences. By this reason, the latter notion is sometimes strengthened to the notion of the R-leftmost-outermost redex occurrence which is always unique in every λ-term (if any).

As to the $\beta\eta$-reduction, the notion of a $\beta\eta$-leftmost redex occurrence is not deterministic as well. However this is a small problem, since the only possibility for the ambiguity in this case is when considering terms with subterm occurrences of the form $(\lambda x.px)q$, where $x \notin p$ (having two distinct leftmost $\beta\eta$-redex occurrences: $\lambda x.px$ and $(\lambda x.px)q$, the both contractions of which lead to the same result pq).

By $left_{\beta\eta}$ denote the so-called $\beta\eta$-leftmost strategy which in every λ-term always contracts the $\beta\eta$-leftmost-outermost redex occurrence (if any). For each term t, it determines a certain finite or infinite $\beta\eta$-reduction sequence starting with t which is also called $\beta\eta$-leftmost. Analogously, by $left_{\beta}$ denote the β-leftmost strategy.

We write $t \xrightarrow[\beta\eta\text{-}left]{} t'$ and $t \xrightarrow[\beta\text{-}left]{} t'$ instead of, resp., $t' = left_{\beta\eta}(t)$ and $t' = left_{\beta}(t)$. The notation $t \xrightarrow[\beta\eta\text{-}left]{\triangle}_{\eta} t'$ means that $t \xrightarrow[\beta\eta\text{-}left]{} t'$ and $t \xrightarrow{\triangle}_{\eta} t'$. Therefore, $\xrightarrow[\beta\eta\text{-}left]{}_{\eta}$ can be considered as a binary relation on the set of λ-terms; by $\xrightarrow[\beta\eta\text{-}left,\neq\varnothing]{}\!\!\!\!\!\rightarrow_{\eta}$ denote its transitive closure. The relations $\xrightarrow[\beta\eta\text{-}left]{}_{\beta}$ and $\xrightarrow[\beta\eta\text{-}left,\neq\varnothing]{}\!\!\!\!\!\rightarrow_{\beta}$ are introduced analogously, the same concerns $\xrightarrow[\beta\text{-}left]{}_{\beta}$ and $\xrightarrow[\beta\text{-}left,\neq\varnothing]{}\!\!\!\!\!\rightarrow_{\beta}$.

Lemma 2. *The following diagram holds:*

$$
\begin{tikzcd}
& \cdot \arrow[dl, dashed, "\beta\text{-}left, \neq\varnothing"', "\beta" near end] \arrow[dr, "\beta\eta\text{-}left, \neq\varnothing", "\eta"' near end] & \\
\cdot \arrow[dr, dashed, "\beta\eta\text{-}left, \neq\varnothing"', "\eta" near end] & & \cdot \arrow[dl, "\beta\text{-}left, \neq\varnothing", "\beta"' near end] \\
& \cdot &
\end{tikzcd}
$$

Proof. First consider the case of a two-step reduction sequence of the form $t_1 \xrightarrow[\beta\eta\text{-}left]{\triangle_\eta}_{\eta} t_2 \xrightarrow[\beta\text{-}left]{\triangle_\beta}_{\beta} t_3$. Since $\triangle_\eta$ is the $\beta\eta$-leftmost-outermost redex, it follows that $\triangle_\eta$ is strictly to the left of $\overleftarrow{\triangle}_\beta$ in t_1. Obviously, this is possible only in the cases 1 and 2 from the proof of the η-Postponement theorem, which, as noted there, leads to the following diagram:

$$
\begin{tikzcd}
& t_1 \arrow[dl, dashed, "\beta\text{-}left"', "\overleftarrow{\triangle}_\beta", "\beta"' near end] \arrow[dr, "\beta\eta\text{-}left", "\triangle_\eta"', "\eta" near end] & \\
t_2' \arrow[dr, dashed, "\beta\eta\text{-}left"', "\overrightarrow{\triangle}_\eta", "\eta"' near end] & & t_2 \arrow[dl, "\beta\text{-}left", "\triangle_\beta"', "\beta" near end] \\
& t_3 &
\end{tikzcd}
$$

(at that, evidently, $\overleftarrow{\triangle}_\beta$ is indeed the β-leftmost redex occurrence in t_1 and $\overrightarrow{\triangle}_\eta$ the $\beta\eta$-leftmost-outermost redex occurrence in t_2').

Now the general case can be concluded from the following typical example of a diagram can arise under such conditions (in which all the labels are omitted due to the triviality of its construction):

$$
\begin{tikzcd}
\cdot \arrow[r] \arrow[d, dashed] & \cdot \arrow[r] \arrow[d, dashed] & \cdot \arrow[r] \arrow[d, dashed] & \cdot \arrow[r] \arrow[d, dashed] & \cdot \arrow[d] \\
\cdot \arrow[r, dashed] \arrow[d, dashed] & \cdot \arrow[r, dashed] \arrow[d, dashed] & \cdot \arrow[r, dashed] \arrow[d, dashed] & \cdot \arrow[r, dashed] \arrow[d, dashed] & \cdot \arrow[d] \\
\cdot \arrow[r, dashed] \arrow[d, dashed] & \cdot \arrow[r, dashed] \arrow[d, dashed] & \cdot \arrow[r, dashed] \arrow[d, dashed] & \cdot \arrow[r, dashed] \arrow[d, dashed] & \cdot \arrow[d] \\
\cdot \arrow[r, dashed] & \cdot \arrow[r, dashed] & \cdot \arrow[r, dashed] & \cdot \arrow[r, dashed] & \cdot
\end{tikzcd}
$$

□

Recall that a $\beta\eta$-strategy f is called normalizing if whenever a term t has a $\beta\eta$-normal form, $f^n(t)$ is a $\beta\eta$-normal form for some natural number n.

Normalization Theorem for $\beta\eta$-Reduction. [4; 5] *The strategy* $left_{\beta\eta}$ *is normalizing.*

Proof. Supposing the contrary, there is a term t having a $\beta\eta$-normal form, the $\beta\eta$-leftmost reduction sequence σ of which is infinite. It will be proved, by means of postponing η-reduction with the help of Lemma 2, that σ can be reconstructed into the infinite β-leftmost reduction sequence (starting with t), which leads to a contradiction: indeed, by the Normalization theorem for β-reduction ([1; 3]), then t does not have a β-normal form and by the $\beta\eta$-Normal form theorem, t does not have a $\beta\eta$-normal form as well.

It can be assumed without loss of generality that σ contracts at least one η-redex (otherwise, σ is already β-leftmost) and, moreover, that it contracts the infinite number of η-redeces (otherwise, exclude from σ such an initial segment that the resulting sequence contracts β-redeces only, which sends back to the previous case). On the other hand, σ should contract the infinite number of β-redeces, since the notion of η-reduction is strongly normalizing. In addition to all these conditions, it can also be assumed, for definiteness, that σ starts with an η-redex contraction (otherwise, exclude from σ its β-prefix). Then σ can be represented in a form of the following infinite sequence:

$$\sigma\colon t_0 \xrightarrow[\beta\eta\text{-}left,\neq\varnothing]{}\!\!\!\!\!\rightarrow_\eta u_0 \xrightarrow[\beta\eta\text{-}left,\neq\varnothing]{}\!\!\!\!\!\rightarrow_\beta t_1 \xrightarrow[\beta\eta\text{-}left,\neq\varnothing]{}\!\!\!\!\!\rightarrow_\eta u_1 \xrightarrow[\beta\eta\text{-}left,\neq\varnothing]{}\!\!\!\!\!\rightarrow_\beta \cdots,$$

where $t_0 \equiv t$ (i.e. σ is being divided into alternating η- and β-segments).

Finally, notice that every $\beta\eta$-leftmost reduction sequence that contracts β-redeces only is evidently the β-leftmost reduction sequence as well. Now the following infinite diagram can be constructed with the help of Lemma 2:

$$
\begin{tikzcd}
t_0 \arrow[r, two heads, "\beta\eta\text{-}left,\neq\varnothing"', "\eta"' near end] \arrow[d, dashed, two heads, "\beta\text{-}left,\neq\varnothing"', "\beta"' near end] & u_0 \arrow[d, two heads, "\beta\text{-}left,\neq\varnothing", "\beta" near end] & & & \\
\cdot \arrow[r, dashed, two heads, "\beta\eta\text{-}left,\neq\varnothing"', "\eta"' near end] \arrow[d, dashed, two heads, "\beta\text{-}left,\neq\varnothing"', "\beta"' near end] & t_1 \arrow[r, two heads, "\beta\eta\text{-}left,\neq\varnothing"', "\eta"' near end] & u_1 \arrow[d, two heads, "\beta\text{-}left,\neq\varnothing", "\beta" near end] & & \\
\cdot \arrow[rr, dashed, two heads, "\beta\eta\text{-}left,\neq\varnothing"', "\eta"' near end] \arrow[d, dashed, two heads, "\beta\text{-}left,\neq\varnothing"', "\beta"' near end] & & t_2 \arrow[r, two heads, "\beta\eta\text{-}left,\neq\varnothing"', "\eta"' near end] & u_2 \arrow[d, two heads, "\beta\text{-}left,\neq\varnothing", "\beta" near end] & \\
\cdot \arrow[rrr, dashed, two heads, "\beta\eta\text{-}left,\neq\varnothing"', "\eta"' near end] \arrow[d, dotted, two heads] & & & t_3 \arrow[r, two heads, "\beta\eta\text{-}left,\neq\varnothing"', "\eta"' near end] & u_3 \arrow[d, dotted, two heads] \\
\cdots & & & & \cdots
\end{tikzcd}
$$

Its vertical line determines the β-leftmost reduction sequence starting with $t_0 \equiv t$. Thus, the latter is indeed infinite, just as expected. $\square$

Note that this proof is rather close to that from [5]. (Both the proofs are based on the idea to reduce the Normalization theorem for $\beta\eta$-reduction to its analogue for β-reduction by means of postponing η-reduction and taking into account that η-reduction is strongly normalizing, both of them are proceeded by contradiction and exploit Lemma 2, however in the other proof the contradiction is obtained in a different way.)

An interested reader is invited to compare the constructed proofs with their original or known analogues. The detailed references to the latter are contained in the following table compiled for his convenience:

η-Postponement theorem	[1, pp. 384 – 386] [2, pp. 132 – 135]
$\beta\eta$-Normal form theorem	[1, pp. 384 – 386] [3, pp. 313 – 314]
Normalization theorem for $\beta\eta$-reduction	[4, pp. 279 – 290] [5, pp. 529 – 537]

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