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Semantic Properties of T-consequence Relation in Logics of Quasiary Predicates*

Mykola Nikitchenko, Stepan Shkilniak

Abstract

In the paper we investigate semantic properties of program-oriented algebras and logics defined for classes of quasiary predicates. Informally speaking, such predicates are partial predicates defined over partial states (partial assignments) of variables. Conventional n-ary predicates can be considered as a special case of quasiary predicates. We define first-order logics of quasiary non-deterministic predicates and investigate semantic properties of T-consequence relation for such logics. Specific properties of T-consequence relation for the class of deterministic predicates are also considered. Obtained results can be used to prove logic validity and completeness.

Keywords: First-order logic, quasiary predicate, partial predicate, non-deterministic predicate.

1 Introduction

Mathematical logic is one of the basic disciplines for computer science. To use effectively mathematical logic it is important to construct logical systems that are adequate for problems considered in computer science. Classical logic, despite its numerous advantages, has some restrictions for its use in this area. For example, classical logic is based on the class of total n-ary predicates, while in computer science partial and non-deterministic predicates often appear. Therefore many logical

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systems which better reflect properties of such kind were constructed [1, 2]. One of specific features for computer science is quasiarity of predicates. Such predicates are partial predicates defined over partial states (partial assignments) of variables. Conventional n-ary predicates can be considered as a special case of quasiary predicates. In our previous works [3, 4, 5] we investigated the class of partial deterministic (single-valued) predicates and constructed corresponding logics. For such logic a natural extension of conventional logical consequence relation, called irrefutability relation, was used.

This paper aims to develop a semantic basis for construction of logics of non-deterministic (many-valued) quasiary predicates. To realize this idea we first construct predicate algebras using *composition-nominative approach* [6]. Terms of such algebras specify the language of logic. Then we define interpretation mappings. At last, we construct calculi of sequent type for defined logics. It is important to admit that constructed logics better reflect specifics of computer science problems, but the opposite side of this feature is that the methods of logic investigation turn out to be more complicated. In particular, the irrefutability relation collapses for the class of non-deterministic predicates. Therefore in this paper which is an extended version of [7] we concentrate on semantic properties of a special T-consequence relation. We also formulate properties of this relation for logics of deterministic predicates.

The rest of the paper is structured as follows. In Section 2 we define first-order algebras of quasiary predicates. In Sections 3 we define logics of quasiary predicates. Section 4 is devoted to semantic properties of such logics. In Section 5 T-consequence relation is specified and its main properties are studied. Section 6 is devoted to properties of T-consequence relation for the class of deterministic predicates. In Section 7 conclusions are formulated.

Arrows $\xrightarrow{t}$, $\xrightarrow{p}$, and $\xrightarrow{r}$ specify total, partial, and relational mappings respectively. Notations not defined in this paper are understood in a sense of [4].

2 First-order algebras of quasiary predicates

Let V be a nonempty *set of names*. According to tradition, names from V are also called *variables*. Let A be a set of basic values ($A \neq \emptyset$). Given V and A, the class ${}^{V}\!A$ of *nominative sets* is defined as the class of all partial mappings from V to A, thus, ${}^{V}\!A = V \xrightarrow{p} A$. Informally speaking, nominative sets represent states of variables.

Though nominative sets are defined as mappings, we follow mathematical tradition and also use set-like notation for these objects. In particular, the notation $d = [v_i \mapsto a_i \mid i \in I]$ describes a nominative set d; the notation $v_i \mapsto a_i \in_n d$ means that $d(v_i)$ is defined and its value is a_i ($d(v_i) \downarrow= a_i$). The main operation for nominative sets is a total unary parametric *renomination* $r^{v_1,\ldots,v_n}_{x_1,\ldots,x_n} : {}^{V}\!A \xrightarrow{t} {}^{V}\!A$, where $v_1, ..., v_n, x_1, ..., x_n \in V$, $v_1, ..., v_n$ are distinct names, $n \geq 0$, which is defined by the following formula:

$$r^{v_1,\ldots,v_n}_{x_1,\ldots,x_n}(d) =$$
$$= [v \mapsto a \in_n d \mid v \notin \{v_1, ..., v_n\}] \cup [v_i \mapsto d(x_i) \mid d(x_i) \downarrow, i \in \{1, ..., n\}].$$

Intuitively, given d this operation yields a new nominative set changing the values of $v_1, ..., v_n$ to the values of $x_1, ..., x_n$ respectively. We also use simpler notation for this formula: $r^{\bar{v}}_{\bar{x}}(d) = d\nabla\bar{v} \mapsto d(\bar{x})$. Also note that we treat a parameter ${}^{v_1,\ldots,v_n}_{x_1,\ldots,x_n}$ as a total mapping from $\{v_1, ..., v_n\}$ into $\{x_1, ..., x_n\}$ thus parameters obtained by pairs permutations are identical.

Operation of deleting a component with a name v from a nominative set d is denoted $d|_{-v}$. Notation $d =_{-v} d'$ means that $d|_{-v} = d'|_{-v}$. The set of *assigned names* (*variables*) in d is defined by the formula

$$asn(d) = \{v \in V \mid v \mapsto a \in_n d \text{ for some } a \in A\}.$$

Let $Bool = \{F, T\}$ be a set of Boolean values.

Let $PrR^V_A = {}^{V}\!A \xrightarrow{r} Bool$ be the set of all non-deterministic (relational) predicates over ${}^{V}\!A$. Such predicates are called *non-deterministic (relational) quasiary predicates*. The term 'relational' means that graphs of such predicates are binary relations from ${}^{V}\!A \times Bool$. Note that non-determinism in logic was intensively studied, see, for example, [8].

We will also use set-theoretic notations for quasiary predicates.

Full image of $d \in {}^V\!A$ under $p \in PrR_A^V$ is defined by the formula $p[d] = \{b \in Bool \mid (d, b) \in p\}$.

For $p \in PrR_A^V$ the truth and falsity domains of p are respectively

$$T(p) = \{d \in {}^V\!A \mid (d, T) \in p\} \text{ and } F(p) = \{d \in {}^V\!A \mid (d, F) \in p\}.$$

Considering predicates from PrR_A^V in set-theoretic style we can speak about such operations as union $\cup$ and intersection $\cap$. The following statement is obvious.

Lemma 1. *The set* $< PrR_A^V; \cup, \cap >$ *is a complete distributive lattice.*

The greatest and the least elements of this lattice are denoted $\top_A^V$ and $\perp_A^V$ respectively. For these elements $T(\top_A^V) = {}^V\!A$, $F(\top_A^V) = {}^V\!A$, $T(\perp_A^V) = \emptyset$, $F(\perp_A^V) = \emptyset$.

Operations over PrR_A^V are called *compositions*. The set $C(V)$ of first-order compositions is $\{\vee, \neg, R_{\bar{x}}^{\bar{v}}, \exists x\}$. Compositions have the following types:

$\vee : PrR_A^V \times PrR_A^V \xrightarrow{t} PrR_A^V$; $\neg, R_{x_1,\ldots,x_n}^{v_1,\ldots,v_n}, \exists x : PrR_A^V \xrightarrow{t} PrR_A^V$ and are defined by the following formulas ($p, q \in PrR_A^V$):

- $T(p \vee q) = T(p) \cup T(q)$; $F(p \vee q) = F(p) \cap F(q)$;
- $T(\neg p) = F(p)$; $F(\neg p) = T(p)$;
- $T(R_{\bar{x}}^{\bar{v}}(p)) = \{d \in {}^V\!A \mid r_{\bar{x}}^{\bar{v}}(d) \in T(p)\}$;
  $F(R_{\bar{x}}^{\bar{v}}(p)) = \{d \in {}^V\!A \mid r_{\bar{x}}^{\bar{v}}(d) \in F(p)\}$;
- $T(\exists xp) = \{d \in {}^V\!A \mid d\nabla x \mapsto a \in T(p) \text{ for some } a \in A\}$;
  $F(\exists xp) = \{d \in {}^V\!A \mid d\nabla x \mapsto a \in F(p) \text{ for all } a \in A\}$.

Here $d\nabla x \mapsto a = [v \mapsto c \in_n d \mid v \neq x] \cup [x \mapsto a]$. Conventional notation is $d[v \mapsto a]$.

Please note that definitions of compositions are similar to strong Kleene's connectives and quantifiers.

Also note that parametric compositions of existential quantification and renomination can also represent classes of compositions. Thus, notation $\exists x$ can represent one composition, when x is fixed, or a class $\{\exists x \mid x \in V\}$ of such compositions for various names.

A pair $AQR(V, A) =< PrR_A^V; C(V) >$ is called *a first-order algebra of non-deterministic quasiary predicates.*

It is not difficult to prove the following statement.

Lemma 2. *Singleton sets* $\{\top_A^V\}$ *and* $\{\perp_A^V\}$ *are sub-algebras of algebra* $AQR(V, A)$.

Algebras $AQR(V, A)$ (for various A) form a semantic base for the constructed first-order pure quasiary predicate logic L^{QR} (called also quasiary logic). Let us now proceed with formal definitions.

3 First-order pure quasiary logic

To define a logic we should first specify its semantic component, syntactic component, and interpretational component [3, 4, 5]. Then a consequence relation should be defined. Semantics of the logic under consideration is specified by algebras of the type $AQR(V, A)$ (for various A), so, we proceed with syntactic component of the logic.

3.1 Syntactic component

A syntactic component specifies the language of L^{QR}. Let $Cs(V)$ be *a set of composition symbols* that represent compositions in algebras defined above – $Cs(V) = \{\vee, \neg, R_{\bar{x}}^{\bar{v}}, \exists x\}$. For simplicity, we use the same notation for symbols of compositions and compositions themselves.

Let Ps be a set of *predicate symbols.* A triple $\Sigma^Q = (V, Cs(V), Ps)$ is a language *signature.* Given Σ^Q , we inductively define the language of L^{QR} – *the set of formulas* $Fr(\Sigma^Q)$:

1) if $P \in Ps$, then $P \in Fr(\Sigma^Q)$; such formulas are called atomic;

2) if Φ, $\Psi \in Fr(\Sigma^Q)$, then $(\Phi \vee \Psi) \in Fr(\Sigma^Q)$;

3) if $\Phi \in Fr(\Sigma^Q)$, then $(\neg\Phi) \in Fr(\Sigma^Q)$;

4) if $\Phi \in Fr(\Sigma^Q)$, $v_1, ..., v_n$, $x_1, ..., x_n \in V$, $v_1, ..., v_n$ are distinct names, $n \geq 0$, then $(R^{v_1,...,v_n}_{x_1,...,x_n}(\Phi)) \in Fr(\Sigma^Q)$;
for such formulas notation $R^{v_1,...,v_n}_{x_1,...,x_n}\Phi$ or $R^{\bar{v}}_{\bar{x}}\Phi$ can be also used;

5) if $\Phi \in Fr(\Sigma^Q)$, $x \in V$, then $(\exists x\Phi) \in Fr(\Sigma^Q)$.

Extra brackets can be omitted using conventional rules of operation priorities. Derived operations like conjunction $\wedge$, implication $\rightarrow$ etc. are defined in a usual way.

3.2 Interpretational component

Given Σ^Q and nonempty set A we can consider an algebra of quasiary predicates $AQR(V, A) =< PrR^V_A; C(V) >$. Composition symbols have fixed interpretation, but we additionally need interpretation $I^{Ps} : Ps \xrightarrow{t} PrR^V_A$ of predicate symbols; obtained predicates are called *basic predicates*. A tuple $J = (\Sigma^Q, A, I^{Ps})$ is called *an interpretation*.

Formulas and interpretations in L^{QR} are called L^{QR}*-formulas* and L^{QR}*-interpretations* respectively. Usually the prefix L^{QR} is omitted. Given a formula Φ and an interpretation J we can speak of an *interpretation of* Φ in J. It is denoted by Φ_J.

3.3 Extensions of L^{QR}

The logic L^{QR} being a rather powerful logic still is not expressive enough to represent transformations required for proving its completeness. Therefore we introduce its two extensions: L^{UR} — *a logic with unessential variables*, and L^{UR}_{ε} — a logic with unessential variables and a parametric total deterministic *variable unassignment predicate* εz which checks if a variable z is unassigned in a given nominative set.

To define L^{UR} we should specify its semantic, syntactic, and interpretational components.

Let U be an infinite set of variables such that $V \cap U = \emptyset$. Variables from U are called *unessential variables* (analogs of fresh variables in classical logic) that should not affect the formula meanings.

Algebras

$$AQR(V \cup U, A) =< Pr_A^{V \cup U}; C(V \cup U) >$$

(for different A) form a semantic base for L^{UR}.

A syntactic component is specified by the set of formulas $Fr(\Sigma^U)$, where $\Sigma^U = (V \cup U, Cs(V \cup U), Ps)$ is the *signature* of L^{UR}.

An interpretational component restricts the class of L^{UR}-interpretations in such a way that interpretations of predicate symbols are neither sensitive to the values of the component with an unessential variable u in nominative sets, nor to presence of such components. Formally, a variable $u \in U$ is *unessential in an interpretation of predicate symbols* I^{Ps} if $I^{Ps}(P)[d] = I^{Ps}(P)[d']$ for all $P \in Ps$, $d, d' \in {}^{V \cup U}A$ such that $d =_{-u} d'$.

The following statement is obvious.

Lemma 3. *L^{UR} is a model-theoretic conservative extension of L^{QR}.*

Note that given $p \in PrR_A^{V \cup U}$ and $v \in V \cup U$ we say that v is unessential for p if $p[d] = p[d']$ for any $d, d' \in {}^{V \cup U}A$ such that $d =_{-v} d'$.

The next logic L_ε^{UR} is an extension of L^{UR} by a null-ary parametric composition (predicate) εz ($z \in V \cup U$) defined in interpretation J by the following formulas:

$$T(\varepsilon z_J) = \{d \in {}^{V \cup U}A \mid z \notin asn(d)\},$$
$$F(\varepsilon z_J) = \{d \in {}^{V \cup U}A \mid z \in asn(d)\}.$$

Thus, for this logic the set of compositions is equal to $\{\vee, \neg, R_{\bar{x}}^{\bar{v}}, \exists x, \varepsilon z\}$.

Note that in free logic [9] $E!z$ corresponds to negation of εz.

Algebras of the form

$$ARE(V \cup U, A) =< Pr_A^{V \cup U}; \vee, \neg, R_{\bar{x}}^{\bar{v}}, \exists x, \varepsilon z >$$

(for different A) constitute a semantic base for L_ε^{UR}.

A syntactic component is specified by the set of formulas $Fr(\Sigma_\varepsilon^U)$, where $\Sigma_\varepsilon^U = (V \cup U, \{\vee, \neg, R_{\bar{x}}^{\bar{v}}, \exists x, \varepsilon z\}, Ps)$ is the signature of L_ε^{UR}.

An interpretational component of L_ε^{UR} is defined in the same way as for L^{UR}.

By construction of L_ε^{UR} we get the following statement.

Lemma 4. *L_ε^{UR} is a model-theoretic conservative extension of L^{UR}.*

Predicates εz specify cases when z is assigned or unassigned. This property can be used for construction of sequent rules for quantifiers.

For a formula Φ and a set of formulas Γ let $nm(\Phi)$ denote all names (variables) that occur in Φ, $nm(\Gamma)$ denote all names that occur in formulas of Γ. Names from $U\backslash nm(\Phi)$ are called fresh unessential variables for Φ and their set is denoted $fu(\Phi)$, in the same way $fu(\Gamma) = U\backslash nm(\Gamma)$ is the set of fresh unessential variables for Γ. We also use natural extensions of this notation for a case of several formulas and sets of formulas like $nm(\Gamma, \Delta, R_{\bar{v}}^{\bar{u}}(\exists x\Phi))$ and $fu(\Gamma, \Delta, R_{\bar{v}}^{\bar{u}}(\exists x\Phi))$. Such notation is also used when we consider properties of predicate algebras. We write $x \in \bar{v}$ to denote that x is a variable from $\bar{v}$. We write $\{\bar{v}, \bar{x}\}$ to denote the set of variables that occur in the sequences $\bar{v}$ and $\bar{x}$.

In the sequel we adopt the following convention: a, b denote elements from A; x, y, z, v, w (maybe with indexes) denote variables (names) from $V \cup U$; d, d', d_1, d_2 denote nominative sets from ${}^{V\cup U}A$; p, q denote predicates from $ARE(V \cup U, A)$; Φ, Ψ, Ξ denote L_{ε}^{UR}-formulas, Γ, Δ denote sets of L_{ε}^{UR}-formulas, J denotes L_{ε}^{UR}-interpretation.

4 Semantic properties of quasiary logics

The set of compositions $\{\vee, \neg, R_{\bar{x}}^{\bar{v}}, \exists x, \varepsilon z\}$ of quasiary logics specifies four types of properties related to propositional compositions $\vee$ and $\neg$, to renomination composition $R_{\bar{x}}^{\bar{v}}$, to unassignment composition (predicate) εz, and to existential quantifier $\exists x$.

4.1 Properties related to propositional compositions

Properties of propositional compositions are traditional. In particular, disjunction composition is associative, commutative, and idempotent; negation composition is involutive

$$\neg\neg\text{: } \neg\neg p = p.$$

4.2 Properties related to renomination composition

Renomination composition is a new composition specific for logics of quasiary predicates. Its properties are not well-known therefore we describe them in more detail. The main attention will be paid to distributivity properties.

Lemma 5. *For every algebra $ARE(V\cup U, A)$ the following properties related to renomination composition hold:*

$R\vee$: $R_{\bar{x}}^{\bar{v}}(p\vee q) = R_{\bar{x}}^{\bar{v}}(p)\vee R_{\bar{x}}^{\bar{v}}(q)$;
$R\neg$: $R_{\bar{x}}^{\bar{v}}(\neg p) = \neg R_{\bar{x}}^{\bar{v}}(p)$;
RI: $R_{z,\bar{x}}^{z,\bar{v}}(p) = R_{\bar{x}}^{\bar{v}}(p)$;
RU: $R_{z,\bar{x}}^{y,\bar{v}}(p) = R_{\bar{x}}^{\bar{v}}(p)$, y is unessential for $R_{\bar{x}}^{\bar{v}}(p)$;
RR: $R_{\bar{x}}^{\bar{v}}(R_{\bar{y}}^{\bar{w}}(p)) = R_{\bar{x}}^{\bar{v}}\circ_{\bar{y}}^{\bar{w}}(p)$;
R: $R(p) = p$;
$R\exists s$: $R_{\bar{x}}^{\bar{v}}(\exists yp) = \exists y(R_{\bar{x}}^{\bar{v}}(p))$, $y\notin\{\bar{v},\bar{x}\}$;
$R\exists r$: $\exists yp = \exists zR_z^y(p)$, z is unessential for p;
$R\exists$: $R_{\bar{x}}^{\bar{v}}(\exists yp) = \exists zR_{\bar{x}}^{\bar{v}}(R_z^y(p))$, z is unessential for $R_{\bar{x}}^{\bar{v}}(\exists yp)$;
$R\exists R$: $R_{z,\bar{x}}^{y,\bar{v}}(\exists yp) = R_{\bar{x}}^{\bar{v}}(\exists yp)$.

Here $R_{\bar{x}}^{\bar{v}}\circ_{\bar{y}}^{\bar{w}}$ represents two successive renominations $R_{\bar{y}}^{\bar{w}}$ and $R_{\bar{x}}^{\bar{v}}$.

Proof. We prove the lemma by showing that truth and falsity domains of predicates in the left- and right-hand sides of equalities coincide. Let us consider properties R$\exists s$, R$\exists r$, and R$\exists$ only.

For R$\exists s$ we have:

$d\in T(R_{\bar{x}}^{\bar{v}}(\exists yp)) \Leftrightarrow r_{\bar{x}}^{\bar{v}}(d)\in T(\exists yp) \Leftrightarrow r_{\bar{x}}^{\bar{v}}(d)\nabla y\mapsto a\in T(p)$ for some $a\in A \Leftrightarrow$ (since $y\notin\{\bar{v},\bar{x}\}$) $r_{\bar{x}}^{\bar{v}}(d\nabla y\mapsto a)\in T(p)$ for some $a\in A$ $\Leftrightarrow d\nabla y\mapsto a\in T(R_{\bar{x}}^{\bar{v}}(p))$ for some $a\in A \Leftrightarrow d\in T(\exists y(R_{\bar{x}}^{\bar{v}}(p)))$;

$d\in F(R_{\bar{x}}^{\bar{v}}(\exists yp)) \Leftrightarrow r_{\bar{x}}^{\bar{v}}(d)\in F(\exists yp) \Leftrightarrow r_{\bar{x}}^{\bar{v}}(d)\nabla y\mapsto a\in F(p)$ for all $a\in A \Leftrightarrow$ (since $y\notin\{\bar{v},\bar{x}\}$) $r_{\bar{x}}^{\bar{v}}(d\nabla y\mapsto a)\in F(p)$ for all $a\in A$ $\Leftrightarrow d\nabla y\mapsto a\in F(R_{\bar{x}}^{\bar{v}}(p))$ for all $a\in A \Leftrightarrow d\in F(\exists y(R_{\bar{x}}^{\bar{v}}(p)))$.

For R$\exists r$ we have:

$d\in T(\exists z(R_z^y(p))) \Leftrightarrow d\nabla z\mapsto a\in T(R_z^y(p))$ for some $a\in A \Leftrightarrow$ $r_z^y(d\nabla z\mapsto a)\in T(p)$ for some $a\in A \Leftrightarrow (d\nabla z\mapsto a)\nabla y\mapsto a\in T(p)$ for some $a\in A \Leftrightarrow$ (since z is unessential for p) $d\nabla y\mapsto a\in T(p)$ for some $a\in A \Leftrightarrow d\in T(\exists yp)$.

In the same way we demonstrate coincidence of the falsity domains for R$\exists r$.

By R$\exists s$ and R$\exists r$ we obtain R$\exists$. □

4.3 Properties related to unassignment composition

Here we formulate only that null-ary unassignment composition (predicate) is total deterministic predicate, i.e.

$$T(\varepsilon y) \cup F(\varepsilon y) = {}^{V}\!A \text{ and } T(\varepsilon y) \cap F(\varepsilon y) = \emptyset.$$

4.4 Properties related to quantifier composition

The following lemmas describe properties of quantifiers.

Lemma 6. *For every algebra $ARE(V \cup U, A)$ and every $p \in PrR_A^{V \cup U}$ the following properties hold ($x \neq y$):*

$$T\exists v : T(R_y^x(p)) \cap F(\varepsilon y) \subseteq T(\exists xp);$$
$$F\exists v : F(\exists xp) \cap F(\varepsilon y) \subseteq F(R_y^x(p));$$
$$T\exists u : T(R_y^x(p)) \subseteq T(\varepsilon y) \cup T(\exists xp);$$
$$F\exists u : F(\exists xp) \subseteq T(\varepsilon y) \cup F(R_y^x(p)).$$

Proof. To prove $T\exists v$ consider arbitrary $d \in T(R_y^x(p)) \cap F(\varepsilon y)$. This means that y is assigned in d with some value a and $d\nabla x \mapsto a \in T(p)$, therefore $d \in T(\exists xp)$.

Property $F\exists v$ is proved in the same manner.

Properties $T\exists u$ and $F\exists u$ are obtained from $T\exists v$ and $F\exists v$ using the following property of Boolean algebra of sets:

$$\text{SI: } S_1 \cap S_2 \subseteq S_3 \Leftrightarrow S_1 \subseteq \overline{S_2} \cup S_3,$$

where $\overline{S_2}$ denotes supplement of S_2 and the properties that $\overline{T(\varepsilon y)} = F(\varepsilon y)$ and $\overline{F(\varepsilon y)} = T(\varepsilon y)$. □

Lemma 7. *For every algebra $ARE(V \cup U, A)$ the following property holds ($x \neq y$):*

$\exists e_L$: $T(\exists xp) =_{-y} (T(R_y^x(p)) \cap F(\varepsilon y))$ if *y is unessential for p.*

Proof. Let $d \in T(\exists xp)|_{-y}$. It means that there exists $d' \in {}^{V \cup U}\!A$ and $a \in A$ such that $d'\nabla x \mapsto a \in T(p)$ and $d' =_{-y} d$. Since y is not essential for p, then $(d'\nabla x \mapsto a)\nabla y \mapsto a \in T(p)$. By definition, we get that $d'\nabla y \mapsto a \in T(R_y^x(p)) \cap F(\varepsilon y)$. But $d = d'|_{-y}$. Thus,

$$T(\exists xp)|_{-y} \subseteq (T(R_y^x(p)) \cap F(\varepsilon y))|_{-y}.$$

The inverse follows from $T\exists v$. □

5 T-consequence relation for sets of formulas

Traditionally, for logics of quasiary predicates a conventional logical consequence is considered [3, 4].

Let $\Gamma \subseteq Fr(\Sigma_\varepsilon^U)$ and $\Delta \subseteq Fr(\Sigma_\varepsilon^U)$ be sets of formulas. Δ is a *consequence* of Γ in an interpretation J (denoted $\Gamma_J \models \Delta$), if $\bigcap_{\Phi\in\Gamma} T(\Phi_J) \cap \bigcap_{\Psi\in\Delta} F(\Psi_J) = \emptyset$.

Δ is a logical consequence of Γ (denoted $\Gamma \models \Delta$), if $\Gamma_J \models \Delta$ in every interpretation J. The introduced relation of logical consequence specifies irrefutability.

For the class of non-deterministic predicates the logical consequence relation collapses, i.e. it is empty. Indeed, for any Γ and Δ we have that $\Gamma_J \not\models \Delta$ if we in J interpret predicate symbols as non-deterministic predicate $\top_A^V$ (Lemma 2).

Therefore we introduce another consequence relation which arises naturally in Computer Science [10].

Δ is a *T-consequence* of Γ in an interpretation J (denoted by $\Gamma_J \models_T \Delta$), if $\bigcap_{\Phi\in\Gamma} T(\Phi_J) \subseteq \bigcup_{\Psi\in\Delta} T(\Psi_J)$. Δ is a T-consequence of Γ (denoted by $\Gamma \models_T \Delta$), if $\Gamma_J \models_T \Delta$ in every interpretation J.

We will also use the following notation: $T^\wedge(\Gamma_J) = \bigcap_{\Phi\in\Gamma} T(\Phi_J)$ and $T^\vee(\Gamma_J) = \bigcup_{\Phi\in\Gamma} T(\Phi_J)$.

Now we describe the main properties of T-consequence relation.

First, let us give the following definitions for arbitrary consequence relation $\models_*$ [11]:

- $\models_*$ is called *paraconsistent* if there exist Γ, Δ, and Φ, such that $\Gamma, \Phi \wedge \neg\Phi \not\models_* \Delta$;

- $\models_*$ is called *paracomplete* if there exist Γ, Δ, and Ψ such that $\Gamma \not\models_* \Psi \vee \neg\Psi, \Delta$;
- $\models_*$ is called *paranormal* if there exist Γ, Δ, Φ, and Ψ such that $\Gamma, \Phi \wedge \neg\Phi \not\models_* \Psi \vee \neg\Psi, \Delta$.

We say that $\models_*$ is consistent, complete, and normal if it is not paraconsistent, not paracomplete, and not paranormal respectively.

It is easy to see that paranormality implies paraconsistency and paracompleteness; consistency or completeness implies normality.

Theorem 1. *T-consequence relation is paraconsistent, paracomplete, and paranormal.*

Proof. To prove the theorem it is sufficient do demonstrate only paranormality of T-consequence relation. Indeed, let $\Gamma = \emptyset$, $\Delta = \emptyset$, Φ be $P_1 \in Ps$, Ψ be $P_2 \in Ps$ such that $P_1 \neq P_2$. Then it is easy to check that interpreting P_1 as predicate $\top_A^V$ and P_2 as predicate $\bot_A^V$ we get that $P_1 \wedge \neg P_1 \not\models_T P_2 \vee \neg P_2$. □

In a similar way we can prove the following statement.

Lemma 8. *For T-consequence relation the following properties hold:*

- $\Gamma, \neg\Phi \models_T \Delta$ *and* $\Gamma \not\models_T \Phi, \Delta$ *for some* Γ, Δ, *and* Φ*;*
- $\Gamma, \Phi \models_T \Delta$ *and* $\Gamma \not\models_T \neg\Phi, \Delta$ *for some* Γ, Δ, *and* Φ*;*
- $\Gamma \models_T \neg\Phi, \Delta$ *and* $\Gamma, \Phi \not\models_T \Delta$ *for some* Γ, Δ, *and* Φ*;*
- $\Gamma \models_T \Phi, \Delta$ *and* $\Gamma, \neg\Phi \not\models_T \Delta$ *for some* Γ, Δ, *and* Φ.

This lemma states that rules of sequent calculi permitting moving (negated) formulas from one side of a sequent to its another side are not valid for T-consequence relations. Consequently, sequent calculi for $\models_T$ will be more complicated.

Still, such transformations are possible for a formula interpreted as total deterministic predicate (see Theorem 2(4)).

Lemma 9. *Let Φ be a formula, $\Gamma, \Gamma', \Delta, \Delta'$ be sets of formulas. Then*

(M) *if* $\Gamma \subseteq \Gamma'$ *and* $\Delta \subseteq \Delta'$*, then* $\Gamma \models_T \Delta \Rightarrow \Gamma' \models_T \Delta'$*;*

(C) $\Phi, \Gamma \models_T \Delta, \Phi$*.*

Proof of the lemma follows immediately from definitions.

Now we continue with those properties of T-consequence relation which induce sequent rules for the logic under consideration. Such properties are constructed upon semantic properties of compositions. To do this the following lemma is often used.

Theorem 2. *Let* Φ, Ψ, *and* Ξ *be formulas,* Γ *and* Δ *be sets of formulas,* J *be* L_ε^{UR}*-interpretation. Then*

(1) *if* $T(\Phi_J) = T(\Psi_J)$, *then*
$$\Phi, \Gamma\ {}_J{\models_T}\ \Delta \Leftrightarrow \Psi, \Gamma\ {}_J{\models_T}\ \Delta \text{ and } \Gamma\ {}_J{\models_T}\ \Phi, \Delta \Leftrightarrow \Gamma\ {}_J{\models_T}\ \Psi, \Delta;$$

(2) *if* $T(\Phi_J) = T(\Psi_J) \cap T(\Xi_J)$, *then*
$$\Phi, \Gamma\ {}_J{\models_T}\ \Delta \Leftrightarrow \Psi, \Xi, \Gamma\ {}_J{\models_T}\ \Delta,$$
$$\Gamma\ {}_J{\models_T}\ \Phi, \Delta \Leftrightarrow (\Gamma\ {}_J{\models_T}\ \Psi, \Delta \text{ and } \Gamma\ {}_J{\models_T}\ \Xi, \Delta);$$

(3) *if* $T(\Phi_J) = T(\Psi_J) \cup T(\Xi_J)$, *then*
$$\Gamma\ {}_J{\models_T}\ \Phi, \Delta \Leftrightarrow \Gamma\ {}_J{\models_T}\ \Psi, \Xi, \Delta,$$
$$\Phi, \Gamma\ {}_J{\models_T}\ \Delta \Leftrightarrow (\Psi, \Gamma\ {}_J{\models_T}\ \Delta \text{ and } \Xi, \Gamma\ {}_J{\models_T}\ \Delta);$$

(4) *if* $T(\Phi_J) \cup F(\Phi_J) = {}^V A$ *and* $T(\Phi_J) \cap F(\Phi_J) = \emptyset$, *then*
$$\Phi, \Gamma\ {}_J{\models_T}\ \Delta \Leftrightarrow \Gamma\ {}_J{\models_T}\ \neg\Phi, \Delta \text{ and} \neg\Phi, \Gamma\ {}_J{\models_T}\ \Delta \Leftrightarrow \Gamma\ {}_J{\models_T}\ \Phi, \Delta;$$
$$\Gamma \models_T \Delta \Leftrightarrow (\Phi, \Gamma \models_T \Delta \text{ and } \Gamma \models_T \Delta, \Phi);$$

(5) *if* $y \in fu(\Gamma, \Delta)$, *then* $\Gamma\ {}_J{\models_T}\ \Delta \Leftrightarrow \Gamma\ {}_J{\models_T}\ \Delta, \varepsilon y$*;*

(6) *if* $T(\Phi_J) =_{-y} T(\Psi_J)$ *for* $y \in fu(\Psi, \Gamma, \Delta)$, *then*
$$\Phi, \Gamma\ {}_J{\models_T}\ \Delta \Leftrightarrow \Psi, \Gamma\ {}_J{\models_T}\ \Delta.$$

Proof. Property (1) is obvious. For (2) we have
$\Phi, \Gamma\ {}_J{\models_T}\ \Delta \Leftrightarrow T(\Phi_J) \cap T^\wedge(\Gamma_J) \subseteq T^\vee(\Delta_J) \Leftrightarrow$
$\Leftrightarrow T(\Psi_J) \cap T(\Xi_J) \cap T^\wedge(\Gamma_J) \subseteq T^\vee(\Delta_J) \Leftrightarrow \Psi, \Xi, \Gamma\ {}_J{\models_T}\ \Delta$.

In the same way the second part of (2) and property (3) are proved. Let us consider (4). We have
$\Phi, \Gamma\ {}_J{\models_T}\ \Delta \Leftrightarrow T(\Phi_J) \cap T^\wedge(\Gamma_J) \subseteq T^\vee(\Delta_J) \Leftrightarrow$

$\Leftrightarrow T^{\wedge}(\Gamma_J) \subseteq \overline{T(\Phi_J)} \cup T^{\vee}(\Delta_J) \Leftrightarrow T^{\wedge}(\Gamma_J) \subseteq T(\neg\Phi_J) \cup T^{\vee}(\Delta_J) \Leftrightarrow$
$\Leftrightarrow \Gamma\,_J{\models_T} \neg\Phi, \Delta$.

In the same way other properties of (4) are proved.

Let us consider (5). By Lemma 9(M) we have that $\Gamma\,_J{\models_T}\ \Delta \Rightarrow \Gamma\,_J{\models_T}\ \Delta, \varepsilon y$. We need to prove that $\Gamma\,_J{\models_T}\ \Delta, \varepsilon y \Rightarrow \Gamma\,_J{\models_T}\ \Delta$. It is equivalent to
$T^{\wedge}(\Gamma_J) \subseteq T^{\vee}(\Delta_J) \cup T(\varepsilon y) \Leftrightarrow T^{\wedge}(\Gamma_J) \cap F(\varepsilon y) \subseteq T^{\vee}(\Delta_J)$.

Let $d \in T^{\wedge}(\Gamma_J) \cap F(\varepsilon y)$. Since y is unessential for Γ, it means that $d|_{-y} \in T^{\wedge}(\Gamma_J)$. From this follows that $d|_{-y} \in T^{\vee}(\Delta_J)$. Since y is unessential for Δ, it means that $d \in T^{\vee}(\Delta_J)$. Thus, $T^{\wedge}(\Gamma_J) \subseteq T^{\vee}(\Delta_J)$ that proves the property under consideration.

Let us consider (6). We should prove that

$$T(\Phi_J) \cap T^{\wedge}(\Gamma_J) \subseteq T^{\vee}(\Delta_J) \Leftrightarrow T(\Psi_J) \cap T^{\wedge}(\Gamma_J) \subseteq T^{\vee}(\Delta_J).$$

Let $d \in T(\Psi_J) \cap T^{\wedge}(\Gamma_J)$. Since $T(\Phi_J) =_{-y} T(\Psi_J)$, there exists $d' \in T(\Phi_J)$ such that $d' =_{-y} d$. Since y is unessential for Γ, we have that $d' \in T^{\wedge}(\Gamma_J)$. Hence $d' \in T^{\vee}(\Delta_J)$. Again, y is also unessential for Δ therefore $d \in T^{\vee}(\Delta_J)$. This proves the direct implication.

Let us prove the inverse implication. First, we prove that $T(\Phi_J) \subseteq T(\Psi_J)$. Indeed, let $d \in T(\Phi_J)$. Since $T(\Phi_J) =_{-y} T(\Psi_J)$, there exists $d' \in T(\Psi_J)$ such that $d' =_{-y} d$. Since y is unessential for Ψ, $d \in T(\Psi_J)$. Thus, $T(\Phi) \subseteq T(\Psi)$.

From this follows that $\Psi, \Gamma\,_J{\models_T}\ \Delta \Rightarrow \Phi, \Gamma\,_J{\models_T}\ \Delta$.

This completes the proof of (6). □

Theorem 3. *The following properties hold for T-consequence relation.*

– *Properties related to propositional compositions:*

$\neg\neg_{\mathrm{L}})$ $\neg\neg\Phi, \Gamma \models_T \Delta \Leftrightarrow \Phi, \Gamma \models_T \Delta$.

$\neg\neg_{\mathrm{R}})$ $\Gamma \models_T \Delta, \neg\neg\Phi \Leftrightarrow \Gamma \models_T \Delta, \Phi$.

$\vee_{\mathrm{L}})$ $\Phi \vee \Psi, \Gamma \models_T \Delta \Leftrightarrow (\Phi, \Gamma \models_T \Delta$ *and* $\Psi, \Gamma \models_T \Delta)$.

$\neg\vee_{\mathrm{L}})$ $\neg(\Phi \vee \Psi), \Gamma \models_T \Delta \Leftrightarrow \neg\Phi, \neg\Psi, \Gamma \models_T \Delta$.

$\vee_{\mathrm{R}})$ $\Gamma \models_T \Delta, \Phi \vee \Psi \Leftrightarrow \Gamma \models_T \Delta, \Phi, \Psi$.

$\neg\vee_{\mathrm{R}})$ $\Gamma \models_T \Delta, \neg(\Phi \vee \Psi) \Leftrightarrow (\Gamma \models_T \Delta, \neg\Phi$ *and* $\Gamma \models_T \Delta, \neg\Psi)$.

– *Properties related to renomination compositions:*

$R\vee_{\mathrm{L}}$) $R_{\bar{x}}^{\bar{v}}(\Phi \vee \Psi), \Gamma \models_T \Delta \Leftrightarrow R_{\bar{x}}^{\bar{v}}(\Phi) \vee R_{\bar{x}}^{\bar{v}}(\Psi), \Gamma \models_T \Delta.$

$\neg R\vee_{\mathrm{L}}$) $\neg R_{\bar{x}}^{\bar{v}}(\Phi \vee \Psi), \Gamma \models_T \Delta \Leftrightarrow \neg(R_{\bar{x}}^{\bar{v}}(\Phi) \vee R_{\bar{x}}^{\bar{v}}(\Psi)), \Gamma \models_T \Delta.$

$R\vee_{\mathrm{R}}$) $\Gamma \models_T \Delta, R_{\bar{x}}^{\bar{v}}(\Phi \vee \Psi) \Leftrightarrow \Gamma \models_T \Delta, R_{\bar{x}}^{\bar{v}}(\Phi) \vee R_{\bar{x}}^{\bar{v}}(\Psi).$

$\neg R\vee_{\mathrm{R}}$) $\Gamma \models_T \Delta, \neg R_{\bar{x}}^{\bar{v}}(\Phi \vee \Psi) \Leftrightarrow \Gamma \models_T \Delta, \neg(R_{\bar{x}}^{\bar{v}}(\Phi) \vee R_{\bar{x}}^{\bar{v}}(\Psi)).$

$\mathrm{R_L}$) $R(\Phi), \Gamma \models_T \Delta \Leftrightarrow \Phi, \Gamma \models_T \Delta.$

$\mathrm{R_R}$) $\Gamma \models_T \Delta, R(\Phi) \Leftrightarrow \Phi, \Gamma \models_T \Delta, \Phi.$

$\mathrm{RI_L}$) $R_{z,\bar{x}}^{z,\bar{v}}(\Phi), \Gamma \models_T \Delta \Leftrightarrow R_{\bar{x}}^{\bar{v}}(\Phi), \Gamma \models_T \Delta.$

$\neg\mathrm{RI_L}$) $\neg R_{z,\bar{x}}^{z,\bar{v}}(\Phi), \Gamma \models_T \Delta \Leftrightarrow \neg R_{\bar{x}}^{\bar{v}}(\Phi), \Gamma \models_T \Delta.$

$\mathrm{RI_R}$) $\Gamma \models_T \Delta, R_{z,\bar{x}}^{z,\bar{v}}(\Phi) \Leftrightarrow \Gamma \models_T \Delta, R_{\bar{x}}^{\bar{v}}(\Phi).$

$\neg\mathrm{RI_R}$) $\Gamma \models_T \Delta, \neg R_{z,\bar{x}}^{z,\bar{v}}(\Phi) \Leftrightarrow \Gamma \models_T \Delta, \neg R_{\bar{x}}^{\bar{v}}(\Phi).$

$\mathrm{RU_L}$) $R_{z,\bar{x}}^{y,\bar{v}}(\Phi), \Gamma \models_T \Delta \Leftrightarrow R_{\bar{x}}^{\bar{v}}(\Phi), \Gamma \models_T \Delta,$ *if* $y \in fu(\Phi).$

$\neg\mathrm{RU_L}$) $\neg R_{z,\bar{x}}^{y,\bar{v}}(\Phi), \Gamma \models_T \Delta \Leftrightarrow \neg R_{\bar{x}}^{\bar{v}}(\Phi), \Gamma \models_T \Delta,$ *if* $y \in fu(\Phi).$

$\mathrm{RU_R}$) $\Gamma \models_T \Delta, R_{z,\bar{x}}^{y,\bar{v}}(\Phi) \Leftrightarrow \Gamma \models_T \Delta, R_{\bar{x}}^{\bar{v}}(\Phi),$ *if* $y \in fu(\Phi).$

$\neg\mathrm{RU_R}$) $\Gamma \models_T \Delta, \neg R_{z,\bar{x}}^{y,\bar{v}}(\Phi), \Leftrightarrow \Gamma \models_T \Delta, \neg R_{\bar{x}}^{\bar{v}}(\Phi),$ *if* $y \in fu(\Phi).$

$\mathrm{RR_L}$) $R_{\bar{x}}^{\bar{v}}(R_{\bar{y}}^{\bar{w}}(\Phi)), \Gamma \models_T \Delta \Leftrightarrow R_{\bar{x}}^{\bar{v}} \circ_{\bar{y}}^{\bar{w}} (\Phi), \Gamma \models_T \Delta.$

$\neg\mathrm{RR_L}$) $\neg R_{\bar{x}}^{\bar{v}}(R_{\bar{y}}^{\bar{w}}(\Phi)), \Gamma \models_T \Delta \Leftrightarrow \neg R_{\bar{x}}^{\bar{v}} \circ_{\bar{y}}^{\bar{w}} (\Phi), \Gamma \models_T \Delta.$

$\mathrm{RR_R}$) $\Gamma \models_T \Delta, R_{\bar{x}}^{\bar{v}}(R_{\bar{y}}^{\bar{w}}(\Phi)) \Leftrightarrow \Gamma \models_T \Delta, R_{\bar{x}}^{\bar{v}} \circ_{\bar{y}}^{\bar{w}} (\Phi).$

$\neg\mathrm{RR_R}$) $\Gamma \models_T \Delta, \neg R_{\bar{x}}^{\bar{v}}(R_{\bar{y}}^{\bar{w}}(\Phi)) \Leftrightarrow \Gamma \models_T \Delta, \neg R_{\bar{x}}^{\bar{v}} \circ_{\bar{y}}^{\bar{w}} (\Phi).$

$\mathrm{R}\neg_{\mathrm{L}}$) $R_{\bar{x}}^{\bar{v}}(\neg\Phi)), \Gamma \models_T \Delta \Leftrightarrow \neg R_{\bar{x}}^{\bar{v}}(\Phi), \Gamma \models_T \Delta.$

$\neg\mathrm{R}\neg_{\mathrm{L}}$) $\neg R_{\bar{x}}^{\bar{v}}(\neg\Phi)), \Gamma \models_T \Delta \Leftrightarrow \neg\neg R_{\bar{x}}^{\bar{v}}(\Phi), \Gamma \models_T \Delta.$

$\mathrm{R}\neg_{\mathrm{R}}$) $\Gamma \models_T \Delta, R_{\bar{x}}^{\bar{v}}(\neg\Phi)) \Leftrightarrow \Gamma \models_T \Delta, \neg R_{\bar{x}}^{\bar{v}}(\Phi).$

$\neg\mathrm{R}\neg_{\mathrm{R}}$) $\Gamma \models_T \Delta, \neg R_{\bar{x}}^{\bar{v}}(\neg\Phi)) \Leftrightarrow \Gamma \models_T \Delta, \neg\neg R_{\bar{x}}^{\bar{v}}(\Phi).$

$\mathrm{R}\exists\mathrm{R_L}$) $R_{\bar{v},y}^{\bar{u},x}(\exists x\Phi), \Gamma \models_T \Delta \Leftrightarrow R_{\bar{v}}^{\bar{u}}(\exists x\Phi), \Gamma \models_T \Delta.$

$\neg\mathrm{R}\exists\mathrm{R_L}$) $\neg R_{\bar{v},y}^{\bar{u},x}(\exists x\Phi), \Gamma \models_T \Delta \Leftrightarrow \neg R_{\bar{v}}^{\bar{u}}(\exists x\Phi), \Gamma \models_T \Delta.$

$\mathrm{R}\exists\mathrm{R_R}$) $\Gamma \models_T \Delta, R_{\bar{v},y}^{\bar{u},x}(\exists x\Phi) \Leftrightarrow \Gamma \models_T \Delta, R_{\bar{v}}^{\bar{u}}(\exists x\Phi).$

$\neg$R$\exists$R$_{\text{R}}$) $\Gamma \models_T \Delta, \neg R^{\bar{u},x}_{\bar{v},y}(\exists x\Phi) \Leftrightarrow \Gamma \models_T \Delta, \neg R^{\bar{u}}_{\bar{v}}(\exists x\Phi)$.

R$\exists$s$_{\text{L}}$) $R^{\bar{v}}_{\bar{x}}(\exists y\Phi), \Gamma \models_T \Delta \Leftrightarrow \exists y R^{\bar{v}}_{\bar{x}}(\Phi), \Gamma \models_T \Delta$, *if* $y \notin \{\bar{v}, \bar{x}\}$.

$\neg$R$\exists$s$_{\text{L}}$) $\neg R^{\bar{v}}_{\bar{x}}(\exists y\Phi), \Gamma \models_T \Delta \Leftrightarrow \exists y \neg R^{\bar{v}}_{\bar{x}}(\Phi), \Gamma \models_T \Delta$, *if* $y \notin \{\bar{v}, \bar{x}\}$.

R$\exists$s$_{\text{R}}$) $\Gamma \models_T \Delta, R^{\bar{v}}_{\bar{x}}(\exists y\Phi) \Leftrightarrow \Gamma \models_T \Delta, \exists y R^{\bar{v}}_{\bar{x}}(\Phi)$, *if* $y \notin \{\bar{v}, \bar{x}\}$.

$\neg$R$\exists$s$_{\text{R}}$) $\Gamma \models_T \Delta, \neg R^{\bar{v}}_{\bar{x}}(\exists y\Phi) \Leftrightarrow \Gamma \models_T \Delta, \exists y \neg R^{\bar{v}}_{\bar{x}}(\Phi)$, *if* $y \notin \{\bar{v}, \bar{x}\}$.

R$\exists_{\text{L}}$) $R^{\bar{v}}_{\bar{x}}(\exists y\Phi), \Gamma \models_T \Delta \Leftrightarrow \exists z R^{\bar{v}}_{\bar{x}} \circ^y_z (\Phi), \Gamma \models_T \Delta$,
if $z \in fu(R^{\bar{v}}_{\bar{x}}(\exists y\Phi))$.

$\neg$R$\exists_{\text{L}}$) $\neg R^{\bar{v}}_{\bar{x}}(\exists y\Phi), \Gamma \models_T \Delta \Leftrightarrow \neg\exists z R^{\bar{v}}_{\bar{x}} \circ^y_z (\Phi), \Gamma \models_T \Delta$,
if $z \in fu(R^{\bar{v}}_{\bar{x}}(\exists y\Phi))$.

R$\exists_{\text{R}}$) $\Gamma \models_T \Delta, R^{\bar{v}}_{\bar{x}}(\exists y\Phi) \Leftrightarrow \Gamma \models_T \Delta, \exists z R^{\bar{v}}_{\bar{x}} \circ^y_z (\Phi)$,
if $z \in fu(R^{\bar{v}}_{\bar{x}}(\exists y\Phi))$.

$\neg$R$\exists_{\text{R}}$) $\Gamma \models_T \Delta, \neg R^{\bar{v}}_{\bar{x}}(\exists y\Phi) \Leftrightarrow \Gamma \models_T \Delta, \neg\exists z R^{\bar{v}}_{\bar{x}} \circ^y_z (\Phi)$,
if $z \in fu(R^{\bar{v}}_{\bar{x}}(\exists y\Phi))$.

– *Properties related to unassignment predicates:*

ε_{LR}) $\Gamma \models_T \Delta \Leftrightarrow \varepsilon y, \Gamma \models_T \Delta$ *and* $\Gamma \models_T \Delta, \varepsilon y$.

ε_{R}) $\Gamma \models_T \Delta \Leftrightarrow \Gamma \models_T \Delta, \varepsilon z$, *if* $z \in fu(\Gamma, \Delta)$.

– *Properties related to quantifiers:*

$\exists$e$_{\text{L}}$) $\exists x\Phi, \Gamma \models_T \Delta \Leftrightarrow R^x_z(\Phi), \Gamma \models_T \Delta, \varepsilon z$, *if* $z \in fu(\Gamma, \Delta, \exists x\Phi)$.

$\neg\exists$e$_{\text{L}}$) $\neg\exists x\Phi, \Gamma \models_T \Delta, \varepsilon y \Leftrightarrow \neg\exists x\Phi, \neg R^x_y(\Phi), \Gamma \models_T \Delta, \varepsilon y$.

$\exists$e$_{\text{R}}$) $\Gamma \models_T \Delta, \exists x\Phi, \varepsilon y \Leftrightarrow \Gamma \models_T \Delta, \exists x\Phi, R^x_y(\Phi), \varepsilon y$.

$\neg\exists$e$_{\text{R}}$) $\Gamma \models_T \Delta, \neg\exists x\Phi \Leftrightarrow \Gamma \models_T \Delta, \neg Rx_z(\Phi), \varepsilon z$,
if $z \in fu(\Gamma, \Delta, \exists x\Phi)$.

Proof. Proof of the formulated properties is based on semantic properties of compositions and properties of T-consequence relation. Consider, for instance, properties $\vee_{\text{L}}$ and $\neg\vee_{\text{L}}$. For $\vee_{\text{L}}$ we have that $\Phi \vee \Psi, \Gamma\ {}_J\!\models_T \Delta \Leftrightarrow (T(\Phi_J) \cup T(\Psi_J)) \cap T^{\wedge}(\Gamma_J) \subseteq T^{\vee}(\Delta_J) \Leftrightarrow (T(\Phi_J) \cap T^{\wedge}(\Gamma_J)) \cup (T(\Psi_J) \cap T^{\wedge}(\Gamma_J)) \subseteq T^{\vee}(\Delta_J) \Leftrightarrow (T(\Phi_J) \cap T^{\wedge}(\Gamma_J) \subseteq T^{\vee}(\Delta_J)$ and $T(\Psi_J) \cap T^{\wedge}(\Gamma_J) \subseteq T^{\vee}(\Delta_J)) \Leftrightarrow (\Phi, \Gamma\ {}_J\!\models_T \Delta$ and $\Psi, \Gamma\ {}_J\!\models_T \Delta)$

for any interpretation J. Thus, $\Phi \vee \Psi, \Gamma \models_T \Delta \Leftrightarrow (\Phi, \Gamma \models_T \Delta$ and $\Psi, \Gamma \models_T \Delta)$.

For $\neg\vee_{\mathrm{L}}$ we have that $\neg(\Phi \vee \Psi), \Gamma\,_J \models_T \Delta \Leftrightarrow T(\neg(\Phi \vee \Psi)_J) \cap T^{\wedge}(\Gamma_J) \subseteq T^{\vee}(\Delta_J) \Leftrightarrow F(\Phi_J) \cap F(\Psi_J) \cap T^{\wedge}(\Gamma_J) \subseteq T^{\vee}(\Delta_J) \Leftrightarrow T(\neg\Phi_J) \cap T(\neg\Psi_J) \cap T^{\wedge}(\Gamma_J) \subseteq T^{\vee}(\Delta_J) \Leftrightarrow \neg\Phi, \neg\Psi, \Gamma\,_J{\models_T} \Delta$ for any interpretation J. Thus, $\neg(\Phi \vee \Psi), \Gamma \models_T \Delta \Leftrightarrow \neg\Phi, \neg\Psi, \Gamma \models_T \Delta$.

Properties related to renomination composition hold by Lemma 5.

Property $\varepsilon_{\mathrm{LR}}$ follows from Theorem 2(4); property ε_{R} follows from Theorem 2(5).

Properties related to quantifiers are consequences of Lemma 6, Lemma 7, and Theorem 2(6). Detailed proof is omitted here. □

Properties presented in Theorem 3 induce sequent rules for calculus formalizing $\models_T$. For example, property $\exists e_{\mathrm{L}}$ induces a rule

$$\frac{R_z^x(\Phi), \Gamma \to \Delta, \varepsilon z}{\exists x\Phi, \Gamma \to \Delta},\ z \in fu(\Gamma, \Delta, \exists x\Phi).$$

Detailed construction of such a calculus will be presented in forthcoming papers.

6 T-consequence relation for the class of deterministic predicates

A class of deterministic quasiary predicates is an important subclass of the class of non-deterministic predicates.

Predicate $p \in PrR_A^V$ is called *deterministic (partial single-valued)* if $T(p) \cap F(p) = \emptyset$. The class of such predicates is denoted Pr_A^V.

Lemma 10. *Class Pr_A^V is a sub-algebra of algebra $AQR(V, A)$.*

The lemma is proved by direct checking that all compositions preserve the class of deterministic predicates. Defined algebra is denoted $AQ(V, A)$.

Such algebras form a semantic base for a logic of quasiary deterministic predicates L^Q. The extended logics L^U and L_ε^U are defined in the same way as for non-deterministic predicates. In this section a sign $\models_T$ denotes a T-consequence relation for the class of deterministic quasiary predicates.

Here we present only those properties of logics of deterministic predicates that differ from corresponding properties for non-deterministic predicates. In particular, a formula from the right side of T-consequence relation can be placed as a negated formula into the left side of the relation. Such property does not hold for the class of non-deterministic predicates.

Lemma 11. *The following properties hold for T-consequence relation for the logic of deterministic quasiary predicates:*

- $\Gamma \models_T \Delta, \Phi \Rightarrow \neg\Phi, \Gamma \models_T \Delta$;
- $\Gamma \models_T \Delta, \neg\Phi \Rightarrow \Phi, \Gamma \models_T \Delta$.

Proof. For any Γ, Δ, Φ, and interpretation J we have that
$\Gamma\ {}_J\!\models_T \Delta, \Phi \Leftrightarrow T^{\wedge}(\Gamma_J) \subseteq T(\Phi) \vee T^{\vee}(\Delta_J) \Leftrightarrow T^{\wedge}(\Gamma_J) \cap \overline{T(\Phi)} \subseteq T^{\vee}(\Delta_J)$.
Since $T(\neg\Phi) \subseteq \overline{T(\Phi)}$ for deterministic predicates, we have that

$$T^{\wedge}(\Gamma_J) \cap \overline{T(\Phi)} \subseteq T^{\vee}(\Delta_J) \Rightarrow T^{\wedge}(\Gamma_J) \cap T(\neg\Phi) \subseteq T^{\vee}(\Delta_J).$$

Thus, $\Gamma\ {}_J\!\models_T \Delta, \Phi \Rightarrow \neg\Phi, \Gamma\ {}_J\!\models_T \Delta$. Therefore

$$\Gamma \models_T \Delta, \Phi \Rightarrow \neg\Phi, \Gamma \models_T \Delta.$$

The second property is proved in the same manner. □

Properties concerning paraconsistency, paracompleteness, and paranormality also differ for the class of deterministic predicates.

Theorem 4. *T-consequence relation for the logic of deterministic quasiary predicates is consistent, paracomplete, and normal.*

Proof. For the class of deterministic predicates we have that $\Phi \wedge \neg\Phi, \Gamma\ {}_J\!\models_T \Psi, \Delta \Leftrightarrow (T(\Phi_J) \cap T(\neg\Phi_J)) \cap T^{\wedge}(\Gamma_J) \subseteq T(\Psi_J) \vee T^{\vee}(\Delta_J)$ for any Γ, Δ, Φ, Ψ, and interpretation J. This inclusion holds because for deterministic predicates $T(\Phi_J) \cap T(\neg\Phi_J) = \emptyset$. Thus, $\models_T$ is consistent; consequently, $\models_T$ is normal. To demonstrate paracompleteness of $\models_T$ we take an interpretation J such that $T(\Phi_J) \neq \emptyset$ and $\Psi_J = \perp_A^V$. Then $\Phi\ {}_J\!\not\models_T \Psi \vee \neg\Psi$. □

In general, comparing properties of T-consequence relation for classes of deterministic and non-deterministic predicates, we can say that the latter is poorer and the former is richer. In particular, if we consider F-consequence relation, which is dual to T-consequence relation, and a combined TF-consequence relation, then all three relations coincide for the class of non-deterministic predicates, but they are different for the class of deterministic predicates. Also, the irrefutability consequence relation is quite natural for the class of deterministic predicates while it collapses for the class of non-deterministic predicates.

7 Conclusion

In the paper we have investigated a special kind of program-oriented algebras and logics defined for classes of non-deterministic and deterministic quasiary predicates. We have considered the main semantic properties of T-consequence relations for such logics. We have presented properties related to propositional compositions, renomination, variable unassignment predicates, and quantifier compositions. These properties form a basis for construction of sequent calculi for logics of non-deterministic and deterministic quasiary predicates. We plan to present such calculi and prove their validity and completeness in forthcoming papers.

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Mykola Nikitchenko, Stepan Shkilniak, Received September 20, 2015

Mykola Nikitchenko
Taras Shevchenko National University of Kyiv
01601, Kyiv, Volodymyrska st, 60
Phone: +38044 2590519
E–mail: nikitchenko@unicyb.kiev.ua

Stepan Shkilniak
Taras Shevchenko National University of Kyiv
01601, Kyiv, Volodymyrska st, 60
Phone: +38044 2590519
E–mail: sssh@unicyb.kiev.ua