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Bounds on Global Total Domination in Graphs

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Abstract

A subset S of vertices in a graph G is a *global total dominating set*, or just GTDS, if S is a *total dominating set* of both G and $\overline{G}$. The *global total domination number* $\gamma_{gt}(G)$ of G is the minimum cardinality of a GTDS of G. We present bounds for the global total domination number in graphs.

Keywords: Domination; Total domination; Global total domination.

1 Introduction

We consider finite, undirected and simple graphs G with vertex set $V(G)$ and edge set $E(G)$. The number of vertices $|V(G)|$ of a graph G is called the order of G and is denoted by $n = n(G)$. We denote the *open neighborhood* of a vertex v of G by $N_G(v)$, or just $N(v)$, and its *closed neighborhood* by $N_G[v]$ or $N[v]$. For a vertex set $S \subseteq V(G)$, we denote $N(S) = \cup_{v\in S}N(v)$ and $N[S] = \cup_{v\in S}N[v]$. The *degree* of a vertex x, $\deg(x)$ (or $\deg_G(x)$ to refer G) in a graph G denotes the number of neighbors of x in G. We refer $\delta = \delta(G)$ as the *minimum degree* of the vertices of G. If S is a subset of $V(G)$, then we denote by $G[S]$ the subgraph of G induced by S. A set of vertices S in G is a *dominating set*, if $N[S] = V(G)$. The *domination number*, $\gamma(G)$, of G is the minimum cardinality of a dominating set of G. A set of vertices S in G is a *total dominating set*, or just TDS, if $N(S) = V(G)$. The *total domination number*, $\gamma_t(G)$, of G is the minimum cardinality of a total dominating set of G. For references and also terminology on domination and total domination in graphs see for example [9, 10].

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Global domination in graphs was introduced by Sampathkumar in [12], and further has been studied by Brigham et al. [3, 4], Dutton et al. [7, 8] and Arumugam et al. [2]. A subset S of vertices of a graph G is a *global dominating set* if S is a dominating set of both G and $\overline{G}$. The *global domination number* of a graph G, $\gamma_g(G)$, is the minimum cardinality of a global dominating set of G. *Global total domination* in graphs was introduced by Kulli et al. in [11]. A subset S of vertices in a graph G is a *global total dominating set*, or just GTDS, if S is a TDS of both G and $\overline{G}$. The *global total domination number* of G, $\gamma_{gt}(G)$, is the minimum cardinality of a GTDS of G. If a graph G of order n has a GTDS, then $\delta(G) \geq 1$ and $\Delta(G) \leq n-2$. That is neither G nor $\overline{G}$ have an isolated vertex.

In this paper, we present probabilistic bounds for the global total domination number in graphs. We adopt the methods of [1]. We make use of the following.

Theorem 1 (Cockayne et al. [6])**.** *If G is a connected graph of order $n \geq 3$, then $\gamma_t(G) \leq 2n/3$.*

Theorem 2 (Brigham et al. [5])**.** *Let G be a connected graph of order $n \geq 3$. Then $\gamma_t(G) = 2n/3$ if and only if G is C_3, C_6 or 2-corona of a connected graph.*

Note that the *corona* of a graph G, denoted by $cor(G)$, is a graph obtained from G by adding a leaf for every vertex of G, and the 2-*corona* of G is a graph obtained from G by adding a leaf of a path P_2 for every vertex of G.

2 Bounds

Let $\bar{\delta} = \delta(\overline{G})$ and $\delta' = \min\{\delta, \bar{\delta}\}$.

Theorem 3. *For any graph G with $\delta' > 3$,*

$$\gamma_{gt}(G) \leq n\Big(1 - \frac{\delta'}{3^{\frac{1}{\delta'}}(1+\delta')^{1+\frac{1}{\delta'}}}\Big).$$

Proof. Let A be a set formed by an independent choice of vertices of G, where each vertex is selected with probability

$$p = 1 - \frac{1}{(3(1+\delta'))^{\frac{1}{\delta'}}}.$$

The condition on $\delta' > 3$ implies that $p < \frac{1}{2}$. Let us denote $B = V(G) \setminus N_G[A]$. We consider the following cases.

Case 1. There exists a vertex $v \in V(G) \setminus A \cup B$ such that v is adjacent to every vertex of $A \cup B$.

Let C be the set of vertices of G that are dominated by no vertex of $A \cup B$ in graph $\overline{G}$. Then $C \neq \emptyset$, since $v \in C$. Furthermore, each vertex of C is adjacent to every vertex of $A \cup B$ in G. Let $C' = \{x \in C, N_{\overline{G}}(x) \cap C = \emptyset\}$. For each vertex $x \in C'$, we choose a vertex $x^* \in N_{\overline{G}}(x)$. Let $C^* = \{x^*, x \in C'\}$. For the expectation of $|B|$ and $|C|$, it is easy to show that

$$\begin{aligned} E(|B|) = \textstyle\sum_{v \in V(G)} Pr(v \in B) &= n(1-p)^{1+\deg_G(v)} \\ &\leq n(1-p)^{1+\delta} \leq n(1-p)^{1+\delta'}, \\ E(|C|) = \textstyle\sum_{v \in V(G)} Pr(v \in C) &= n(1-p)^{1+\deg_{\overline{G}}(v)} \\ &\leq n(1-p)^{1+\bar{\delta}} \leq n(1-p)^{1+\delta'}. \end{aligned}$$

Since $|C^*| \leq |C|$, we have

$$E(|C^*|) \leq E(|C|) \leq n(1-p)^{1+\delta'}.$$

It is obvious that the set $D_1 = A \cup B \cup C \cup C^*$ is a total global dominating set. Clearly $C^* \cap (A \cup B) = \emptyset$. Thus any vertex of C^* is adjacent to some vertex of A. Thus $G[D_1]$ contains no isolated vertex. Furthermore, $\overline{G}[D_1]$ contains no isolated vertex, since in $\overline{G}$ any vertex of A is adjacent to every vertex of B and any vertex of C is adjacent to some vertex of $C \cup C^*$. Let $d \in V(G) - D_1$. Then clearly $d \in N_G(A)$. Since $d \notin C$, d is not adjacent to all vertices of $A \cup B$. Thus d is dominated by some vertex of A in G, and is dominated by some vertex of $A \cup B$ in $\overline{G}$. Thus D_1 is a TDS of both G and $\overline{G}$. Consequently D_1

is a global total dominating set. The expectation of $|D_1|$ is

$$\begin{aligned}E(|D_1|) &\le E(|A|) + E(|B|) + E(|C|) + E(|C^*|)\\ &\le np + n(1-p)^{1+\delta'} + n(1-p)^{1+\delta'} + n(1-p)^{1+\delta'}\\ &= n(p + 3(1-p)^{1+\delta'}).\end{aligned}$$

Case 2. No vertex of $V \setminus A \cup B$ is adjacent to every vertex of $A \cup B$.

Let A' be the set of vertices $a \in A$ such that a is an isolated vertex in $G[A]$, and B' be the set of vertices $b \in B$ such that b is an isolated vertex in $G[B]$. For each $a \in A'$ we choose a vertex $a^* \in N_G(a)$, and for each $b \in B'$ we choose a vertex $b^* \in N_G(b)$. Let $A^* = \{a^* | a \in A'\}$ and $B^* = \{b^* | b \in B'\}$. It follows that

$$E(|A^*|) = \sum_{v \in V(G)} Pr(v \in A^*) = np(1-p)^{\deg(v)} \le np(1-p)^{\delta'}.$$

Since $|B^*| \le |B|$, thus we have

$$E(|B^*|) \le E(|B|) \le n(1-p)^{1+\delta'}.$$

Any vertex of A is adjacent to some vertex of $A \cup A^*$ in G and is adjacent to every vertex of B in $\overline{G}$. Similarly any vertex of B is adjacent to some vertex of $B \cup B^*$ in G and is adjacent to every vertex of A in $\overline{G}$. Let $a \in A^* - A$. By hypothesis a is not adjacent to every vertex of $A \cup B$. Similarly for every vertex $b \in B^* - B$, b is not adjacent to every vertex of $A \cup B$. Thus the graphs induced by $D_2 = A \cup B \cup A^* \cup B^*$ in G and $\overline{G}$ have no isolated vertex. Let $c \in V(G) - D_2$. Then by hypothesis c is dominated by some vertex of D_2. Thus D_2 is a TDS for both G and $\overline{G}$. Consequently D_2 is a global total dominating set. The expectation of $|D_2|$ is

$$\begin{aligned}E(|D_2|) &\le E(|A|) + E(|B|) + E(|A^*|) + E(|B^*|)\\ &\le np + n(1-p)^{1+\delta'} + np(1-p)^{\delta'} + n(1-p)^{1+\delta'}\\ &= n(p + 2(1-p)^{1+\delta'} + p(1-p)^{\delta'}).\end{aligned}$$

Since $p < \frac{1}{2}$, $p(1-p)^{\delta'} \leq (1-p)^{1+\delta'}$ and thus $E(|D_2|) \leq n(p + 3(1-p)^{1+\delta'})$. Therefore in both cases there is a global total dominating set D with

$$E(|D|) \leq n(p + 3(1-p)^{1+\delta'}). \tag{1}$$

By the pigeonhole property of expectation we obtain that

$$\begin{aligned}\gamma_{gt}(G) &\leq n(p + 3(1-p)^{1+\delta'}) \\ &= n\Big(1 - \frac{\delta'}{3^{\frac{1}{\delta'}}(1+\delta')^{1+\frac{1}{\delta'}}}\Big).\end{aligned}$$

□

Corollary 1. *For any graph G with $\delta' > 3$,*

$$\gamma_{gt}(G) \leq \Big(\frac{\ln(1+\delta') + \ln 3 + 1}{1+\delta'}\Big)n.$$

Proof. We follow the proof of Theorem 3 considering $p = \frac{\ln(1+\delta')+\ln 3}{1+\delta'}$. Using the inequality $1 - p \leq e^{-p}$, we obtain the following estimation of (1):

$$E(|D|) \leq n(p + 3(1-p)^{1+\delta'}) \leq np + 3ne^{-p(1+\delta')}.$$

A simple calculation implies that

$$E(|D|) \leq \Big(\frac{\ln(1+\delta') + \ln 3 + 1}{1+\delta'}\Big)n.$$

Now the result follows by the pigeonhole property of expectation. □

Zverovich and Poghosyan [13] proved that when n is large there exists a graph G such that

$$\gamma_g(G) \geq \Big(\frac{\ln(1+\delta') + \ln 2 + 1}{1+\delta'}\Big)n(1+o(1)).$$

With an identical proof of them we can obtain that when n is large there exists a graph G such that

$$\gamma_{gt}(G) \geq \Big(\frac{\ln(1+\delta') + \ln 3 + 1}{1+\delta'}\Big)n(1+o(1)).$$

Thus the upper bound of Corollary 1 is asymptotically best possible.

Theorem 4. *For any graph G with $\delta' = 3$, $\gamma_{gt}(G) \leq 0.683n$.*

Proof. It is a routine matter using calculus to see that the equation

$$4x^3 - 15x^2 + 18x - 6 = 0$$

has a root x_0 with $\frac{1}{2} < x_0 < 1$. We follow the proof of Theorem 3 with $p = x_0$. Since $p > \frac{1}{2}$, we conclude that $E(|D_1|) \leq n(p + 2(1-p)^4 + p(1-p)^3)$. Thus in both cases we obtain a global total dominating set D with

$$E(|D|) \leq n(p + 2(1-p)^4 + p(1-p)^3).$$

With the estimation $p = 0.545$ and the pigeonhole property of expectation we obtain the desired bound. □

Theorem 5. *For any graph G with $\delta' = 2$, $\gamma_{gt}(G) \leq \frac{22}{27}n$.*

Proof. We follow the proof of Theorem 3 with $p = \frac{2}{3}$. Since $p > \frac{1}{2}$, we conclude that $E(|D_1|) \leq n\big(p + 2(1-p)^3 + p(1-p)^2\big)$. Thus in both cases we obtain a global total dominating set D with

$$E(|D|) \leq n(p + 2(1-p)^3 + p(1-p)^2) = \frac{22}{27}n.$$

Now the proof follows by the pigeonhole property of expectation. □

Theorem 6. *For any graph G with $\delta' = 1$, $\gamma_{gt}(G) \leq \frac{2}{3}n + 1$, and this bound is sharp.*

Proof. Without loss of generality assume that $\delta(G) = 1$. Let a be a vertex with $\deg(a) = 1$ and b be the unique neighbor of a. Let S be a $\gamma_t(G)$-set. If $\gamma_t(G) = \frac{2}{3}n$ then by Theorem 2 G is 2-corona of a connected graph H. Then clearly S is a TDS of $\overline{G}$, and thus $\gamma_{gt}(G) \leq \frac{2}{3}n$. Thus by Theorem 1, $\gamma_t(G) \leq 2n/3 - 1$. Assume that $G[S]$ is not a complete graph. Let $x \in S$ be a vertex that is not adjacent to every vertex of S, and let $y \in N(x) - S$. Then $S \cup \{y\}$ is a TDS for both G and $\overline{G}$, and thus $\gamma_{gt}(G) \leq \frac{2}{3}n$. We thus assume that $G[S]$ is a complete graph. Let $y \in V(G) - (S \cup \{a\})$. Then $S \cup \{a, y\}$ is a TDS for both G and $\overline{G}$, and thus $\gamma_{gt}(G) \leq \frac{2}{3}n + 1$. To see the sharpness consider $G = cor(C_3)$. □

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