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A Note on Solvable Polynomial Algebras

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Abstract

In terms of their defining relations, solvable polynomial algebras introduced by Kandri-Rody and Weispfenning [J. Symbolic Comput., 9(1990)] are characterized by employing Gröbner bases of ideals in free algebras, thereby solvable polynomial algebras are completely determinable and constructible in a computational way.

Keywords: PBW basis, Monomial ordering, Gröbner basis, Solvable polynomial algebra.

1 Introduction

In the late 1980s, the Gröbner basis theory invented by Bruno Buchberger ([2], [3]) for commutative polynomial ideals was successfully generalized to one-sided ideals in enveloping algebras of Lie algebras by Apel and Lassner [1], to one-sided ideals in Weyl algebras (including algebras of partial differential operators with polynomial coefficients over a field of characteristic 0) by Galligo [6], and more generally, to one-sided and two-sided ideals in solvable polynomial algebras (or algebras of solvable type) by Kandri-Rody and Weispfenning [8]. In particular, the noncommutative Buchberger Algorithm for computing Gröbner bases of one-sided and two-sided ideals in solvable polynomial algebras has been implemented in some well-developed computer algebra systems such as Singular [4].

Originally, a noncommutative solvable polynomial algebra R' was defined in [8] by first fixing a monomial ordering $\prec$ on the standard K-basis $\mathscr{B} = \{X_1^{\alpha_1}\cdots X_n^{\alpha_n} \mid \alpha_i \in \mathbb{N}\}$ of the commutative polynomial algebra $R = K[X_1,\dots,X_n]$ in n variables $X_1,\dots,X_n$ over a field K, and then introducing a new multiplication $*$ on R, such that certain axioms ([8], AXIOMS 1.2) are satisfied. In the formal language of associative K-algebras, a solvable polynomial algebra can actually be defined as a finitely generated associative K-algebra $A = K[a_1, \ldots, a_n]$, that has the PBW K-basis $\mathcal{B} = \{a_1^{\alpha_1} \cdots a_n^{\alpha_n} \mid \alpha_i \in \mathbb{N}\}$ and a monomial ordering $\prec$ on $\mathcal{B}$ such that for $1 \le i < j \le n$, $a_j a_i = \lambda_{ji} a_i a_j + f_{ji}$ and $\mathbf{LM}(f_{ji}) \prec a_i a_j$, where $\lambda_{ji} \in K - \{0\}$, $f_{ji} \in K$-span$\mathcal{B}$ and $\mathbf{LM}(f_{ji})$ is the leading monomial of f_{ji} with respect to $\prec$ ([11], Definition 2.1). Full details on this definition will be recalled in the next section. In the literature, some results on the construction of solvable polynomial algebras by means of Gröbner bases for ideals in a free K-algebra $K\langle X\rangle = K\langle X_1, \ldots, X_n\rangle$ were given (see [8], Theorem 1.11; [9], CH.III, Proposition 2.2, Proposition 2.3; [10], Ch.4, Proposition 4.2), but a complete constructive characterization of solvable polynomial algebras has not been reached.

By employing Gröbner bases of ideals in free algebras, in this note we give a characterization of solvable polynomial algebras in terms of their defining relations (Section 2, Theorem 2.1), which shows that solvable polynomial algebras are completely determinable and constructible in a computational way.

Throughout this note, K denotes a field, $K^* = K - \{0\}$; $\mathbb{N}$ denotes the set of all nonnegative integers. Moreover, the Gröbner basis theory for ideals of free algebras is referred to [12] and [7].

2 The main result

We first briefly recall from ([8], [11], [9]) some basics concerning solvable polynomial algebras. Let $A = K[a_1, \ldots, a_n]$ be a finitely generated K-algebra with the set of generators $\{a_1, \ldots, a_n\}$. If, for some permutation $\tau = i_1 i_2 \cdots i_n$ of $1, 2, \ldots, n$, the set $\mathcal{B} = \{a^\alpha = a_{i_1}^{\alpha_1} \cdots a_{i_n}^{\alpha_n} \mid \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n\}$ forms a K-basis of A, then $\mathcal{B}$ is referred to as a *PBW K-basis* of A. It is clear that if A has a PBW K-basis, then we can always assume that $i_1 = 1, \ldots, i_n = n$. Thus, we make the following convention once for all.

Convention From now on in this paper, if we say that the algebra A has the PBW K-basis $\mathcal{B}$, then it always means that

$$\mathcal{B} = \{a^\alpha = a_1^{\alpha_1} \cdots a_n^{\alpha_n} \mid \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n\}.$$

Moreover, adopting the commonly used terminology in computational algebra, elements of $\mathcal{B}$ are referred to as *monomials* of A.

Suppose that A has the PBW K-basis $\mathcal{B}$ as presented above and that $\prec$ is a total ordering on $\mathcal{B}$. Then every nonzero element $f \in A$ has a unique expression $f = \lambda_1 a^{\alpha(1)} + \lambda_2 a^{\alpha(2)} + \cdots + \lambda_m a^{\alpha(m)}$ with $\lambda_j \in K^*$ and $a^{\alpha(j)} = a_1^{\alpha_{1j}} a_2^{\alpha_{2j}} \cdots a_n^{\alpha_{nj}} \in \mathcal{B}$, $1 \leq j \leq m$, in which $a^{\alpha(1)} \prec a^{\alpha(2)} \prec \cdots \prec a^{\alpha(m)}$. It follows that the *leading monomial*, the *leading coefficient*, and the *leading term* of f are respectively defined as $\mathbf{LM}(f) = a^{\alpha(m)}$, $\mathbf{LC}(f) = \lambda_m$, and $\mathbf{LT}(f) = \lambda_m a^{\alpha(m)}$.

Definition 1. *Suppose that the K-algebra $A = K[a_1, \ldots, a_n]$ has the PBW K-basis $\mathcal{B}$. If $\prec$ is a total ordering on $\mathcal{B}$ that satisfies the following three conditions:*

(1) *$\prec$ is a well-ordering;*

(2) *For any $a^\gamma, a^\alpha, a^\beta, a^\eta \in \mathcal{B}$, if $a^\alpha \prec a^\beta$ and $\mathbf{LM}(a^\gamma a^\alpha a^\eta)$, $\mathbf{LM}(a^\gamma a^\beta a^\eta) \notin K$, then $\mathbf{LM}(a^\gamma a^\alpha a^\eta) \prec \mathbf{LM}(a^\gamma a^\beta a^\eta)$;*

(3) *For any $a^\gamma, a^\alpha, a^\beta, a^\eta \in \mathcal{B}$, if $a^\beta \neq a^\gamma$, and $a^\gamma = \mathbf{LM}(a^\alpha a^\beta a^\eta)$, then $a^\beta \prec a^\gamma$ (thereby $1 \prec a^\gamma$ for all $a^\gamma \neq 1$),*

then $\prec$ is called a monomial ordering on $\mathcal{B}$ (or a monomial ordering on A).

Definition 2. *If the K-algebra $A = K[a_1, \ldots, a_n]$ satisfies the following two conditions:*

(S1) *A has the PBW K-basis $\mathcal{B}$;*

(S2) *There is a monomial ordering $\prec$ on $\mathcal{B}$ such that for all $a^\alpha = a_1^{\alpha_1} \cdots a_n^{\alpha_n}$, $a^\beta = a_1^{\beta_1} \cdots a_n^{\beta_n} \in \mathcal{B}$, $a^\alpha a^\beta = \lambda_{\alpha,\beta} a^{\alpha+\beta} + f_{\alpha,\beta}$, where $\lambda_{\alpha,\beta} \in K^*$, $a^{\alpha+\beta} = a_1^{\alpha_1+\beta_1} \cdots a_n^{\alpha_n+\beta_n}$, and either $f_{\alpha,\beta} = 0$ or $f_{\alpha,\beta} \in K$-span$\mathcal{B}$ with $\mathbf{LM}(f_{\alpha,\beta}) \prec a^{\alpha+\beta}$,*

then A is said to be a solvable polynomial algebra.

The results of the next proposition are summarized from ([8], Sections 2 – 5).

Proposition 2.1. *Let $A = K[a_1, \ldots, a_n]$ be a solvable polynomial algebra with the monomial ordering $\prec$ on the PBW K-basis $\mathcal{B}$ of A. The following statements hold.*

(i) *A is a (left and right) Noetherian domain.*

(ii) *Every left ideal I of A has a finite left Gröbner basis $\mathcal{G} = \{g_1, \ldots, g_t\}$ in the sense that if $0 \neq f \in I$, then there is some $g_i \in \mathcal{G}$ such that $\mathbf{LM}(g_i)|\mathbf{LM}(f)$, i.e., there is some $a^\gamma \in \mathcal{B}$ such that $\mathbf{LM}(f) = \mathbf{LM}(a^\gamma \mathbf{LM}(g_i))$, or equivalently, with $\gamma(i_j) = (\gamma_{i_{1j}}, \gamma_{i_{2j}}, \ldots, \gamma_{i_{nj}}) \in \mathbb{N}^n$, f has a left Gröbner representation:*

$$\begin{array}{rcl} f & = & \sum_{i,j} \lambda_{ij} a^{\gamma(i_j)} g_j, \text{ where } \lambda_{ij} \in K^*,\ a^{\gamma(i_j)} \in \mathcal{B},\ g_j \in \mathcal{G}, \\ & & \text{satisfying } \mathbf{LM}(a^{\gamma(i_j)} g_j) \preceq \mathbf{LM}(f) \text{ for all } (i,j). \end{array}$$

(iii) *The Buchberger's Algorithm, that computes a finite Gröbner basis for a finitely generated commutative polynomial ideal, has a complete noncommutative version that computes a finite left Gröbner basis for a finitely generated left ideal $I = \sum_{i=1}^m Af_i$ of A.*

(iv) *Similar results of (ii) and (iii) hold for right ideals and two-sided ideals of A.*

□

It follows from Definition 2 that the two conditions (S1) and (S2) satisfied by a solvable polynomial algebra $A = K[a_1, \ldots, a_n]$ are completely *independent* factors. To reach the main result of this note, we also recall from the literature a constructive result for getting PBW bases.

Let $K\langle X\rangle = K\langle X_1, \ldots, X_n\rangle$ be the free K-algebra on $X = \{X_1, \ldots, X_n\}$ and $\mathbb{B} = \{1,\ X_{i_1} \cdots X_{i_s} \mid X_{i_j} \in X,\ s \geq 1\}$ the standard K-basis of $K\langle X\rangle$. For convenience, we use capital letters $U, V, W, S, \ldots$ to denote elements (monomials) of $\mathbb{B}$. Recall that a monomial ordering $\prec_X$ on $\mathbb{B}$ is a well-ordering such that for any $W, U, V, S \in \mathbb{B}$, $U \prec_X V$ implies $WU \prec_X WV$, $US \prec_X VS$ (or equivalently, $WUS \prec_X WVS$); and moreover, if $U \neq V$, then $V = WUS$ implies $U \prec_X V$ (thereby $1 \prec_X W$ for all $1 \neq W \in \mathbb{B}$). Let I be an ideal of $K\langle X\rangle$ and $\mathcal{G} \subset I$. If, with respect to some monomial ordering $\prec_X$ on $\mathbb{B}$, $\langle \mathbf{LM}(I)\rangle = \langle \mathbf{LM}(\mathcal{G})\rangle$, then $\mathcal{G}$ is said to be a Gröbner basis of I, where $\langle \mathbf{LM}(I)\rangle$ and $\langle \mathbf{LM}(\mathcal{G})\rangle$ are respectively the ideals generated by the sets of leading monomials of I and $\mathcal{G}$. The reduced Gröbner basis of I is defined in a similar way as in the commutative case. Concerning the relation between Gröbner bases of I and the existence of a PBW K-basis for the quotient algebra $A = K\langle X\rangle/I$, the following result is a generalization of ([7], Proposition 2,14; [9], CH.III, Theorem 1.5).

Proposition 2.2. ([10], Ch.4, Theorem 3.1) *Let $A = K\langle X\rangle/I$ be as above. Suppose that I contains a subset of $\frac{n(n-1)}{2}$ elements*

$$G = \{g_{ji} = X_jX_i - F_{ji} \mid F_{ji} \in K\langle X\rangle,\ 1 \le i < j \le n\}$$

such that with respect to some monomial ordering $\prec_X$ on $\mathbb{B}$, $\mathbf{LM}(g_{ji}) = X_jX_i$ holds for all the g_{ji}, where $\mathbf{LM}(g_{ji})$ denotes the leading monomial of g_{ji} with respect to $\prec_X$. The following two statements are equivalent:

(i) *A has the PBW K-basis $\mathscr{B} = \{\overline{X}_1^{\alpha_1}\overline{X}_2^{\alpha_2}\cdots\overline{X}_n^{\alpha_n} \mid \alpha_j \in \mathbb{N}\}$, where each $\overline{X}_i$ denotes the coset of I represented by X_i in $K\langle X\rangle/I$.*

(ii) *Any subset $\mathcal{G}$ of I containing G is a Gröbner basis for I with respect to $\prec_X$.*

□

Remark 1. Obviously, Proposition 2.2 holds true if we use any permutation $\{X_{k_1}, \ldots, X_{k_n}\}$ of $\{X_1, \ldots, X_n\}$ (see an example given in the end of this note). So, in what follows we conventionally use only $\{X_1, \ldots, X_n\}$.

We note that if $G = \{g_{ji} = X_jX_i - F_{ji} \mid F_{ji} \in K\langle X\rangle,\ 1 \le i < j \le n\}$ is a Gröbner basis of the ideal I such that $\mathbf{LM}(g_{ji}) = X_jX_i$ for all the g_{ji}, then the *reduced Gröbner basis* of I is of the form

$$\mathcal{G} = \left\{ \begin{array}{rcl} g_{ji} & = & X_jX_i - \sum_q \mu_q^{ji} X_1^{\alpha_{1q}}X_2^{\alpha_{2q}}\cdots X_n^{\alpha_{nq}} \text{ with } \mu_q^{ji} \in K, \\ & & \text{and } \mathbf{LM}(g_{ji}) = X_jX_i,\ 1 \le i < j \le n \end{array} \right\}.$$

Bearing in mind Definition 2 and combining this fact, we are now able to present the main result of this note.

Theorem 2.1. *Let $A = K[a_1, \ldots, a_n]$ be a finitely generated algebra over the field K, and let $K\langle X\rangle = K\langle X_1, \ldots, X_n\rangle$ be the free K-algebras* *with the standard K-basis $\mathbb{B} = \{1,\ X_{i_1} \cdots X_{i_s} \mid X_{i_j} \in X,\ s \geq 1\}$. With notation as before, the following two statements are equivalent:*

(i) *A is a solvable polynomial algebra in the sense of Definition 2.*

(ii) *$A \cong \overline{A} = K\langle X\rangle / I$ via the K-algebra epimorphism π_1: $K\langle X\rangle \rightarrow A$ with $\pi_1(X_i) = a_i$, $1 \leq i \leq n$, $I = Ker\pi_1$, satisfying*

(a) *with respect to some monomial ordering $\prec_X$ on $\mathbb{B}$, the ideal I has a finite Gröbner basis G and the reduced Gröbner basis of I is of the form*

$$\mathcal{G} = \left\{ \begin{array}{l} g_{ji} = X_jX_i - \lambda_{ji}X_iX_j - F_{ji} \text{ with } \lambda_{ji} \in K^*, \\ \qquad F_{ji} = \sum_q \mu_q^{ji} X_1^{\alpha_{1q}} X_2^{\alpha_{2q}} \cdots X_n^{\alpha_{nq}}, \mu_q^{ji} \in K, \\ \qquad \text{and } \mathbf{LM}(g_{ji}) = X_jX_i,\ 1 \leq i < j \leq n \end{array} \right\},$$

thereby $\mathscr{B} = \{\overline{X}_1^{\alpha_1}\overline{X}_2^{\alpha_2} \cdots \overline{X}_n^{\alpha_n} \mid \alpha_j \in \mathbb{N}\}$ forms a PBW K-basis for $\overline{A}$, where each $\overline{X}_i$ denotes the coset of I represented by X_i in $\overline{A}$; and

(b) *there is a monomial ordering $\prec$ on $\mathscr{B}$ such that*

$$\mathbf{LM}(\overline{F}_{ji}) \prec \overline{X}_i\overline{X}_j \text{ whenever } \overline{F}_{ji} \neq 0,$$

where $\overline{F}_{ji} = \sum_q \mu_q^{ji} \overline{X}_1^{\alpha_{1i}} \overline{X}_2^{\alpha_{2i}} \cdots \overline{X}_n^{\alpha_{ni}}$, $1 \leq i < j \leq n$.

Proof. (i) $\Rightarrow$ (ii) Let $\mathcal{B} = \{a^\alpha = a_1^{\alpha_1} \cdots a_n^{\alpha_n} \mid \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n\}$ be the PBW K-basis of the solvable polynomial algebra A and $\prec$ a monomial ordering on $\mathcal{B}$. By Definition 2, the generators of A satisfy the relations:

$$a_ja_i = \lambda_{ji}a_ia_j + f_{ji}, \quad 1 \leq i < j \leq n, \tag{$*$}$$

where $\lambda_{ji} \in K^*$ and $f_{ji} = \sum_q \mu_q^{ji} a^{\alpha(q)} \in K$-span$\mathcal{B}$ with $\mathbf{LM}(f_{ji}) \prec a_ia_j$. Consider in the free K-algebra $K\langle X\rangle = K\langle X_1, \ldots, X_n\rangle$ the subset

$$\mathcal{G} = \{g_{ji} = X_jX_i - \lambda_{ji}X_iX_j - F_{ji} \mid 1 \leq i < j \leq n\},$$

where if $f_{ji} = \sum_q \mu_q^{ji} a_1^{\alpha_{1q}} \cdots a_n^{\alpha_{nq}}$, then $F_{ji} = \sum_q \mu_q^{ji} X_1^{\alpha_{1q}} \cdots X_n^{\alpha_{nq}}$ for $1 \leq i < j \leq n$. We write $J = \langle \mathcal{G} \rangle$ for the ideal of $K\langle X\rangle$ generated by $\mathcal{G}$ and put $\overline{A} = K\langle X\rangle / J$. Let π_1: $K\langle X\rangle \rightarrow A$ be the K-algebra epimorphism with $\pi_1(X_i) = a_i$, $1 \leq i \leq n$, and let π_2: $K\langle X\rangle \rightarrow \overline{A}$ be the canonical algebra epimorphism. It follows from the universal property of the canonical homomorphism that there is an algebra epimorphism φ: $\overline{A} \to A$ defined by $\varphi(\overline{X}_i) = a_i$, $1 \le i \le n$, such that the following diagram of algebra homomorphisms is commutative:

$$\begin{array}{ccc} K\langle X\rangle & \xrightarrow{\pi_2} & \overline{A} \\ {\scriptstyle\pi_1}\Big\downarrow & \swarrow_{\varphi} & \\ A & & \end{array} \qquad \varphi\circ\pi_2 = \pi_1\ .$$

On the other hand, by the definition of each g_{ji} we see that every element $\overline{H} \in \overline{A}$ may be written as $\overline{H} = \sum_j \mu_j \overline{X}_1^{\beta_{1j}}\overline{X}_2^{\beta_{2j}}\cdots\overline{X}_n^{\beta_{nj}}$ with $\mu_j \in K$ and $(\beta_{1j},\ldots,\beta_{nj}) \in \mathbb{N}^n$, where each $\overline{X}_i$ is the coset of J represented by X_i in $\overline{A}$. Noticing the relations presented in $(*)$, it is straightforward to check that the correspondence

$$\begin{array}{cccc} \psi: & A & \longrightarrow & \overline{A} \\ & \displaystyle\sum_i \lambda_i a_1^{\alpha_{1i}}\cdots a_n^{\alpha_{ni}} & \mapsto & \displaystyle\sum_i \lambda_i \overline{X}_1^{\alpha_{1i}}\cdots\overline{X}_n^{\alpha_{ni}} \end{array}$$

is an algebra homomorphism such that $\varphi\circ\psi = 1_A$ and $\psi\circ\varphi = 1_{\overline{A}}$, where 1_A and $1_{\overline{A}}$ denote the identity maps of A and $\overline{A}$ respectively. This shows that $A \cong \overline{A}$, thereby $\mathrm{Ker}\pi_1 = I = J$; moreover, $\mathscr{B} = \{\overline{X}_1^{\alpha_1}\overline{X}_2^{\alpha_2}\cdots\overline{X}_n^{\alpha_n} \mid \alpha_j \in \mathbb{N}\}$ forms a PBW K-basis for $\overline{A}$, and $\prec$ is a monomial ordering on $\mathscr{B}$.

We next show that $\mathcal{G}$ forms the reduced Gröbner basis for I as described in (a). To this end, we first show that the monomial ordering $\prec$ on $\mathcal{B}$ induces a monomial ordering $\prec_X$ on the standard K-basis $\mathbb{B}$ of $K\langle X\rangle$. For convenience, as before we use capital letters $U, V, W, S, \ldots$ to denote elements (monomials) in $\mathbb{B}$. We also fix a graded lexicographic ordering $\prec_{grlex}$ on $\mathbb{B}$ (with respect to a fixed positively weighted gradation of $K\langle X\rangle$) such that

$$X_1 \prec_{grlex} X_2 \prec_{grlex} \cdots \prec_{grlex} X_n.$$

Then, for $U, V \in \mathbb{B}$ we define

$$U \prec_X V \text{ if } \begin{cases} \mathbf{LM}(\pi_1(U)) \prec \mathbf{LM}(\pi_1(V)), \\ \text{or} \\ \mathbf{LM}(\pi_1(U)) = \mathbf{LM}(\pi_1(V)) \text{ and } U \prec_{grlex} V. \end{cases}$$

Since A is a domain (Proposition 2.1(i)) and π_1 is an algebra homomorphism with $\pi_1(X_i) = a_i$ for $1 \le i \le n$, it follows that $\mathbf{LM}(\pi_1(W)) \ne 0$ for all $W \in \mathbb{B}$. We also note from Definition 2 that if $f, g \in A$ are nonzero elements, then $\mathbf{LM}(fg) = \mathbf{LM}(\mathbf{LM}(f)\mathbf{LM}(g))$. Thus,

if $U, V, W \in \mathbb{B}$ and $U \prec_X V$ subject to $\mathbf{LM}(\pi_1(U)) \prec \mathbf{LM}(\pi_1(V))$, then

$$\begin{array}{rcl} \mathbf{LM}(\pi_1(WU)) & = & \mathbf{LM}(\mathbf{LM}(\pi_1(W))\mathbf{LM}(\pi_1(U))) \\ & \prec & \mathbf{LM}(\mathbf{LM}(\pi_1(W))\mathbf{LM}(\pi_1(V))) \\ & = & \mathbf{LM}(\pi_1(WV)) \end{array}$$

implies $WU \prec_X WV$;

if $U, V, W \in \mathbb{B}$ and $U \prec_X V$ subject to $\mathbf{LM}(\pi_1(U)) = \mathbf{LM}(\pi_1(V))$ and $U \prec_{grlex} V$, then

$$\begin{array}{rcl} \mathbf{LM}(\pi_1(WU)) & = & \mathbf{LM}(\mathbf{LM}(\pi_1(W))\mathbf{LM}(\pi_1(U))) \\ & = & \mathbf{LM}(\mathbf{LM}(\pi_1(W))\mathbf{LM}(\pi_1(V))) \\ & = & \mathbf{LM}(\pi_1(WV)), \end{array}$$

and $WU \prec_{grlex} WV$ implies $WU \prec_X WV$.

Similarly, if $U \prec_X V$, then $US \prec_X VS$ for all $S \in \mathbb{B}$. Moreover, if $W, U, V, S \in \mathbb{B}$, $W \ne V$, such that $W = UVS$, then $\mathbf{LM}(\pi_1(W)) = \mathbf{LM}(\pi_1(UVS))$ and clearly $V \prec_{grlex} W$, thereby $V \prec_X W$. Since $\prec$ is a well-ordering on $\mathcal{B}$ and $\prec_{grlex}$ is a well-ordering on $\mathbb{B}$, the above argument shows that $\prec_X$ is a monomial ordering on $\mathbb{B}$. With this monomial ordering $\prec_X$ in hand, by the definition of F_{ji} we see that $\mathbf{LM}(F_{ji}) \prec_X X_iX_j$. Furthermore, since $\mathbf{LM}(\pi_1(X_jX_i)) = a_ia_j = \mathbf{LM}(\pi_1(X_iX_j))$ and $X_iX_j \prec_{grlex} X_jX_i$, we see that $X_iX_j \prec_X X_jX_i$. It follows that $\mathbf{LM}(g_{ji}) = X_jX_i$ for $1 \le i < j \le n$. Now, by Proposition 2.2 we conclude that $\mathcal{G}$ forms a Gröbner basis for I with respect to $\prec_X$. Finally, by the definition of $\mathcal{G}$, it is clear that $\mathcal{G}$ is the reduced Gröbner basis of I with respect to $\prec_X$, as desired.

(ii) $\Rightarrow$ (i) Note that (a) + (b) tells us that the generators of $\overline{A}$ satisfy the relations $\overline{X}_j\overline{X}_i = \lambda_{ji}\overline{X}_i\overline{X}_j + \overline{F}_{ji}$, $1 \le i < j \le n$, and that if $\overline{F}_{ji} \neq 0$, then $\mathbf{LM}(\overline{F}_{ji}) \prec \overline{X}_i\overline{X}_j$ with respect to the given monomial ordering $\prec$ on $\mathscr{B}$. It follows that $\overline{A}$ and hence A is a solvable polynomial algebra in the sense of Definition 2. $\square$

Remark 2. The monomial ordering $\prec_X$ we defined in the proof of Theorem 2.1 is a modification of the *lexicographic extension* defined in [5]. But our definition of $\prec_X$ involves a graded monomial ordering $\prec_{grlex}$ on the standard K-basis $\mathbb{B}$ of the free K-algebra $K\langle X\rangle = K\langle X_1, \dots, X_n\rangle$. The reason is that *the monomial ordering $\prec_X$ on $\mathbb{B}$ must be compatible with the usual rule of division*, namely, $W, U, V, S \in \mathbb{B}$, $W \neq V$, and $W = UVS$ implies $V \prec_X W$. While it is clear that if we use any lexicographic ordering $\prec_{lex}$ in the definition of $\prec_X$, then this rule will not work in general.

We end this note by an example illustrating Theorem 2.1, in particular, illustrating that the monomial ordering $\prec_X$ used in the condition (a) and the monomial ordering $\prec$ used in the condition (b) may be mutually independent, namely $\prec$ may not necessarily be the restriction of $\prec_X$ on $\mathscr{B}$, and the choice of $\prec$ is indeed quite flexible.

Example 1. Considering the $\mathbb{N}$-graded structure of the free K-algebra $K\langle X\rangle = K\langle X_1, X_2, X_3\rangle$ by assigning X_1 the degree 2, X_2 the degree 1 and X_3 the degree 4, let I be the ideal of $K\langle X\rangle$ generated by the elements

$$\begin{aligned} g_1 &= X_1X_2 - X_2X_1, \\ g_2 &= X_3X_1 - \lambda X_1X_3 - \mu X_3X_2^2 - f(X_2), \\ g_3 &= X_3X_2 - X_2X_3, \end{aligned}$$

where $\lambda \in K^*$, $\mu \in K$, $f(X_2)$ is a polynomial in X_2 which has degree ≤ 6, or $f(X_2) = 0$. The following properties hold.

(1) If we use the graded lexicographic ordering $X_2 \prec_{grlex} X_1 \prec_{grlex} X_3$ on $K\langle X\rangle$, then the three generators have the leading monomials $\mathbf{LM}(g_1) = X_1X_2$, $\mathbf{LM}(g_2) = X_3X_1$, and $\mathbf{LM}(g_3) = X_3X_2$. It is

straightforward to verify that $\mathcal{G} = \{g_1, g_2, g_3\}$ forms a Gröbner basis for I.

(2) With respect to the fixed $\prec_{grlex}$ in (1), the reduced Gröbner basis $\mathcal{G}'$ of I consists of

$$\begin{aligned} g_1 &= X_1X_2 - X_2X_1, \\ g_2 &= X_3X_1 - \lambda X_1X_3 - \mu X_2^2X_3 - f(X_2), \\ g_3 &= X_3X_2 - X_2X_3, \end{aligned}$$

(3) Writing $A = K[a_1, a_2, a_3]$ for the quotient algebra $K\langle X\rangle/I$, where a_1, a_2 and a_3 denote the cosets $X_1 + I$, $X_2 + I$ and $X_3 + I$ in $K\langle X\rangle/I$ respectively, it follows that A has the PBW basis $\mathcal{B} = \{a^\alpha = a_2^{\alpha_2}a_1^{\alpha_1}a_3^{\alpha_3} \mid \alpha = (\alpha_2, \alpha_1, \alpha_3) \in \mathbb{N}^3\}$. Noticing that $a_2a_1 = a_1a_2$, it is clear that $\mathcal{B}' = \{a^\alpha = a_1^{\alpha_1}a_2^{\alpha_2}a_3^{\alpha_3} \mid \alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3\}$ is also a PBW basis for A. Since $a_3a_1 = \lambda a_1a_3 + \mu a_2^2a_3 + f(a_2)$, where $f(a_2) \in K$-span$\{1, a_2, a_2^2, \ldots, a_2^6\}$, we see that A has the monomial ordering $\prec_{lex}$ on $\mathcal{B}'$ such that $a_3 \prec_{lex} a_2 \prec_{lex} a_1$ and $\mathbf{LM}(\mu a_2^2a_3 + f(a_2)) \prec_{lex} a_1a_3$, thereby A is turned into a solvable polynomial algebra with respect to $\prec_{lex}$.

Moreover, one easily checks that if a_1 is assigned the degree 2, a_2 is assigned the degree 1 and a_3 is assigned the degree 4, then A has another monomial ordering on $\mathcal{B}'$, namely the graded lexicographic ordering $\prec_{grlex}$ such that $a_3 \prec_{grlex} a_2 \prec_{grlex} a_1$ and $\mathbf{LM}(\mu a_2^2a_3 + f(a_2)) \prec_{grlex} a_1a_3$, thereby A is turned into a solvable polynomial algebra with respect to $\prec_{grlex}$.

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