Computer Science Journal of Moldova, vol.**22**, no.1(64), 2014

Chromatic Polynomials Of Some $(m, l)-$Hyperwheels

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Abstract

In this paper, using a standard method of computing the chromatic polynomial of hypergraphs, we introduce a new reduction theorem which allows us to find explicit formulae for the chromatic polynomials of some (complete) non-uniform $(m, l)-$ hyperwheels and non-uniform $(m, l)-$hyperfans. These hypergraphs, constructed through a "join" graph operation, are some generalizations of the well-known wheel and fan graphs, respectively. Further, we revisit some results concerning these graphs and present their chromatic polynomials in a standard form that involves the Stirling numbers of the second kind.

Keywords: chromatic polynomial, hyperfan, hyperwheel, Stirling numbers.

1 Basic definitions and notations

For basic definitions of graphs and hypergraphs we refer the reader to [1, 4, 10, 17, 20]. A *hypergraph* $\mathcal{H}$ of order n is an ordered pair $\mathcal{H} = (X, \mathcal{E})$, where $|X| = n$ is a finite nonempty set of vertices and $\mathcal{E}$ is a collection of not necessarily distinct non empty subsets of X called *(hyper)edges*. In this paper, all hypergraphs discussed are considered *simple* and *Sperner*, i.e., they have distinct hyperedges and no hyperedge is a subset of another.

A hypergraph $\mathcal{H}$ is $r-$*uniform*, if $|e| = r$ for each $e \in \mathcal{E}$; otherwise, $\mathcal{H}$ is said to be *non-uniform*. In the case when $r = 2$, the resulting hypergraph is called a *graph* which is often defined by $H = (V, E)$. A hypergraph is said to be *linear* if each pair of hyperedges has at most one vertex in common. The *degree* of a vertex v, denoted by $d(v)$, is the number of hyperedges that contain v. Hypergraphs in this paper are assumed to be connected and linear unless stated otherwise.

Given a hypergraph $\mathcal{H} = (X, \mathcal{E})$, we define the *deletion* of e by $\mathcal{H} - e$, which is the hypergraph obtained from $\mathcal{H}$ by deleting some hyperedge $e \in \mathcal{E}$. The *contraction* of e defined by $\mathcal{H}.e$, is the hypergraph obtained from $\mathcal{H}$ by identifying all the vertices in e by a single vertex and removing e from $\mathcal{E}$ (clearing).

A hyperedge $e_1 \in \mathcal{E}$ is called a *hyperleaf* if there exists $e_2 \in \mathcal{E} - e_1$ such that $e \cap e_2 \subseteq e_1 \cap e_2$ for every $e \in \mathcal{E} - e_1$. In the case of linear hypergraphs, a hyperleaf is simply a hyperedge with exactly one vertex of degree greater than 1. If $P^l := v_1, e_1, v_2, e_2, \ldots, v_l, e_l, v_{l+1}$ denotes an alternating sequence of distinct hyperedges e_i and distinct intersecting vertices v_i, then $\mathcal{P}^l$ is called an $l-$*hyperpath*, for all $l \geq 1$. In the event $v_1 = v_{l+1}$ for all $l \geq 2$, the resulting hypergraph is called an $l-$*hypercycle* which we denote by $\mathcal{C}^l$. It causes no confusion to say that $\mathcal{C}^l$ is induced by the sequence of hyperedges $(e_1, e_2, \ldots, e_l)$, for all $l \geq 2$. We note that the term *elementary* hypercycle has also been used by Tomescu [15] to describe $l-$hypercycle and yet, for simplicity, we choose to use the former term. Moreover, we point out that a $2-$hypercycle induced by (e_1, e_2) when $2 < |e_1| \leq |e_2|$, is not linear, and in fact, is a $2-$hyperpath with $|e_1 \cap e_2| = 2$. For this paper, we do not make such a distinction in name, since it does not affect the results.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two hypergraphs. The *join* of $\mathcal{H}_1$ and $\mathcal{H}_2$, denoted by $\mathcal{H}_1 \vee \mathcal{H}_2$, is the hypergraph $\mathcal{H}$ whose vertex set is $X(\mathcal{H}) = X(\mathcal{H}_1) \cup X(\mathcal{H}_2)$, a disjoint union, and whose hyperedge set is $\mathcal{E}(\mathcal{H}) = \mathcal{E}(\mathcal{H}_1) \cup \mathcal{E}(\mathcal{H}_2) \cup \{x_1x_2 \mid x_1 \in X(\mathcal{H}_1),\ x_2 \in X(\mathcal{H}_2)\}$. For example, $\overline{K}_{n_1} \vee \overline{K}_{n_2} \vee \ldots \vee \overline{K}_{n_k} = K(n_1, n_2, \ldots, n_k)$ is a complete $k-$partite graph with part sizes $n_1, \ldots, n_k$. We denote a *wheel* graph by $W^l = C^l \vee v$, where C^l is a cycle on $l = n$ vertices. C^l is the *rim* of the wheel and the edges not in the rim are called *spokes*. We will call a wheel on l rim edges (or on $n+1$ vertices), an $l-$wheel, for short. For instance, when $l = 2$, a $2-$wheel graph is a cycle $C^3 \simeq K_3$; for this reason, it is customary to define a $3-$wheel instead. Although a wheel and a cycle are both traditionally defined on n vertices (see [20] for instance), we think it causes no confusion to substitute (where it is convenient) the number of vertices n for l, the number of edges. Further, the notation of W^l and C^l (instead of W_n and C_n, respectively) will be particularly important for us when handling hypergraphs. In each of the formula presented in this paper, one can easily replace l with the appropriate number of vertices by a simple substitution. We also denote the falling factorial $\lambda^{\underline{t}} = \lambda(\lambda-1)(\lambda-2)\dots(\lambda-t+1)$ with $\lambda^{\underline{0}} = 1$. Further, the Stirling number of the second kind is denoted by $\left\{ {n \atop k} \right\}$; it counts the number of partitions of a set of n elements into k nonempty subsets. Clearly $\left\{ {n \atop 0} \right\} = \left\{ {0 \atop n} \right\} = 0$ and $\left\{ {n \atop 1} \right\} = \left\{ {n \atop n} \right\} = \left\{ {0 \atop 0} \right\} = 1$. These notations and other combinatorial identities can be found in [12].

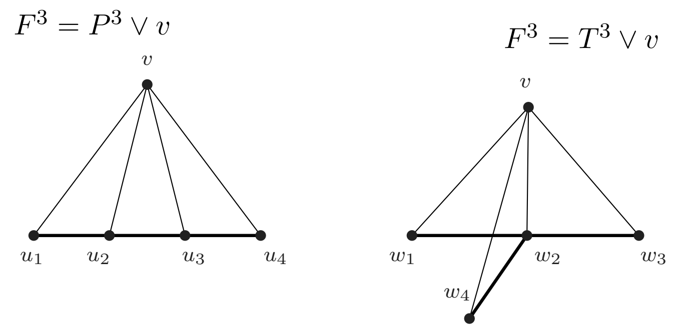
2 Chromatic polynomial of some graphs

The notion of coloring the vertices of a graph has been widely studied [10, 20]. A given graph G on n vertices can be properly colored in many different ways using a sufficiently large number of colors. This property of a graph is expressed elegantly by means of a polynomial. This polynomial is called the *chromatic polynomial* of G. It is well-known that Birkhoff [5] first introduced this polynomial in 1912 in an attempt to prove the four color theorem. The value of the chromatic polynomial $P(G, \lambda) = P(G)$ of a graph with n vertices gives the number of ways to properly color the graph G, using λ or fewer colors. For instance, the chromatic polynomials of a complete graph on n vertices, a tree, and a cycle with l edges are respectively given by $P(K_n) = \lambda^{\underline{n}}$, $P(T^l) = \lambda(\lambda-1)^l$ and $P(C^l) = (\lambda-1)^l + (-1)^l(\lambda-1)$.

The following theorem of Whitney [22], which gives the chromatic polynomial of a graph in terms of "broken circuits", is often used as a standard form; this form explicitly gives a basic property of the chromatic polynomial, namely, the powers of the chromatic polynomial are consecutive and their coefficients alternate in sign.

Theorem 2.1. *(Whitney's "Broken Circuits" Theorem). Let* $P(G, \lambda) = \lambda^n - a_1\lambda^{n-1} + a_2\lambda^{n-2} - \dots + (-1)^{n-1}a_{n-1}\lambda$. *The coefficient* a_i *is equal* *to the number of $i-$subsets of edges of the graph G which contain no broken cycles, for each $i = 1, 2, \ldots, n-1$.*

Although any chromatic polynomial can be written in this form, it is shown in [13] that chromatic polynomials written in terms of the Stirling numbers of the second kind have many applications. Still, we can always rewrite our results given in terms of Stirling numbers into a more standard basis, using the combinatorial identity (see [12] for instance) that $\left\{ {n \atop k} \right\} = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j} j^n$.



(a) Gem graph

(b) $\Theta(1,2,2,2)$ graph

Figure1. Two non-isomorphic fan graphs

Theorem 2.2. *Suppose $\mathcal{H}$ is a (hyper)graph. Then* $P(\mathcal{H} \vee \overline{K}_n, \lambda) = \sum_{i=0}^{n} \left\{ {\lambda \atop i} \right\} \lambda^{\underline{i}} P(\mathcal{H}, \lambda - i)$.

Proof. Recall that $\left\{ {\lambda \atop i} \right\}$ counts the number of ways of partioning n vertices into i distinct classes of colors. Now, there are $\lambda^{\underline{i}}$ ways to color the vertices of each class. Since a color used in a class cannot occur on any vertex in $\mathcal{H}$, the result follows. □

Using a similar argument, we derive an alternative version of the previous theorem as:

Theorem 2.3. *(Alternative Version) Suppose $\mathcal{H}^l$ is a hypergraph with l hyperedges. Then $P(\mathcal{H}^l \vee K_n, \lambda) = \lambda^{\underline{n}} P(\mathcal{H}^l, \lambda - n)$.*

Corollary 2.3.1. *Let $G = \mathcal{T}^l \vee K_n$. The chromatic polynomial of G is given by $P(G) = \lambda^{\underline{n}}(\lambda - n)(\lambda - n - 1)^l$.*

It is easy to verify that when $l = 1$ and $n = 2$, $P(G) = \lambda^{\underline{2}}(\lambda - 2)(\lambda - 3) = \lambda^{\underline{4}} = P(K_4)$.

Corollary 2.3.2. *The chromatic polynomial of a complete $k-$partite graph $G = K(n, 1 \ldots, 1)$ is $P(G) = \displaystyle\sum_{i=0}^{n} \left\{ \lambda \atop i \right\} \lambda^{\underline{i}}(\lambda - i)^{\underline{k-1}}$.*

Proof. Clearly $G \simeq \overline{K}_n \vee K_{k-1}$ and $P(K_{k-1}, \lambda) = \lambda^{\underline{k-1}}$. The result follows from Theorem 2.2. □

Observe that if we choose to count the proper colorings of the vertices of K_{k-1} first, we can easily establish an equivalent formula that $P(G) = \lambda^{\underline{k-1}}(\lambda - k + 1)^n$, a formula that is also supported by Theorem 2.3.

Corollary 2.3.3. *Suppose $G = K(n_1, n_2)$ is a complete bipartite graph. Then $P(G) = \displaystyle\sum_{i=0}^{n_1} \left\{ \lambda \atop i \right\} \lambda^{\underline{i}}(\lambda - i)^{n_2}$.*

Proof. This result also follows from Theorem 2.2, since $G \simeq \overline{K}_{n_1} \vee \overline{K}_{n_2}$. □

We define an $(m, l)-$*wheel* and $(m, l)-$*fan* graphs respectively by $W^{m,l} \simeq \overline{K}_m \vee C^l$ and $F^{m,l} \simeq \overline{K}_m \vee P^l$, where P^l is an $l-$path, C^l an $l-$cycle, and $\overline{K}_m$ is an empty graph on m vertices. We note that though P^l may not be isomorphic to T^l for $l \geq 3$, their chromatic polynomials are the same for all l. For this reason, it even makes sense to define an $(m, l)-$*fan* graph by $F^{m,l} \simeq \overline{K}_m \vee T^l$, where T^l is an $l-$tree. When $m = 1$, a $(1, l)-$*wheel* graph is simply the usual *wheel* graph and a

$(1,l)-$*fan* graph is simply called a *fan* graph (also known as a $2-$*tree* graph in [20]); in which case, we let $W^{1,l} = W^l$ and $F^{1,l} = F^l$. Thus, $(m,l)-$wheel and $(m,l)-$fan graphs are some generalizations of the ordinary wheel and fan graphs (Figure 1), for $m \geq 1$. In section 4, we provide further generalizations of these graphs to hypergraphs.

Corollary 2.3.4. *The chromatic polynomial of an $(m,l)-$wheel graph is given by* $P(W^{m,l}) = \sum_{i=0}^{m} \left\{ \begin{matrix} \lambda \\ i \end{matrix} \right\} \Big((\lambda - i - 1)^l + (-1)^l (\lambda - i - 1) \Big)$.

Corollary 2.3.5. *The chromatic polynomial of an $(m,l)-$fan graph is given by* $P(F^{m,l}) = \sum_{i=0}^{m} \left\{ \begin{matrix} \lambda \\ i \end{matrix} \right\} \lambda^{\underline{i}} (\lambda - i)(\lambda - i - 1)^l$.

Corollary 2.3.6. *The chromatic polynomial of an $l-$wheel graph is given by* $P(W^l) = \lambda((\lambda-2)^l + (-1)^l(\lambda-2))$.

Corollary 2.3.7. *The chromatic polynomial of a fan graph is* $P(F^l) = \lambda(\lambda-1)(\lambda-2)^l$.

3 Chromatic polynomial of some hypergraphs

In 1966 P. Erdös and A. Hajnal extended the notion of *proper coloring* of a graph to the coloring of a hypergraph [11]. Thus, the *chromatic polynomial* of a hypergraph $\mathcal{H}$, first denoted in [16] by $P(\mathcal{H}, \lambda) = P(\mathcal{H})$, is the function that counts the number of *proper* $\lambda-$*colorings*, which are mappings, $f \colon X \to \{1, 2, \ldots, \lambda\}$ with the condition that every hyperedge has at least two vertices with distinct colors. We encourage the reader to refer to [1, 6, 13, 14, 17] for detailed information about chromatic polynomials, research, and applications of hypergraph colorings.

This next theorem will be instrumental to streamline the arguments we make in several upcoming proofs for the remaining of this paper. Similar versions of this theorem can be found in [6, 19]. However, we think this particular version (Reduction Theorem) is unknown as it generalizes the Fundamental Reduction Theorem for graphs found in [10, 20] for instance.

Theorem 3.1. *(Reduction Theorem) Suppose $\mathcal{H}$ is a hypergraph with $l \geq 2$ hyperedges. Then $P(\mathcal{H}) = \lambda^{|e|-2}P(\mathcal{H}-e) - P(\mathcal{H}.e)$, where e is a hyperedge with exactly two vertices of degree 2 or greater.*

Proof. Note that if $|e| = 2$, then the relation satisfies the reduction for graphs.

Let u_1 and u_2 be the 2 vertices of degree 2 in e. In any proper coloring of the hyperedge e using λ colors, the following is true:

Either (i) u_1 and u_2 have the same color, or (ii) u_1 and u_2 have different colors. We therefore count the number of such colorings for each case in turn.

Case (i) There are $\lambda^{|e|-2}-1$ ways to color the vertices in $e\backslash\{u_1,u_2\}$ so that not all receive the same color, and there are $P(\mathcal{H}.e)$ ways to color the remaining vertices so that $f(u_1) = f(u_2)$ (to see this, delete e and identify u_1 and u_2). Hence, there are $(\lambda^{|e|-2}-1)P(\mathcal{H}.e)$ colorings.

Case (ii) There are $\lambda^{|e|-2}$ colorings of the vertices in $e\backslash\{u_1,u_2\}$. For each such coloring, the number of colorings of the remaining vertices is $P(\mathcal{H}-e)-P(\mathcal{H}.e)$, since the first term counts the number of colorings where u_1 and u_2 may have the same or different colors, and the second term counts the number of colors where u_1 and u_2 may have the same color. So there are $\lambda^{|e|-2}(P(\mathcal{H}-e)-P(\mathcal{H}.e))$ colorings altogether.

By combining (i) and (ii), we obtain the result for all $|e| \geq 2$.

□

Dohmen [8] extended Whitney's "Broken Circuits" Theorem to hypergraphs with the next proposition. It denotes by $n(\mathcal{H})$, the number of vertices of $\mathcal{H}$, $m(\mathcal{H})$, the number of hyperedges of $\mathcal{H}$ and by $c(\mathcal{H})$, the number of connected components of $\mathcal{H}$.

Proposition 3.1. *Let $\mathcal{H}$ be a hypergraph. Then*

$$P(\mathcal{H}) = \sum_{S\subseteq\mathcal{H}} (-1)^{m(S)}\lambda^{n(\mathcal{H})-n(S)+c(S)}. \tag{1}$$

Later, Tomescu [15] also presented a similar result using an inclusion-exclusion principle argument in

Lemma 3.1. *Let $\mathcal{H} = (X, \mathcal{E})$ be a connected hypergraph with $|X| = n$. Denote by $N(i,j)$ the number of spanning subhypergraphs of $\mathcal{H}$ with n vertices, i components, and j hyperedges. Then*

$$P(\mathcal{H}) = \sum_{i=1}^{n} a_i \lambda^i, \tag{2}$$

where $a_i = \sum_{j \geq 0} (-1)^j N(i,j)$.

We now present some known results (see [1, 6, 19] for instance), although in different forms. Our results are more in line with the standard form (2) as they can have some added benefits for further analysis, namely, finding generating functions. In addition, these results will not only help to illustrate the Reduction Theorem but also later serve the purpose of comparing the effect of a certain "join" operation on the chromatic polynomial of some hypergraphs.

The next theorem was first presented by Walter [19] as a generalization of Dohmen's result for $r-$uniform hypertree [9]. Though the proof can be obtained by a recursion using the Reduction Theorem, it is quite simple by an induction on $l \geq 1$.

Theorem 3.2. *If $\mathcal{T}^l = (X, \mathcal{E})$ is an $l-$hypertree, then $P(\mathcal{T}^l) = \lambda \prod_{i=1}^{l} (\lambda^{|e_i|-1} - 1)$, for all $l \geq 1$.*

Proof. When $l = 1$, it is clear that there are $\lambda^{|e|} - \lambda = \lambda(\lambda^{|e|-1} - 1)$ ways to color the vertices of a hyperedge so that not all of them have the same color. So, we assume $l \geq 2$. Let e_l be a hyperleaf (there is at least one since $\mathcal{T}^l$ is acyclic). Then, for each proper coloring of $\mathcal{T}^l - e_l$, there are exactly $\lambda^{|e_l|-1} - 1$ ways to properly color the remaining (pendant) vertices of e_l, giving $(\lambda^{|e_l|-1} - 1)P(\mathcal{T}^l - e_l)$ proper colorings. Since $P(\mathcal{T}^l - e_l) = P(\mathcal{T}^{l-1})$, the result follows from the inductive hypothesis. ☐

Theorem 3.3. *The chromatic polynomial of an $l-$hypercycle is given by*

$$P(\mathcal{C}^l) = \sum_{i=0}^{l-1}(-1)^i \lambda^{|e_{l-i}|-2} P(\mathcal{T}^{l-i-1}),\ \textit{with}\ P(\mathcal{T}^0) = \lambda(1-\lambda^{2-|e_1|}).$$

Proof. When $l = 2$, we apply the Reduction Theorem on e_2 to obtain that $P(\mathcal{C}^2) = \lambda^{|e_2|-2}P(\mathcal{T}^1) - P(\mathcal{T}^1_*)$, where $\mathcal{T}^1_*$ is a (loop) hyperedge on $|e_1| - 1$ vertices which chromatic polynomial is $\lambda^{|e_1|-1} - \lambda = \lambda^{|e_1|-2}P(\mathcal{T}^0)$.

Further, when $l = 3$, we have that $P(\mathcal{C}^3) = \lambda^{|e_3|-2}P(\mathcal{T}^2) - P(\mathcal{C}^2) = \lambda^{|e_3|-2}P(\mathcal{T}^2) - \lambda^{|e_2|-2}P(\mathcal{T}^1) + P(\mathcal{T}^1_*)$. Using the previous result and a simple recursion, we establish the formula for all $l \geq 2$, with $P(\mathcal{T}^0) = \lambda^{2-|e_1|}P(\mathcal{T}^1_*)$. □

Corollary 3.3.1. *The chromatic polynomial of an $l-$hypercycle is given by*

$$P(\mathcal{C}^l) = \lambda\Big(\sum_{i=0}^{l-2}(-1)^i\lambda^{|e_{l-i}|-2}\prod_{j=1}^{l-i-1}(\lambda^{|e_j|-1}-1) + (-1)^{l-1}(\lambda^{|e_1|-2} - 1)\Big),\ \textit{for all}\ l \geq 2.$$

Proof. This follows directly from Theorems 3.2 and 3.3. □

The following result follows when $|e| = r$, for each $e \in \mathcal{E}$.

Corollary 3.3.2. *The chromatic polynomial of any $r-$uniform $l-$hypercycle is given by*

$$P(\mathcal{C}^l_r) = \lambda\Big(\sum_{i=0}^{l-2}(-1)^i\lambda^{r-2}(\lambda^{r-1}-1)^{l-i-1} + (-1)^{l-1}(\lambda^{r-2}-1)\Big),\ \textit{for all}\ l \geq 2,\ r \geq 2.$$

The case when $r = 2$ follows as

Corollary 3.3.3. *The chromatic polynomial of any $l-$cycle is given by*

$$P(C^l) = \lambda\sum_{i=0}^{l-2}(-1)^i(\lambda-1)^{l-i-1},\ \textit{for all}\ l \geq 2.$$

We observe from this last result that we established the following:

$P(C^l) = (\lambda-1)^l + (-1)^l(\lambda-1) = \lambda\sum_{i=0}^{l-2}(-1)^i(\lambda-1)^{l-i-1}$, for all $l \geq 2$.

We now present some results on some new families of hypergraphs.

4 Chromatic polynomial of some $(m, l)-$hyperwheels

Suppose $\mathcal{C}^l = (X, \mathcal{E})$ is an $l-$hypercycle induced by the set of hyperedges $(e_1, \ldots, e_l)$ and let $\overline{K}_m \simeq \{v_1, \ldots, v_m\}$ be the empty graph on m vertices. We shall call $G_1 = \overline{K}_m \vee \mathcal{C}^l$, a *complete* $(m, l)-$*hyperwheel* and $G_2 = \overline{K}_m \vee \mathcal{P}^l$, a *complete* $(m, l)-$*hyperfan*, where $\mathcal{P}^l = \mathcal{C}^{l+1} - e$ for some hyperedge e. We call G_1 and G_2 "complete" in the sense of the "join" operation. For this reason, their chromatic polynomials are easily obtained from Theorems 2.2, 3.2 and Corollary 3.3.1; these findings are presented in the next two corollaries. Further, we found that removing some of the edges of G_1 (and G_2) yields more interesting families that are less "complete". These families are better generalizations of their graphs counterparts, namely the wheel and the fan graphs. They will be called $(m, l)-$hyperwheels and $(m, l)-$hyperfans. The chromatic polynomials of $(m, l)-$hyperwheels and $(m, l)-$hyperfans remain open for all $m \geq 2$. However, we present the chromatic polynomials of the particular case when $m = 1$, that we call $l-$hyperwheels ($l \geq 2$) and $l-$hyperfans ($l \geq 1$).

Corollary 4.0.4. *The chromatic polynomial of a complete* $(m, l)-$*hyperwheel is given by*

$$P(G_1, \lambda) = \sum_{k=0}^{m}\left\{\begin{matrix}\lambda\\ k\end{matrix}\right\}\lambda^{\underline{k}}(\lambda-k)\left(\sum_{i=0}^{l-2}(-1)^i(\lambda-k)^{|e_{l-i}|-2}\prod_{j=1}^{l-i-1}\Big((\lambda-k)^{|e_j|-1}-1\Big)+(-1)^{l-1}\Big((\lambda-k)^{|e_l|-2}-1\Big)\right),$$ *for all* $m \geq 1$, $l \geq 2$.

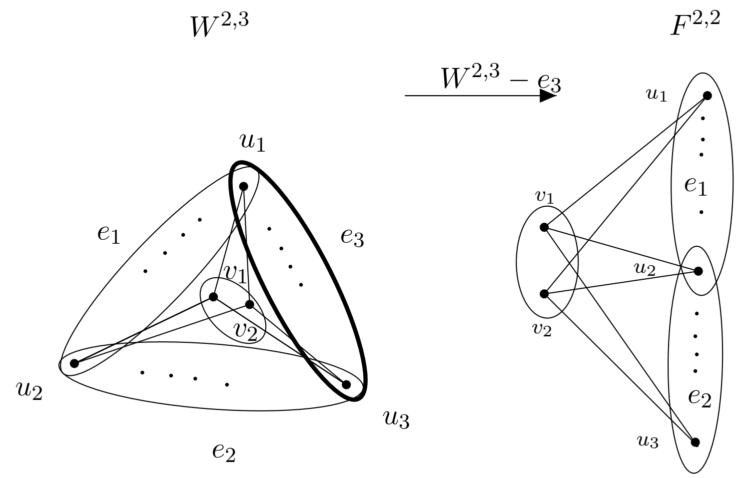


Figure 2. A $(2, 3)-$hyperwheel and a $(2, 2)-$hyperfan

Corollary 4.0.5. *The chromatic polynomial of a complete $(m, l)-$hyperfan is given by* $P(G_2, \lambda) = \sum_{k=0}^{m} \left\{ \begin{matrix} \lambda \\ k \end{matrix} \right\} \lambda^{\underline{k}} (\lambda - k) \prod_{j=1}^{l} \Big((\lambda - k)^{|e_j|-1} - 1 \Big)$, *for all $m \geq 1$, $l \geq 2$.*

Suppose $\mathcal{C}^l = (X, \mathcal{E})$ is an $l-$hypercycle induced by the set of hyperedges $(e_1, \ldots, e_l)$ and let $\overline{K}_m \simeq \{v_1, \ldots, v_m\}$ be the empty graph on m vertices. We define an $(m, l)-$*hyperwheel* by $\mathcal{W}^{m,l} \simeq (\overline{K}_m \vee \mathcal{C}^l) - \{uv | deg(u) = 1\}$ for each $u \in X$ and $v \in V(\overline{K}_m)$. Each edge $\{u, v\}$ is referred to as a *spoke* and its endpoints u and v are called *rim* and *apex* vertices respectively. The hyperedges e_i are referred to as *rim* hyperedges as well. Figure 2 contains an example of a $(2, 3)-$hyperwheel with rim hyperedges of size $|e_i|$, for $i = 1, 2, 3$.

An $(m, l)-$*hyperfan* is defined by $\mathcal{F}^{m,l} = \mathcal{W}^{m,l+1} - e$, where $\mathcal{W}^{m,l+1}$ is an $(m, l+1)-$hyperwheel and e is a rim hyperedge. In the case when $m = 1$, we write $\mathcal{W}^l$ (and $\mathcal{F}^l$) and call it an $l-$*hyperwheel* (and an

$l-$*hyperfan*). Figure 2 contains a representation of a $(2,2)-$hyperfan with rim hyperedges of size $|e_i|$, for $i = 1, 2$.

When $|e| = r$ for each rim hyperedge e, we denote an $(m,l)-$*hyperwheel* and an $(m,l)-$*hyperfan* respectively by $\mathcal{W}_r^{m,l}$ and $\mathcal{F}_r^{m,l}$. For instance $\mathcal{W}_2^l = W^l$, an $l-$wheel ($l \geq 2$) and $\mathcal{F}_2^l = F^l$, an $l-$fan ($l \geq 1$). A 1$-$hyperfan (with 2 spokes) is a 3$-$hypercycle (with one hyperedge and 2 edges).

Theorem 4.1. *The chromatic polynomial of an $l-$hyperfan is given by*

$$P(\mathcal{F}^l) = \lambda(\lambda-1)\prod_{i=1}^{l}(\lambda^{|e_i|-1} - \lambda^{|e_i|-2} - 1), \text{ for all } l \geq 1.$$

Proof. We proceed by induction on l, which is the number of rim hyperedges.

Note that when $l = 1$, $\mathcal{F}^1 = \mathcal{C}_*^3$, which is a 3$-$hypercycle with a single hyperedge that we denote by e_1. From Corollary 3.3.1, when $l = 3$, we have that $P(\mathcal{C}_*^3) = (\lambda-1)^2(\lambda^{|e_1|-1} - 1) + (-1)^3(\lambda-1) = \lambda(\lambda-1)(\lambda^{|e_1|-1} - \lambda^{|e_1|-2} - 1)$.

For $l \geq 2$, let $e_1, \ldots, e_l$ be the rim hyperedges. Let u_l and u_{l+1} be the two vertices of e_l that are incident to v, the apex vertex. We apply the Reduction Theorem on e_l to get that $P(\mathcal{F}^l) = \lambda^{|e_l|-2}P(\mathcal{F}^l - e_l) - P(\mathcal{F}^l.e_l)$. Now, $P(\mathcal{F}^l - e_l) = P(\mathcal{F}^{l-1} \cup \{vu_{l+1}\}) = (\lambda-1)P(\mathcal{F}^{l-1})$ and $P(\mathcal{F}^l.e_l) = P(\mathcal{F}^{l-1})$. Thus, we obtain the relation that

$P(\mathcal{F}^l) = \lambda^{|e_l|-2}(\lambda-1)P(\mathcal{F}^{l-1}) - P(\mathcal{F}^{l-1}) = P(\mathcal{F}^{l-1})(\lambda^{|e_l|-1} - \lambda^{|e_l|-2} - 1)$.

Further, using $P(\mathcal{F}^1)$ as the basis of the recursion, we have that $P(\mathcal{F}^l) = P(\mathcal{F}^1)\prod_{i=2}^{l}(\lambda^{|e_i|-1} - \lambda^{|e_i|-2} - 1)$.

The result follows, since $P(\mathcal{F}^1) = P(\mathcal{C}_*^3) = \lambda(\lambda-1)(\lambda^{|e_1|-1} - \lambda^{|e_1|-2} - 1)$.

$\square$

The following corollary is derived when each rim hyperedge of a hyperfan is of size $r \geq 2$.

Corollary 4.1.1. *The chromatic polynomial of an $l-$hyperfan with $r-$uniform rim hyperedges is* $P(\mathcal{F}_r^l) = \lambda(\lambda-1)(\lambda^{r-1}-\lambda^{r-2}-1)^l$, $l \geq 1$, $r \geq 2$.

We note that when $r = 2$, the previous formula coincides with that of Corollary 2.3.7.

Theorem 4.2. *Suppose $\mathcal{W}^l$ is a hyperwheel with rim hyperedges $e_1, e_2, \ldots, e_l$, $l \geq 2$. Then* $P(\mathcal{W}^l) = \sum_{i=0}^{l-1}(-1)^i \lambda^{|e_{l-i}|-2} P(\mathcal{F}^{l-i-1})$ *with* $P(\mathcal{F}^0) = \lambda(\lambda-1)(1-\lambda^{2-|e_1|})$.

Proof. Apply the Reduction Theorem on e_l, for $l \geq 2$, to obtain that $P(\mathcal{W}^l) = \lambda^{|e_l|-2}P(\mathcal{F}^{l-1}) - P(\mathcal{W}^{l-1})$. From this relation, when $l = 2$, we have that $P(\mathcal{W}^2) = \lambda^{|e_2|-2}P(\mathcal{F}^1) - P(\mathcal{W}^1)$, where $\mathcal{W}^1$ is a $2-$hyperpath with one hyperedge e_1 and one edge. Thus, $P(\mathcal{W}^1) = (\lambda-1)(\lambda^{|e_1|-1} - \lambda)$. Similarly, the result follows by a recursion with $P(\mathcal{F}^0) = \lambda^{2-|e_1|}(\lambda - 1)(\lambda^{|e_1|-1} - \lambda) = \lambda(\lambda-1)(1-\lambda^{2-|e_1|})$. □

Corollary 4.2.1. *The chromatic polynomial of an $l-$hyperwheel is given by*

$$P(\mathcal{W}^l) = \lambda(\lambda-1)\left(\sum_{i=0}^{l-2}(-1)^i \lambda^{|e_{l-i}|-2} \prod_{j=1}^{l-i-1}\left(\lambda^{|e_j|-1} - \lambda^{|e_j|-2} - 1\right) + (-1)^{l-1}(\lambda^{|e_1|-2} - 1)\right),$$

for all $l \geq 2$.

The following result follows when $|e| = r$, for each rim hyperedge e.

Corollary 4.2.2. *The chromatic polynomial of any $l-$hyperwheel with $r-$uniform rim hyperedges is given by*

$$P(\mathcal{W}_r^l) =$$

$$= \lambda(\lambda-1)\Big(\sum_{i=0}^{l-2}(-1)^i \lambda^{r-2}(\lambda^{r-1}-\lambda^{r-2}-1)^{l-i-1} + (-1)^{l-1}(\lambda^{r-2}-1)\Big),$$

for all $l \geq 2$, $r \geq 2$.

The case when $r = 2$ follows as

Corollary 4.2.3. *The chromatic polynomial of any $l-$wheel is given by*

$$P(W^l) = \lambda(\lambda-1)\sum_{i=0}^{l-2}(-1)^i(\lambda-2)^{l-i-1}, \text{ for all } l \geq 2.$$

We observe that from Corollary 2.3.6 and Corollary 4.2.3, together we have

$$P(W^l) = \lambda((\lambda-2)^l + (-1)^l(\lambda-2)) = \lambda(\lambda-1)\sum_{i=0}^{l-2}(-1)^i(\lambda-2)^{l-i-1},$$

for all $l \geq 2$.

5 Conclusion

The purpose of this paper, as it has been for related research on the topic of chromatic polynomials, is to derive the formulae for a number of graphs and hypergraphs with the goal to further classify them. We, once again, discovered that the process of finding these formulae is of a great computational complexity. Further work is needed to determine the efficiency of our proposed Reduction Theorem. Nonetheless, we derived some nice recursive relationships which we hope can serve the purpose of further analysis such as the finding of the roots and the meaning of the coefficients of these polynomials. Our focus has been primarily on graphs and hypergraphs which are obtained as a result of a "join" graph operation. This process and the resulting new class of hypergraphs discussed in this paper are certainly worth extending to other known multivariate polynomials such as the Tutte polynomial [2, 21].

Another variation of hypergraph coloring, is the concept of *mixed* hypergraph coloring, which has been studied extensively by Voloshin [16, 18]. A *mixed hypergraph* $\mathcal{H}$ with vertex set X, is a triple $(X, \mathcal{C}, \mathcal{D})$ such that $\mathcal{C}$ and $\mathcal{D}$ are subsets of X, called $\mathcal{C}-$hyperedges and $\mathcal{D}-$hyperedges, respectively. Given the mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, when $\mathcal{C} = \emptyset$, we write $\mathcal{H} = (X, \mathcal{D})$ and call it a

$\mathcal{D}-$hypergraph. This is the classical hypergraph discussed in this paper. It will be interesting to see these results extended to mixed hypergraphs since the chromatic polynomial of mixed $l-$hypercyles has recently been found [3].

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