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Computing the Pareto-Nash equilibrium set in finite multi-objective mixed-strategy games

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Abstract

The Pareto-Nash equilibrium set (PNES) is described as intersection of graphs of efficient response mappings. The problem of PNES computing in finite multi-objective mixed-strategy games (Pareto-Nash games) is considered. A method for PNES computing is studied.

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1 Introduction

The Pareto-Nash equilibrium set (PNES) may be determined via intersection of graphs of efficient response mappings — an approach which may be considered a generalization of the earlier works [14, 15, 16, 17, 7, 18] and the method initiated by Ungureanu in [16] for Nash equilibrium set (NES) computing in finite mixed-strategy games. By applying the same approach, the method of PNES computing in finite mixed-strategy n-player multi-objective games is constructed.

Consider a finite multi-objective strategic game:

$$\Gamma = \langle \mathbf{N}, \{\mathbf{S_p}\}_{p\in\mathbf{N}}, \{\mathbf{u_p}(\mathbf{s})\}_{p\in\mathbf{N}} \rangle,$$

where

- $\mathbf{N} = \{1, 2, ..., n\}$ is a set of players;
- $\mathbf{S_p} = \{1, 2, ..., m_p\}$ is a set of strategies of player $p \in \mathbf{N}$;
- $\mathbf{u_p}(\mathbf{s}) : \mathbf{S} \mapsto \mathbf{R^{k_p}}$, $\mathbf{u_p}(\mathbf{s}) = \left(u_p^1(\mathbf{s}), u_p^2(\mathbf{s}), ..., u_p^{k_p}(\mathbf{s})\right)$ is the utility vector-function of the player $p \in \mathbf{N}$;
- $\mathbf{s} = (s_1, s_2, \ldots, s_n) \in \mathbf{S} = \underset{\mathbf{p} \in \mathbf{N}}{\times} \mathbf{S_p}$, where $\mathbf{S}$ is the set of profiles;
- $k_p, m_p < +\infty$, $p \in \mathbf{N}$.

Let us associate with the utility vector-function $\mathbf{u_p}(\mathbf{s}), p \in \mathbf{N}$, its matrix representation

$$\mathbf{u_p}(\mathbf{s}) = \mathbf{A^p_s} = \left[a^{pi}_{s_1 s_2 \ldots s_n}\right]^{i=1,\ldots,k_p}_{\mathbf{s} \in \mathbf{S}} \in \mathbf{R^{k_p \times m_1 \times m_2 \times \cdots \times m_n}}.$$

The pure-strategy multi-criteria game defines in an evident manner a mixed-strategy multi-criteria game:

$$\Gamma' = \langle \mathbf{N}, \{\mathbf{X_p}\}_{p \in \mathbf{N}}, \{\mathbf{f_p}(\mathbf{x})\}_{\mathbf{p} \in \mathbf{N}} \rangle,$$

where

- $\mathbf{X_p} = \{\mathbf{x^p} \in \mathbf{R}^{\mathbf{m_p}}_{\geq} : x_1^p + x_2^p + \cdots + x_{m_p}^p = 1\}$ is a set of mixed strategies of player $p \in \mathbf{N}$;
- $\mathbf{f_p}(\mathbf{x}) : \mathbf{X} \mapsto \mathbf{R^{k_p}}$, $\mathbf{f_p}(\mathbf{x}) = \left(f_p^1(\mathbf{x}), f_p^2(\mathbf{x}), ..., f_p^{k_p}(\mathbf{x})\right)$ is the utility vector-function of the player $p \in \mathbf{N}$ defined on the Cartesian product $\mathbf{X} = \underset{\mathbf{p} \in \mathbf{N}}{\times} \mathbf{X_p}$ and

$$f_p^i(\mathbf{x}) = \sum_{s_1=1}^{m_1} \sum_{s_2=1}^{m_2} \ldots \sum_{s_n=1}^{m_n} a^{pi}_{s_1 s_2 \ldots s_n} x^1_{s_1} x^2_{s_2} \ldots x^n_{s_n}.$$

Remark that each player has to solve solely the multi-criteria parametric optimization problem, where the parameters are strategic choices of the other players.

Definition 1. *Strategy* $\mathbf{x^p} \in \mathbf{X_{-p}}$ *is "better" than* $\mathbf{y^p} \in \mathbf{X_{-p}}$ *if*

$$\mathbf{f_p}(\mathbf{x^p}, \mathbf{x^{-p}}) \geq \mathbf{f_p}(\mathbf{y^p}, \mathbf{x^{-p}}), \forall \mathbf{x^{-p}} \in \mathbf{X_{-p}}$$

and there exist an index $i \in \{1, ..., k_p\}$ *and a joint strategy* $\mathbf{x^{-p}} \in \mathbf{X_{-p}}$ *for which*

$$f_p^i(\mathbf{x^p}, \mathbf{x^{-p}}) > f_p^i(\mathbf{y^p}, \mathbf{x^{-p}}).$$

The defined relationship is denoted $\mathbf{x^p} \succeq \mathbf{y^p}$.

Player problem. The player p selects from his set of strategies the strategy $\hat{\mathbf{x}}^\mathbf{p} \in \mathbf{X_p}$, $p \in \mathbf{N}$, for which every component of the utility vector-function $\mathbf{f_p}(\mathbf{x^p}, \hat{\mathbf{x}}^{-\mathbf{p}})$ has a maximum possible value.

2 Pareto optimality

Definition 2. *Strategy* $\hat{\mathbf{x}}^\mathbf{p}$ *is named efficient (optimal in the sense of Pareto [11]), if there does not exist other strategy* $\mathbf{x^p} \in \mathbf{X_p}$ *so that* $\mathbf{x^p} \succeq \hat{\mathbf{x}}^\mathbf{p}$.

Let us denote the set of efficient strategies (solutions) of the player p by $\mathbf{ef\, X_p}$. Any two efficient strategies are equivalent or incomparable.

Theorem 1. *If the sets* $\mathbf{X_p} \subseteq \mathbf{R^{k_p}}$, $p = \overline{1, n}$, *are compact and the cost functions are continuous* ($f_p^i(\mathbf{x}) \in C(\mathbf{X_p})$, $i = \overline{1, m_p}$, $p = \overline{1, n}$), *then the sets* $\mathbf{ef\, X_p}$, $p = \overline{1, n}$, *are non empty.*

The proof follows from the known results [4].

Theorem 2. *Every element* $\hat{\mathbf{x}} = (\hat{x}^1, \hat{x}^2, ..., \hat{x}^n) \in \mathbf{ef\, X} = \underset{\mathbf{p} \in \mathbf{N}}{\times} \mathbf{ef\, X_p}$ *is efficient.*

The proof follows from the definition of efficient strategy.

3 Pareto-Nash equilibrium

Definition 3. *The outcome* $\hat{\mathbf{x}} \in \mathbf{X}$ *of the game is Pareto-Nash equilibrium [1, 2, 12] if*

$$\mathbf{f_p}(\mathbf{x^p}, \hat{\mathbf{x}}^{-\mathbf{p}}) \le \mathbf{f_p}(\hat{\mathbf{x}}^{\mathbf{p}}, \hat{\mathbf{x}}^{-\mathbf{p}}), \forall \mathbf{x^p} \in \mathbf{X_p}, \forall \mathbf{p} \in \mathbf{N},$$

where

$$\hat{\mathbf{x}}^{-\mathbf{p}} = (\hat{x}^1, \hat{x}^2, ..., \hat{x}^{p-1}, \hat{x}^{p+1}, ..., \hat{x}^n),$$

$$\hat{\mathbf{x}}^{-\mathbf{p}} \in \mathbf{X_{-p}} = \mathbf{X_1} \times \mathbf{X_2} \times ... \times \mathbf{X_{p-1}} \times \mathbf{X_{p+1}} \times ... \times \mathbf{X_n},$$

$$\hat{\mathbf{x}} = \hat{\mathbf{x}}^{\mathbf{p}} \parallel \hat{\mathbf{x}}^{-\mathbf{p}} = (\hat{\mathbf{x}}^{\mathbf{p}}, \hat{\mathbf{x}}^{-\mathbf{p}}) = (\hat{x}^1, \hat{x}^2, ..., \hat{x}^{p-1}, \hat{x}^p, \hat{x}^{p+1}, ..., \hat{x}^n) \in \mathbf{X}.$$

It is well known that not all the games in pure strategies have **PNE**, but all the extended games Γ' have **PNE**. The proof based on scalarization technique is presented below. The same scalarization technique may serve as a bases for diverse alternative formulations of a **PNE**, as well as for NE: as a fixed point of the efficient response correspondence, as a fixed point of a synthesis sum of functions, as a solution of a nonlinear complementarity problem, as a solution of a stationary point problem, as a maximum of a synthesis sum of functions on a polytope, as a semi-algebraic set. The **PNES** may be considered as well as an intersection of graphs of efficient response multi-valued mappings [17, 7]:

$$\underset{\mathbf{x^p} \in \mathbf{X_p}}{\operatorname{Arg\,ef\,max}}\, \mathbf{f_p}(\mathbf{x^p}, \mathbf{x^{-p}}) : \mathbf{X_{-p}} \to \mathbf{X_p}, p = \overline{1, n} :$$

$$\mathbf{PNE}(\mathbf{\Gamma'}) = \bigcap_{\mathbf{p} \in \mathbf{N}} \mathbf{Gre_p},$$

$$\mathbf{Gre_p} = \left\{ (\mathbf{x^p}, \mathbf{x^{-p}}) \in \mathbf{X} : \begin{array}{l} \mathbf{x^{-p}} \in \mathbf{X_{-p}}, \\ \mathbf{x^p} \in \underset{\mathbf{x^p} \in \mathbf{X_p}}{\operatorname{Arg\,ef\,max}}\, \mathbf{f_p}(\mathbf{x^p}, \mathbf{x^{-p}}) \end{array} \right\}.$$

The problem of **PNES** determination in the mixed extension of two-person game was studied in [7]. In this paper a method for

PNES computing in two-matrix mixed extended games and multi-matrix mixed extended games is analysed and the method for its computing is proposed.

The complexity of the problem of **PNES** may be established on the bases of the problem of NE computing. Let us remember that according to [13]: *"The computational complexity of finding one equilibrium is unclear... Gilboa and Zemel [5] show that finding an equilibrium of a bi-matrix game with maximum payoff sum is NP-hard, so for this problem no efficient algorithm is likely to exist. The same holds for other problems that amount essentially to examining all equilibria, like finding an equilibrium with maximum support"*. Consequently, the problem of Pareto-Nash equilibria set computing has at least such complexity as the problem of NE computing. Recently, in [3] the fact that the problem of NE computing in two-player game is PPAD-complete was established(PPAD is an abbreviation for Polynomial Parity Argument for Directed graphs [10]). The hardness of the problem of computing Nash equilibria in a two-player normal form (bimatrix) game was established in [6], too, from the perspective of parameterized complexity. **These facts enforce conclusion that the problem of computing PNES is computationally very hard (unless P = NP).**

As it is easy to see, the algorithm for **PNES** computing in multi-matrix mixed-strategy games solves, particularly, the problem of **PNES** computing in $m \times n$ mixed-strategy games. But, two-matrix game has peculiar features that permits to give a more expedient algorithm. Examples have to give the reader the opportunity to easy and prompt grasp of the paper.

4 Scalarization Technique

The solution of multi-criteria problem may be found by applying the scalarization technique (weighted sum method), which may interpret the weighted sum of the player utility functions as the unique utility (synthesis) function of the player p ($p = \overline{1, n}$):

$$F_p(\mathbf{x}, \lambda^{\mathbf{p}}) = \lambda_1^p \sum_{s_1=1}^{m_1} \sum_{s_2=1}^{m_2} ... \sum_{s_n=1}^{m_n} a^{p1}_{s_1 s_2 ... s_n} x^1_{s_1} x^2_{s_2} ... x^n_{s_n} + ... +$$
$$+ \lambda_{k_p}^p \sum_{s_1=1}^{m_1} \sum_{s_2=1}^{m_2} ... \sum_{s_n=1}^{m_n} a^{pk_p}_{s_1 s_2 ... s_n} x^1_{s_1} x^2_{s_2} ... x^n_{s_n},$$

$$\mathbf{x^p} \in \mathbf{X_p},$$
$$\lambda^{\mathbf{p}} = (\lambda_1^p, \lambda_2^p, \ldots, \lambda_{k_p}^p) \in \mathbf{\Lambda_p},$$
$$\mathbf{\Lambda_p} = \left\{ \lambda^{\mathbf{p}} \in \mathbf{R^{k_p}} : \begin{array}{l} \lambda_1^p + \lambda_2^p + \cdots + \lambda_{k_p}^p = 1, \\ \lambda_i^p \geq 0, i = \overline{1, k_p}, \end{array} \right\},$$
$$p = \overline{1, n}.$$

Theorem 3. *Let* $\mathbf{x^{-p}} \in \mathbf{X_{-p}}$.

1. *If* $\hat{\mathbf{x}}^{\mathbf{p}}$ *is the solution of mono-criterion problem* $\max\limits_{\mathbf{x^p} \in \mathbf{X_p}} F_p(\mathbf{x}, \lambda^{\mathbf{p}})$, *for some* $\lambda^{\mathbf{p}} \in \mathbf{\Lambda_p}, \lambda^{\mathbf{p}} > \mathbf{0}$, *then* $\hat{\mathbf{x}}^{\mathbf{p}}$ *is the efficient point for player* $p \in \mathbf{N}$ *for the fixed* $\mathbf{x^{-p}}$.
2. *The solution* $\hat{\mathbf{x}}^{\mathbf{p}}$ *of problem* $\max\limits_{\mathbf{x^p} \in \mathbf{X_p}} F_p(\mathbf{x}, \lambda^{\mathbf{p}})$, *with* $\lambda^{\mathbf{p}} \geq \mathbf{0}$, $\mathrm{p} \in \mathbf{N}$ *is efficient point for player* $p \in \mathbf{N}$, *if it is unique.*

Theorem's proof follows from the sufficient Pareto condition with linear synthesis function [4].

Let us define the mono-criteria game

$$\Gamma''(\lambda^{\mathbf{1}}, \lambda^{\mathbf{2}}, ..., \lambda^{\mathbf{n}}) = \langle \mathbf{N}, \{\mathbf{X_p}\}_{p \in \mathbf{N}}, \{F_p(\mathbf{x}, \lambda^{\mathbf{p}})\}_{p \in \mathbf{N}} \rangle,$$

where

- $\lambda^{\mathbf{p}} \in \mathbf{\Lambda_p}$, $p \in \mathbf{N}$,
- $\mathbf{X_p} = \{\mathbf{x^p} \in \mathbf{R}^{\mathbf{m_p}}_{\geq} : x_1^p + x_2^p + \cdots + x_{m_p}^p = 1\}$ is a set of mixed strategies of player $p \in \mathbf{N}$;

- $F_p(\mathbf{x}, \lambda^{\mathbf{p}}) \,:\, \mathbf{X} \mapsto \mathbf{R}^{\mathbf{k_p}}$, is the utility synthesis function of the player $p \in \mathbf{N}$, described above, defined on $\mathbf{X}$ and $\mathbf{\Lambda_p}$.

For simplicity, let us introduce the notations:

$$\Gamma''(\lambda) = \Gamma''(\lambda^{\mathbf{1}}, \lambda^{\mathbf{2}}, ..., \lambda^{\mathbf{n}}),$$

$$\lambda = (\lambda^{\mathbf{1}}, \lambda^{\mathbf{2}}, ..., \lambda^{\mathbf{n}}) \in \mathbf{\Lambda} = \mathbf{\Lambda_1} \times \mathbf{\Lambda_2} \times ... \times \mathbf{\Lambda_n}.$$

Evidently, the game $\Gamma''(\lambda)$ represents a multi-matrix mixed-strategy mono-criterion game for a fixed $\lambda \in \mathbf{\Lambda}$. It's very well known that such a game has NE [9]. Consequently, from this well known result and from the precedent theorem the next theorems follow.

Theorem 4. *The outcome $\hat{\mathbf{x}} \in \mathbf{X}$ is a PNE in Γ' if and only if there exists such a $\lambda \in \mathbf{\Lambda}, \lambda > \mathbf{0}, \mathbf{p} \in \mathbf{N}$ for which $\hat{\mathbf{x}} \in \mathbf{X}$ is a NE in $\Gamma''(\lambda)$.*

Theorem 5. $\mathbf{PNES}(\mathbf{\Gamma'}) = \bigcup_{\lambda \in \mathbf{\Lambda}, \lambda > \mathbf{0}} \mathbf{NES}(\mathbf{\Gamma''}(\lambda)) \neq \emptyset.$

Let us denote the graphs of best response mappings

$$\operatorname{Arg}\max_{\mathbf{x^p} \in \mathbf{X_p}} F_p(\mathbf{x^p}, \mathbf{x^{-p}}, \lambda^{\mathbf{p}}) : \mathbf{X_{-p}} \to \mathbf{X_p}, p = \overline{1, n},$$

by

$$\mathbf{Gr_p}(\lambda^{\mathbf{p}}) = \left\{ (\mathbf{x^p}, \mathbf{x^{-p}}) \in \mathbf{X} : \begin{array}{l} \mathbf{x^{-p}} \in \mathbf{X_{-p}}, \\ \mathbf{x^p} \in \operatorname{Arg}\max\limits_{\mathbf{x^p} \in \mathbf{X_p}} F_p(\mathbf{x^p}, \mathbf{x^{-p}}, \lambda^{\mathbf{p}}) \end{array} \right\},$$

$$\mathbf{Gr_p} = \bigcup_{\lambda^{\mathbf{p}} \in \mathbf{\Lambda_p}, \lambda^{\mathbf{p}} > \mathbf{0}} \mathbf{Gr_p}(\lambda^{\mathbf{p}}).$$

From the above, we are able to establish the truth of the next theorem, which permits us to compute the **PNES** in Γ'.

Theorem 6. $\mathbf{PNES} = \mathbf{PNES}(\mathbf{\Gamma'}) = \bigcap_{\mathbf{p}=\mathbf{1}}^{\mathbf{n}} \mathbf{Gr_p}.$

5 PNES in two-player mixed-strategy games

Consider a two-player $m \times n$ game Γ with matrices:

$$A^q = (a^q_{ij}), B^r = (b^r_{ij}), i = \overline{1,m}, j = \overline{1,n}, q = \overline{1,k_1}, r = \overline{1,k_2}.$$

Let $A^{iq}, i = \overline{1,m}, q = \overline{1,k_1}$ denote the lines of matrices $A^q, q = \overline{1,k_1}$,
$b^{jr}, j = \overline{1,n}, r = \overline{1,k_2}$, denote the columns of matrices $B^r, r = \overline{1,k_2}$,

$$\mathbf{X} = \{\mathbf{x} \in \mathbf{R}^{\mathbf{m}}_{\geq} : x_1 + x_2 + \cdots + x_m = 1\},$$

$$\mathbf{Y} = \{\mathbf{y} \in \mathbf{R}^{\mathbf{n}}_{\geq} : y_1 + y_2 + \cdots + y_n = 1\}.$$

As above, we consider the mixed-strategy game Γ' and the game $\Gamma''(\lambda^{\mathbf{1}}, \lambda^{\mathbf{2}})$ with synthesis functions of the players:

$$\begin{aligned}
F_1(\mathbf{x}, \mathbf{y}, \lambda^{\mathbf{1}}) &= \lambda^1_1 \sum_{i=1}^{m}\sum_{j=1}^{n} a^1_{ij} x_i y_j + \cdots + \lambda^1_{k_1} \sum_{i=1}^{m}\sum_{j=1}^{n} a^{k_1}_{ij} x_i y_j = \\
&= \left[\left(\lambda^1_1 A^{11} + \lambda^1_2 A^{12} + \cdots + \lambda^1_{k_1} A^{1k_1}\right)\mathbf{y}\right] x_1 + \cdots + \\
&+ \left[\left(\lambda^1_1 A^{m1} + \lambda^1_2 A^{m2} + \cdots + \lambda^1_{k_1} A^{mk_1}\right)\mathbf{y}\right] x_m,
\end{aligned}$$

$$\begin{aligned}
F_2(\mathbf{x}, \mathbf{y}, \lambda^{\mathbf{2}}) &= \lambda^2_1 \sum_{i=1}^{m}\sum_{j=1}^{n} b^1_{ij} x_i y_j + \cdots + \lambda^2_{k_2} \sum_{i=1}^{m}\sum_{j=1}^{n} b^{k_2}_{ij} x_i y_j = \\
&= \left[\mathbf{x}^T\left(\lambda^2_1 b^{11} + \lambda^2_2 b^{12} + \cdots + \lambda^2_{k_2} b^{1k_2}\right)\right] y_1 + \cdots + \\
&+ \left[\mathbf{x}^T\left(\lambda^2_1 b^{n1} + \lambda^2_2 b^{n2} + \cdots + \lambda^2_{k_2} b^{nk_2}\right)\right] y_n,
\end{aligned}$$

$$\lambda^1_1 + \lambda^1_2 + \cdots + \lambda^1_{k_1} = 1, \lambda^1_q \geq 0, q = 1, ..., k_1,$$

$$\lambda^2_1 + \lambda^2_2 + \cdots + \lambda^2_{k_2} = 1. \lambda^2_r \geq 0, r = 1, ..., k_2.$$

The game $\Gamma'' = \langle \mathbf{X}, \mathbf{Y}; F_1, F_2 \rangle$ is a scalarization of the mixed-strategy multi-criteria two-player game Γ'.

If the strategy of the second player is fixed, then the first player has to solve a linear programming parametric problem:

$$F_1(\mathbf{x}, \mathbf{y}, \lambda^{\mathbf{1}}) \to max, \mathbf{x} \in \mathbf{X}, \tag{1}$$

where $\lambda^{\mathbf{1}} \in \mathbf{\Lambda_1}$ and $\mathbf{y} \in \mathbf{Y}$.

Analogically, the second player has to solve the linear programming parametric problem:

$$F_2(\mathbf{x}, \mathbf{y}, \lambda^{\mathbf{2}}) \to max, \mathbf{y} \in \mathbf{Y}, \tag{2}$$

with the parameter-vector $\lambda^{\mathbf{2}} \in \mathbf{\Lambda_2}$ and $\mathbf{x} \in \mathbf{X}$.

Denote $\mathbf{ex}^T = (1, \ldots, 1) \in \mathbf{R^m}, \mathbf{ey}^T = (1, \ldots, 1) \in \mathbf{R^n}$. The solution of linear programming problem is realized on the vertices of polytopes of feasible solutions. In the problems (1) and (2) the sets $\mathbf{X}$ and $\mathbf{Y}$ have m and, respectively, n vertices — the axis unit vectors $\mathbf{e}^{x_i} \in \mathbf{R^m}, i = \overline{1,m}$ and $\mathbf{e}^{y_j} \in \mathbf{R^n}, j = \overline{1,n}$. Thus, in accordance with the simplex method and its optimality criterion, in the parametric problem (1) the parameter set $\mathbf{Y}$ is partitioned in such m subsets

$$Y^i(\lambda^{\mathbf{1}}) = \left\{ \mathbf{y} \in \mathbf{R^n} : \begin{array}{l} \left(\sum\limits_{q=1}^{k_1} \lambda_q^1 (A^{kq} - A^{iq}) \right) \mathbf{y} \le 0, \\ k = \overline{1,m}, \\ \lambda_1^1 + \lambda_2^1 + \cdots + \lambda_{k_1}^1 = 1, \ \lambda^{\mathbf{1}} > 0, \\ \mathbf{ey}^T \mathbf{y} = 1, \\ \mathbf{y} \ge \mathbf{0}. \end{array} \right\}, i = \overline{1,m},$$

for which one of the optimal solution of the linear programming problem (1) is $\mathbf{e}^{x_i}$ – the corresponding x_i axis unit vector.

Let $U = \left\{ i \in \{1, 2, \ldots, m\} : Y^i(\lambda^{\mathbf{1}}) \neq \emptyset \right\}$. In conformity with the optimality criterion of the simplex method $\forall i \in U$ and $\forall I \in 2^{U \setminus \{i\}}$ all the points of

$$\mathbf{Conv}\{\mathbf{e}^{x_k}, k \in I \cup \{i\}\} = \left\{ \mathbf{x} \in \mathbf{R^m} : \begin{array}{l} \mathbf{ex}^T \mathbf{x} = 1, \\ \mathbf{x} \ge 0, \\ x_k = 0, k \notin I \cup \{i\} \end{array} \right\}$$

are optimal for parameters

$$\mathbf{y} \in Y^{iI}(\lambda^{\mathbf{1}}) = \left\{ \mathbf{y} \in \mathbf{R^n} : \begin{array}{l} \left(\sum_{q=1}^{k_1} \lambda_q^1 (A^{kq} - A^{iq}) \right) \mathbf{y} = 0, k \in I, \\ \left(\sum_{q=1}^{k_1} \lambda_q^1 (A^{kq} - A^{iq}) \right) \mathbf{y} \leq 0, k \notin I \cup \{i\}, \\ \lambda_1^1 + \lambda_2^1 + \dots + \lambda_{k_1}^1 = 1, \, \lambda^{\mathbf{1}} > 0, \\ \mathbf{ey}^T \mathbf{y} = 1, \\ \mathbf{y} \geq \mathbf{0}. \end{array} \right\}.$$

Evidently $Y^{i\emptyset}(\lambda^{\mathbf{1}}) = Y^i(\lambda^{\mathbf{1}})$. Hence,

$$\mathbf{Gr_1}(\lambda^{\mathbf{1}}) = \bigcup_{i \in U, I \in 2^{U \setminus \{i\}}} \mathbf{Conv}\{\mathbf{e}^{x_k}, k \in I \cup \{i\}\} \times Y^{iI}(\lambda^{\mathbf{1}}).$$

$$\mathbf{Gr_1} = \bigcup_{\lambda^{\mathbf{1}} \in \mathbf{\Lambda_1}, \, \lambda^{\mathbf{1}} > \mathbf{0}} \mathbf{Gr_1}(\lambda^{\mathbf{1}}).$$

In the parametric problem (2) the parameter set $\mathbf{X}$ is partitioned in such n subsets

$$X^j(\lambda^{\mathbf{2}}) = \left\{ \mathbf{x} \in \mathbf{R^m} : \begin{array}{l} \left(\sum_{r=1}^{k_2} \lambda_r^2 (b^{kr} - b^{jr}) \right) \mathbf{x} \leq 0, k = \overline{1, n}, \\ \lambda_1^2 + \lambda_2^2 + \dots + \lambda_{k_2}^2 = 1, \, \lambda^{\mathbf{2}} > 0, \\ \mathbf{ex}^T \mathbf{x} = 1, \\ \mathbf{x} \geq \mathbf{0}. \end{array} \right\}, j = \overline{1, n},$$

for which one of the optimal solution of the linear programming problem (2) is $\mathbf{e}^{y_j}$ – the corresponding y_j axis unit vector.

Let $V = \{ j \in \{1, 2, \dots, n\} : X^j(\lambda^{\mathbf{2}}) \neq \emptyset \}$. In conformity with the optimality criterion of the simplex method $\forall j \in V$ and $\forall J \in 2^{V \setminus \{j\}}$ all the points of

$$\mathbf{Conv}\{\mathbf{e}^{y_k}, k \in J \cup \{j\}\} = \left\{ \mathbf{y} \in \mathbf{R^n} : \begin{array}{l} \mathbf{ey}^T \mathbf{y} = 1, \\ \mathbf{y} \geq 0, \\ y_k = 0, k \notin J \cup \{j\} \end{array} \right\}$$

are optimal for parameters

$$\mathbf{x} \in X^{jJ}(\lambda^{\mathbf{2}}) = \left\{ \mathbf{x} \in \mathbf{R^m} : \begin{array}{l} \left(\sum_{r=1}^{k_2} \lambda_r^2 (b^{kr} - b^{jr}) \right) \mathbf{x} = 0, k \in J, \\ \left(\sum_{r=1}^{k_2} \lambda_r^2 (b^{kr} - b^{jr}) \right) \mathbf{x} \leq 0, k \notin J \cup \{j\}, \\ \lambda_1^2 + \lambda_2^2 + \cdots + \lambda_{k_2}^2 = 1,\ \lambda^{\mathbf{2}} > 0, \\ \mathbf{ex}^T \mathbf{x} = 1, \\ \mathbf{x} \geq \mathbf{0}. \end{array} \right\}.$$

Evidently $X^{j\emptyset}(\lambda^{\mathbf{2}}) = X^j(\lambda^{\mathbf{2}})$. Hence,

$$\mathbf{Gr_2}(\lambda^{\mathbf{2}}) = \bigcup_{j \in V, J \in 2^{V \setminus \{j\}}} X^{jJ}(\lambda^{\mathbf{2}}) \times \mathbf{Conv}\{\mathbf{e}^{y_k}, k \in J \cup \{j\}\}.$$

$$\mathbf{Gr_2} = \bigcup_{\lambda^{\mathbf{2}} \in \mathbf{\Lambda_2},\, \lambda^{\mathbf{2}} > \mathbf{0}} \mathbf{Gr_2}(\lambda^{\mathbf{2}}).$$

Finally,

$$\mathbf{PNE}(\mathbf{\Gamma}'') = \mathbf{Gr_1} \bigcap \mathbf{Gr_2} =$$

$$= \bigcup_{\substack{\lambda^{\mathbf{1}} \in \mathbf{\Lambda_1},\ \lambda^{\mathbf{1}} > \mathbf{0} \\ \lambda^{\mathbf{2}} \in \mathbf{\Lambda_2},\ \lambda^{\mathbf{2}} > \mathbf{0}}} \ \bigcup_{\substack{i \in U, I \in 2^{U \setminus \{i\}} \\ j \in V, J \in 2^{V \setminus \{j\}}}} X_{iI}^{jJ}(\lambda^{\mathbf{2}}) \times Y_{jJ}^{iI}(\lambda^{\mathbf{1}}),$$

where $X_{iI}^{jJ}(\lambda^{\mathbf{2}}) \times Y_{jJ}^{iI}(\lambda^{\mathbf{1}})$ is a convex component of **PNES**,

$$X_{iI}^{jJ}(\lambda^{\mathbf{2}}) = \mathbf{Conv}\{\mathbf{e}^{x_k}, k \in I \cup \{i\}\} \cap X^{jJ}(\lambda^{\mathbf{2}}),$$
$$Y_{jJ}^{iI}(\lambda^{\mathbf{1}}) = \mathbf{Conv}\{\mathbf{e}^{y_k}, k \in J \cup \{j\}\} \cap Y^{iI}(\lambda^{\mathbf{1}}),$$

$$X_{iI}^{jJ}(\lambda^{\mathbf{2}}) = \left\{ \mathbf{x} \in \mathbf{R^m} : \begin{array}{l} \left(\sum_{r=1}^{k_2} \lambda_r^2 (b^{kr} - b^{jr}) \right) \mathbf{x} = 0, k \in J, \\ \left(\sum_{r=1}^{k_2} \lambda_r^2 (b^{kr} - b^{jr}) \right) \mathbf{x} \leq 0, k \notin J \cup \{j\}, \\ \lambda_1^2 + \lambda_2^2 + \cdots + \lambda_{k_2}^2 = 1,\ \lambda^{\mathbf{2}} > 0, \\ \mathbf{ex}^T \mathbf{x} = 1, \mathbf{x} \geq \mathbf{0}, \\ x_k = 0, k \notin I \cup \{i\} \end{array} \right\}$$

is a set of strategies $\mathbf{x} \in \mathbf{X}$ with support from $\{i\} \cup I$ and for which points of $\mathbf{Conv}\{\mathbf{e}^{y_k}, k \in J \cup \{j\}\}$,

$$Y_{jJ}^{iI}(\lambda^{\mathbf{1}}) = \left\{ \mathbf{y} \in \mathbf{R^n} : \begin{array}{l} \left(\sum_{q=1}^{k_1} \lambda_q^1 (A^{kq} - A^{iq}) \right) \mathbf{y} = 0, k \in I, \\ \left(\sum_{q=1}^{k_1} \lambda_q^1 (A^{kq} - A^{iq}) \right) \mathbf{y} \leq 0, k \notin I \cup \{i\}, \\ \lambda_1^1 + \lambda_2^1 + \cdots + \lambda_{k_1}^1 = 1, \lambda^{\mathbf{1}} > 0, \\ \mathbf{ey}^T \mathbf{y} = 1, \mathbf{y} \geq \mathbf{0}, \\ y_k = 0, k \notin J \cup \{j\} \end{array} \right\}$$

is a set of strategies $\mathbf{y} \in \mathbf{Y}$ with support from $\{j\} \cup J$ and for which points of $\mathbf{Conv}\{\mathbf{e}^{x_k}, k \in I \cup \{i\}\}$ are optimal.

Theorem 7. $\mathbf{PNE}(\mathbf{\Gamma''}) = \mathbf{Gr_1} \bigcap \mathbf{Gr_2} =$

$$= \bigcup_{\substack{\lambda^{\mathbf{1}} \in \mathbf{\Lambda_1},\ \lambda^{\mathbf{1}} > \mathbf{0} \\ \lambda^{\mathbf{2}} \in \mathbf{\Lambda_2},\ \lambda^{\mathbf{2}} > \mathbf{0}}} \bigcup_{\substack{i \in U, I \in 2^{U \setminus \{i\}} \\ j \in V, J \in 2^{V \setminus \{j\}}}} X_{iI}^{jJ}(\lambda^{\mathbf{2}}) \times Y_{jJ}^{iI}(\lambda^{\mathbf{1}}).$$

The proof of the theorem is performed above.

Theorem 8. *If* $X_{iI}^{j\emptyset}(\lambda^{\mathbf{2}}) = \emptyset$*, then* $X_{iI}^{jJ}(\lambda^{\mathbf{2}}) = \emptyset$ *for all* $J \in 2^V$.

For the proof it is sufficient to maintain that $X_{iI}^{jJ}(\lambda^{\mathbf{2}}) \subseteq X_{iI}^{j\emptyset}(\lambda^{\mathbf{2}})$ for $J \neq \emptyset$.

Theorem 9. *If* $Y_{jJ}^{i\emptyset}(\lambda^{\mathbf{1}}) = \emptyset$*, then* $Y_{jJ}^{iI}(\lambda^{\mathbf{1}}) = \emptyset$ *for all* $I \in 2^U$.

From the above the algorithm for **PNES** computing follows:

$$PNE = \emptyset; \quad U = \{i \in \{1, 2, ..., m\} : Y^i(\lambda^{\mathbf{1}}) \neq \emptyset\}; \quad UX = U;$$
$$V = \{j \in \{1, 2, ..., n\} : X^j(\lambda^{\mathbf{2}}) \neq \emptyset\};$$

```
for  i ∈ U  do
  {
      UX = UX \ {i};
      for  I ∈ 2^UX  do
        {
            VY = V;
            for  j ∈ V  do
              {
                  if  (X_iI^j∅(λ²) = ∅)  break;
                  VY = VY \ {j};
                  for  J ∈ 2^VY  do
                      if  (Y_jJ^iI(λ¹) ≠ ∅)
                        PNE = PNE ∪ (X_iI^jJ(λ²) × Y_jJ^iI(λ¹));
              }
        }
  }
```

Algorithm executes the interior **if** no more then

$$2^{m-1}(2^{n-1}+2^{n-2}+\cdots+2^1+2^0)+2^{m-2}(2^{n-1}+2^{n-2}+\cdots+2^1+2^0)+$$
$$\cdots$$
$$+2^1(2^{n-1}+2^{n-2}+\cdots+2^1+2^0)+2^0(2^{n-1}+2^{n-2}+\cdots+2^1+2^0)=$$
$$=(2^m-1)(2^n-1)$$

times. So, the following theorem is true.

Theorem 10. *The algorithm examines no more than* $(2^m-1)(2^n-1)$ *polytopes of the* $X_{iI}^{jJ}(\boldsymbol{\lambda^2}) \times Y_{jJ}^{iI}(\boldsymbol{\lambda^1})$ *type.*

If all the players' strategies are equivalent, then **PNES** consists of $(2^m-1)(2^n-1)$ polytopes.

Evidently, for practical reasons algorithm may be improved by identifying equivalent, dominant and dominated strategies in pure game [4, 14, 15, 16] with the following pure and extended game simplification, but the difficulty is connected with multi-criteria nature of the initial game. "In a nondegenerate game, both players use the same number of pure strategies in equilibrium, so only supports of equal cardinality need to be examined" [13]. This property may be used to minimize essentially the number of components $X_{iI}^{jJ}(\lambda^{\mathbf{2}}) \times Y_{jJ}^{iI}(\lambda^{\mathbf{1}})$ examined in nondegenerate game.

Example 1. *Matrices of the two person game are*

$$A = \begin{bmatrix} 1,0 & 0,2 & 4,1 \\ 0,2 & 2,1 & 3,3 \end{bmatrix}, B = \begin{bmatrix} 0,1 & 2,3 & 3,3 \\ 6,4 & 5,1 & 3,0 \end{bmatrix}$$

The exterior cycle in the above algorithm is executed for the value $i = 1$. *As*

$$X_{1\emptyset}^{1\emptyset}(\lambda^{\mathbf{2}}) = \left\{ \mathbf{x} \in \mathbf{R^2} : \begin{array}{l} (2\lambda_1^2 + 2\lambda_2^2)x_1 + (-\lambda_1^2 - 3\lambda_2^2)x_2 \leq 0, \\ (3\lambda_1^2 + 2\lambda_2^2)x_1 + (-3\lambda_1^2 - 4\lambda_2^2)x_2 \leq 0 \\ \lambda_1^2 + \lambda_2^2 = 1,\ \lambda^{\mathbf{2}} > 0, \\ x_1 + x_2 = 1, \\ x_1 \geq 0, x_2 = 0. \end{array} \right\} = \emptyset,$$

then the cycle for $j = 1$ *is omitted.*
Since

$$X_{1\emptyset}^{2\emptyset}(\lambda^{\mathbf{2}}) = \left\{ \mathbf{x} \in \mathbf{R^2} : \begin{array}{l} (-2\lambda_1^2 - 2\lambda_2^2)x_1 + (\lambda_1^2 + 3\lambda_2^2)x_2 \leq 0, \\ \lambda_1^2 x_1 + (-2\lambda_1^2 - \lambda_2^2)x_2 \leq 0, \\ \lambda_1^2 + \lambda_1^2 = 1,\ \lambda^{\mathbf{2}} > 0, \\ x_1 + x_2 = 1, \\ x_1 \geq 0, x_2 = 0. \end{array} \right\} \neq \emptyset,$$

$$Y_{2\emptyset}^{1\emptyset}(\lambda^{\mathbf{1}}) = \left\{ \mathbf{y} \in \mathbf{R^3} : \begin{array}{l} (-\lambda_1^1 + 2\lambda_2^1)y_1 + (2\lambda_1^1 - \lambda_2^1)y_2 + \\ \quad +(-\lambda_1^1 + 2\lambda_2^1)y_3 \leq 0, \\ \lambda_1^1 + \lambda_2^1 = 1,\ \lambda^{\mathbf{1}} > 0, \\ y_1 + y_2 + y_3 = 1, \\ y_1 = 0, y_2 \geq 0, y_3 = 0. \end{array} \right\} \neq \emptyset,$$

the point $(1, 0) \times (0, 1, 0)$ *is a Pareto-Nash equilibrium for which* $\langle 0, 2\rangle$

and $\langle 2,3\rangle$.

$$X_{1\emptyset}^{2\{3\}}(\lambda^{\mathbf{2}}) = \left\{ \mathbf{x}\in\mathbf{R^2} : \begin{array}{l} (-2\lambda_1^2-2\lambda_2^2)x_1+(\lambda_1^2+3\lambda_2^2)x_2\le 0,\\ \lambda_1^2x_1+(-2\lambda_1^2-\lambda_2^2)x_2=0,\\ \lambda_1^2+\lambda_2^2=1,\ \lambda^{\mathbf{2}}>0,\\ x_1+x_2=1,\\ x_1\ge 0, x_2=0. \end{array} \right\} \ne \emptyset,$$

$$Y_{2\{3\}}^{1\emptyset}(\lambda^{\mathbf{1}}) = \left\{ \mathbf{y}\in\mathbf{R^3} : \begin{array}{l} (-\lambda_1^1+2\lambda_2^1)y_1+(2\lambda_1^1-\lambda_2^1)y_2+\\ \qquad +(-\lambda_1^1+2\lambda_2^1)y_3\le 0,\\ \lambda_1^1+\lambda_2^1=1,\ \lambda^{\mathbf{1}}>0,\\ y_1+y_2+y_3=1,\\ y_1=0, y_2\ge 0, y_3\ge 0. \end{array} \right\} \ne \emptyset,$$

the set $\left\{ \begin{pmatrix} 1\\ 0\end{pmatrix}\times\begin{pmatrix} 0\\ 0\le y_2\le \frac{1}{3}\\ \frac{2}{3}\le y_3\le 1\end{pmatrix}, \begin{pmatrix} 1\\ 0\end{pmatrix}\times\begin{pmatrix} 0\\ \frac{2}{3}\le y_2\le 1\\ 0\le y_3\le \frac{1}{3}\end{pmatrix} \right\}$
is **PNE**.
Since

$$X_{1\emptyset}^{3\emptyset}(\lambda^{\mathbf{2}}) = \left\{ \mathbf{x}\in\mathbf{R^2} : \begin{array}{l} (-3\lambda_1^2-2\lambda_2^2)x_1+(3\lambda_1^2+4\lambda_2^2)x_2\le 0,\\ -\lambda_1^2x_1+(2\lambda_1^2+\lambda_2^2)x_2\le 0,\\ \lambda_1^2+\lambda_2^2=1,\ \lambda^{\mathbf{2}}>0,\\ x_1+x_2=1,\\ x_1\ge 0, x_2=0. \end{array} \right\} \ne \emptyset,$$

$$Y_{3\emptyset}^{1\emptyset}(\lambda^{\mathbf{1}}) = \left\{ \mathbf{y}\in\mathbf{R^3} : \begin{array}{l} (-\lambda_1^1+2\lambda_2^1)y_1+(2\lambda_1^1-\lambda_2^1)y_2+\\ \qquad +(-\lambda_1^1+2\lambda_2^1)y_3\le 0,\\ \lambda_1^1+\lambda_2^1=1,\ \lambda^{\mathbf{1}}>0,\\ y_1+y_2+y_3=1,\\ y_1=0, y_2=0, y_3\ge 0. \end{array} \right\} \ne \emptyset,$$

the point $(1,0)\times(0,0,1)$ *is a Pareto-Nash equilibrium for which* $\langle 4,1\rangle$ *and* $\langle 3,3\rangle$.

Since

$$X^{1\emptyset}_{1\{2\}}(\lambda^{\mathbf{2}}) = \left\{ \mathbf{x} \in \mathbf{R^2} : \begin{array}{l} (2\lambda^2_1 + 2\lambda^2_2)x_1 + (-\lambda^2_1 - 3\lambda^2_2)x_2 \leq 0, \\ (3\lambda^2_1 + 2\lambda^2_2)x_1 + (-3\lambda^2_1 - 4\lambda^2_2)x_2 \leq 0, \\ \lambda^2_1 + \lambda^2_2 = 1,\ \lambda^{\mathbf{2}} > 0, \\ x_1 + x_2 = 1, \\ x_1 \geq 0, x_2 \geq 0. \end{array} \right\} \neq \emptyset,$$

$$Y^{1\{2\}}_{1\emptyset}(\lambda^{\mathbf{1}}) = \left\{ \mathbf{y} \in \mathbf{R^3} : \begin{array}{l} (-\lambda^1_1 + 2\lambda^1_2)y_1 + (2\lambda^1_1 - \lambda^1_2)y_2 + \\ +(-\lambda^1_1 + 2\lambda^1_2)y_3 = 0, \\ \lambda^1_1 + \lambda^1_2 = 1,\ \lambda^{\mathbf{1}} > 0, \\ y_1 + y_2 + y_3 = 1, \\ y_1 \geq 0, y_2 = 0, y_3 = 0. \end{array} \right\} \neq \emptyset,$$

the set $\left\{\left(0 \leq x_1 \leq \frac{1}{3}, \frac{2}{3} \leq x_2 \leq 1\right) \times (1,0,0)\right\}$ *is a Pareto-Nash equilibrium.*
Since

$$X^{1\{2\}}_{1\{2\}}(\lambda^{\mathbf{2}}) = \left\{ \mathbf{x} \in \mathbf{R^2} : \begin{array}{l} (2\lambda^2_1 + 2\lambda^2_2)x_1 + (-\lambda^2_1 - 3\lambda^2_2)x_2 = 0, \\ (3\lambda^2_1 + 2\lambda^2_2)x_1 + (-3\lambda^2_1 - 4\lambda^2_2)x_2 \leq 0, \\ \lambda^2_1 + \lambda^2_2 = 1,\ \lambda^{\mathbf{2}} > 0, \\ x_1 + x_2 = 1, \\ x_1 \geq 0, x_2 \geq 0. \end{array} \right\} \neq \emptyset,$$

$$Y^{1\{2\}}_{1\{2\}}(\lambda^{\mathbf{1}}) = \left\{ \mathbf{y} \in \mathbf{R^3} : \begin{array}{l} (-\lambda^1_1 + 2\lambda^1_2)y_1 + (2\lambda^1_1 - \lambda^1_2)y_2 + \\ +(-\lambda^1_1 + 2\lambda^1_2)y_3 = 0, \\ \lambda^1_1 + \lambda^1_2 = 1,\ \lambda^{\mathbf{1}} > 0, \\ y_1 + y_2 + y_3 = 1 \\ y_1 \geq 0, y_2 \geq 0, y_3 = 0. \end{array} \right\} \neq \emptyset,$$

the set $\left\{\left(\frac{1}{3} \leq x_1 \leq \frac{3}{5}, \frac{2}{5} \leq x_2 \leq \frac{2}{3}\right) \times \left(0 \leq y_1 \leq \frac{1}{3}, \frac{2}{3} \leq y_2 \leq 1, 0\right)\right\} \bigcup$ $\left\{\left(\frac{1}{3} \leq x_1 \leq \frac{3}{5}, \frac{2}{5} \leq x_2 \leq \frac{2}{3}\right) \times \left(\frac{2}{3} \leq y_1 \leq 1, 0 \leq y_2 \leq \frac{1}{3}, 0\right)\right\}$ *is a Pareto-Nash equilibrium.* $X^{1\{3\}}_{1\{2\}}(\lambda^{\mathbf{2}}) = \emptyset$, $X^{1\{2,3\}}_{1\{2\}}(\lambda^{\mathbf{2}}) = \emptyset$.
Since

$$X^{2\emptyset}_{1\{2\}}(\lambda^{\mathbf{2}}) = \left\{ \mathbf{x} \in \mathbf{R^2} : \begin{array}{l} (-2\lambda^2_1 - 2\lambda^2_2)x_1 + (\lambda^2_1 + 3\lambda^2_2)x_2 \leq 0, \\ \lambda^2_1 x_1 + (-2\lambda^2_1 - \lambda^2_2)x_2 \leq 0, \\ \lambda^2_1 + \lambda^2_2 = 1,\ \lambda^{\mathbf{2}} > 0, \\ x_1 + x_2 = 1, \\ x_1 \geq 0, x_2 \geq 0. \end{array} \right\} \neq \emptyset,$$

$$Y_{2\emptyset}^{1\{2\}}(\lambda^{\mathbf{1}}) = \left\{ \mathbf{y} \in \mathbf{R^3} : \begin{array}{l} (-\lambda_1^1 + 2\lambda_2^1)y_1 + (2\lambda_1^1 - \lambda_2^1)y_2 + \\ \quad + (-\lambda_1^1 + 2\lambda_2^1)y_3 = 0, \\ \lambda_1^1 + \lambda_2^1 = 1,\ \lambda^{\mathbf{1}} > 0, \\ y_1 + y_2 + y_3 = 1, \\ y_1 = 0, y_2 \geq 0, y_3 = 0. \end{array} \right\} \neq \emptyset,$$

the set $\left\{\left(\frac{3}{5} \leq x_1 \leq \frac{2}{3}, \frac{1}{3} \leq x_2 \leq \frac{2}{5}\right) \times (0,1,0)\right\}$ *is a Pareto-Nash equilibrium.*
Since

$$X_{1\{2\}}^{2\{3\}}(\lambda^{\mathbf{2}}) = \left\{ \mathbf{x} \in \mathbf{R^2} : \begin{array}{l} (-2\lambda_1^2 - 2\lambda_2^2)x_1 + (\lambda_1^2 + 3\lambda_2^2)x_2 \leq 0, \\ \lambda_1^2 x_1 + (-2\lambda_1^2 - \lambda_2^2)x_2 = 0, \\ \lambda_1^2 + \lambda_2^2 = 1,\ \lambda^{\mathbf{2}} > 0, \\ x_1 + x_2 = 1, \\ x_1 \geq 0, x_2 \geq 0. \end{array} \right\} \neq \emptyset,$$

$$Y_{2\{3\}}^{1\{2\}}(\lambda^{\mathbf{1}}) = \left\{ \mathbf{y} \in \mathbf{R^3} : \begin{array}{l} (-\lambda_1^1 + 2\lambda_2^1)y_1 + (2\lambda_1^1 - \lambda_2^1)y_2 + \\ \quad + (-\lambda_1^1 + 2\lambda_2^1)y_3 = 0, \\ \lambda_1^1 + \lambda_2^1 = 1,\ \lambda^{\mathbf{1}} > 0, \\ y_1 + y_2 + y_3 = 1, \\ y_1 = 0, y_2 \geq 0, y_3 \geq 0. \end{array} \right\} \neq \emptyset,$$

the set $\left\{\left(\frac{2}{3} \leq x_1 \leq 1, 0 \leq x_2 \leq \frac{1}{3}\right) \times \left(0, 0 \leq y_2 \leq \frac{1}{3}, \frac{2}{3} \leq y_3 \leq 1\right)\right\} \bigcup$ $\left\{\left(\frac{2}{3} \leq x_1 \leq 1, 0 \leq x_2 \leq \frac{1}{3}\right) \times \left(0, \frac{2}{3} \leq y_2 \leq 1, 0 \leq y_3 \leq \frac{1}{3}\right)\right\}$ *is a Pareto-Nash equilibrium.*

$$X_{1\{2\}}^{3\emptyset}(\lambda^{\mathbf{2}}) = \left\{ \mathbf{x} \in \mathbf{R^2} : \begin{array}{l} (-3\lambda_1^2 - 2\lambda_2^2)x_1 + (3\lambda_1^2 + 4\lambda_2^2)x_2 \leq 0, \\ -\lambda_1^2 x_1 + (2\lambda_1^2 + \lambda_2^2)x_2 \leq 0, \\ \lambda_1^2 + \lambda_2^2 = 1,\ \lambda^{\mathbf{2}} > 0, \\ x_1 + x_2 = 1, \\ x_1 \geq 0, x_2 \geq 0. \end{array} \right\} \neq \emptyset,$$

$$Y_{3\emptyset}^{1\{2\}}(\lambda^{\mathbf{1}}) = \left\{ \mathbf{y} \in \mathbf{R^3} : \begin{array}{l} (-\lambda_1^1 + 2\lambda_2^1)y_1 + (2\lambda_1^1 - \lambda_2^1)y_2 + \\ \quad + (-\lambda_1^1 + 2\lambda_2^1)y_3 = 0, \\ \lambda_1^1 + \lambda_2^1 = 1,\ \lambda^{\mathbf{1}} > 0, \\ y_1 + y_2 + y_3 = 1, \\ y_1 = 0, y_2 = 0, y_3 \geq 0. \end{array} \right\} \neq \emptyset,$$

the set $\left\{\left(\frac{2}{3} \le x_1 \le 1, 0 \le x_2 \le \frac{1}{3}\right) \times (0,0,1)\right\}$ *is a Pareto-Nash equilibrium.*

The exterior cycle is executed for the value $i = 2$*.*

$$X_{2\emptyset}^{1\emptyset}(\lambda^{\mathbf{2}}) = \left\{ \mathbf{x} \in \mathbf{R^2} : \begin{array}{l} (2\lambda_1^2 + 2\lambda_2^2)x_1 + (-\lambda_1^2 - 3\lambda_2^2)x_2 \le 0, \\ (3\lambda_1^2 + 2\lambda_2^2)x_1 + (-3\lambda_1^2 - 4\lambda_2^2)x_2 \le 0, \\ \lambda_1^2 + \lambda_2^2 = 1,\ \lambda^{\mathbf{2}} > 0, \\ x_1 + x_2 = 1, \\ x_1 = 0, x_2 \ge 0. \end{array} \right\} \ne \emptyset$$

$$Y_{1\emptyset}^{2\emptyset}(\lambda^{\mathbf{1}}) = \left\{ \mathbf{y} \in \mathbf{R^3} : \begin{array}{l} (\lambda_1^1 - 2\lambda_2^1)y_1 + (-2\lambda_1^1 + \lambda_2^1)y_2 + \\ \qquad +(\lambda_1^1 - 2\lambda_2^1)y_3 \le 0, \\ \lambda_1^1 + \lambda_2^1 = 1,\ \lambda^{\mathbf{1}} > 0, \\ y_1 + y_2 + y_3 = 1, \\ y_1 \ge 0, y_2 = 0, y_3 = 0. \end{array} \right\} \ne \emptyset$$

the point $(0,1) \times (1,0,0)$ *is a Pareto-Nash equilibrium for which* $\langle 0, 2\rangle$ *and* $\langle 6, 4\rangle$*.* $X_{2\emptyset}^{1\{2\}}(\lambda^{\mathbf{2}}) = \emptyset$*,* $X_{2\emptyset}^{1\{3\}}(\lambda^{\mathbf{2}}) = \emptyset$*,* $X_{2\emptyset}^{1\{2,3\}}(\lambda^{\mathbf{2}}) = \emptyset$*.*
Because

$$X_{2\emptyset}^{2\emptyset}(\lambda^{\mathbf{2}}) = \left\{ \mathbf{x} \in \mathbf{R^2} : \begin{array}{l} (-2\lambda_1^2 - 2\lambda_2^2)x_1 + (\lambda_1^2 + 3\lambda_2^2)x_2 \le 0, \\ \lambda_1^2 x_1 + (-2\lambda_1^2 - \lambda_2^2)x_2 \le 0, \\ \lambda_1^2 + \lambda_2^2 = 1,\ \lambda^{\mathbf{2}} > 0, \\ x_1 + x_2 = 1, \\ x_1 = 0, x_2 \ge 0. \end{array} \right\} = \emptyset,$$

the cycle for $j = 2$ *is omitted.*

$$X_{2\emptyset}^{3\emptyset}(\lambda^{\mathbf{2}}) = \left\{ \mathbf{x} \in \mathbf{R^2} : \begin{array}{l} (-3\lambda_1^2 - 2\lambda_2^2)x_1 + (3\lambda_1^2 + 4\lambda_2^2)x_2 \le 0, \\ -\lambda_1^2 x_1 + (2\lambda_1^2 + \lambda_2^2)x_2 \le 0, \\ \lambda_1^2 + \lambda_2^2 = 1,\ \lambda^{\mathbf{2}} > 0, \\ x_1 + x_2 = 1, \\ x_1 = 0, x_2 \ge 0. \end{array} \right\} = \emptyset.$$

Thus, the PNES consists of nine elements – three pure and six mixed Pareto-Nash equilibria.

Let us add one more utility function in the above example for each player.

Example 2. *Matrices of the two person game are*

$$A = \begin{bmatrix} 1,0,2 & 0,2,1 & 4,1,3 \\ 0,2,1 & 2,1,0 & 3,3,1 \end{bmatrix}, B = \begin{bmatrix} 0,1,0 & 2,3,1 & 3,3,2 \\ 6,4,5 & 5,1,3 & 3,0,1 \end{bmatrix}.$$

Algorithm examines $(2^2 - 1)(2^3 - 1) = 21$ *cases for this game. The* **PNES** *consists of eleven components. The set of Pareto-Nash equilibria is expanded comparatively with the first example and it coincides with the graph of best response mapping of the second player.*

Corollary. Number of criteria increases the total number of arithmetic operations, but the number of investigated cases remains intact.

Example 3. *Let us examine the game with matrices:*

$$A = \begin{bmatrix} 2,0 & 1,2 & 6,-1 \\ 3,5 & 2,0 & -1,2 \\ -1,3 & 2,3 & 1,1 \end{bmatrix}, B = \begin{bmatrix} 1,2 & 0,1 & 3,2 \\ -1,3 & 1,-1 & -2,0 \\ 2,0 & -1,3 & 2,1 \end{bmatrix}.$$

The algorithm will examine $(2^3-1)(2^3-1) = 49$ *of polyhedra* $X_{iI}^{jJ}(\lambda^{\mathbf{2}}) \times Y_{jJ}^{iI}(\lambda^{\mathbf{1}})$. *In this game thirty-seven components* $X_{iI}^{jJ}(\lambda^{\mathbf{2}})$ *and eighteen components* $Y_{jJ}^{iI}(\lambda^{\mathbf{1}})$ *are nonempty. The* **PNES** *consists of twenty-three elements.*

6 PNES in n-player $m_1 \times m_2 \times \cdots \times m_n$ mixed-strategy games

Consider a n-player $m_1 \times m_2 \times \cdots \times m_n$ mixed-strategy game

$$\Gamma''(\lambda) = \langle \mathbf{N}, \{\mathbf{X_p}\}_{p\in\mathbf{N}}, \{F_p(\mathbf{x}, \lambda^{\mathbf{p}})\}_{p\in\mathbf{N}} \rangle,$$

formulated in Section 4. The utility synthesis functions of the player p are linear if the strategies of the remaining players are fixed:

$$F_p(\mathbf{x},\lambda^{\mathbf{p}}) = (\lambda_1^p \sum_{s_{-p}\in S_{-p}} a^{p1}_{1\|s_{-p}} \prod_{q=\overline{1,n},q\neq p} x^q_{s_q} + \cdots + \lambda^p_{k_p} \sum_{s_{-p}\in S_{-p}} a^{pk_p}_{1\|s_{-p}} \prod_{q=\overline{1,n},q\neq p} x^q_{s_q})x^p_1 + \cdots + (\lambda_1^p \sum_{s_{-p}\in S_{-p}} a^{p1}_{m_p\|s_{-p}} \prod_{q=\overline{1,n},q\neq p} x^q_{s_q} + \cdots + \lambda^p_{k_p} \sum_{s_{-p}\in S_{-p}} a^{pk_p}_{m_p\|s_{-p}} \prod_{q=\overline{1,n},q\neq p} x^q_{s_q})x^p_{m_p},$$

$$\lambda_1^p + \lambda_2^p + \cdots + \lambda^p_{k_p} = 1,\ \lambda^p_i \geq 0, i = \overline{1,k_p},$$

Thus, the player p has to solve a linear parametric problem with parameter vectors $\mathbf{x^{-p}} \in \mathbf{X_{-p}}$ and $\lambda^{\mathbf{p}} \in \mathbf{\Lambda_p}$:

$$F_p(\mathbf{x^p},\mathbf{x^{-p}},\lambda^{\mathbf{p}}) \to max,\ \mathbf{x^p}\in\mathbf{X_p},\ \lambda^{\mathbf{p}}\in\mathbf{\Lambda_p}, p=\overline{1,n}. \qquad (3)$$

The solution of this problem is realized on the vertices of polytope $\mathbf{X_p}$ that has m_p vertices — x^p_i axis unit vectors $\mathbf{e}^{x^p_i} \in \mathbf{R^{m_i}}, i = \overline{1,m_p}$. Thus, in accordance with the simplex method and its optimality criterion, the parameter set $\mathbf{X_{-p}}$ is partitioned in the such m_p subsets $X_{-p}(i_p)(\lambda^{\mathbf{p}})$:

$$\begin{cases} \sum\limits_{s_{-p}\in S_{-p}} \left(\sum\limits_{i=\overline{1,k_p}} \lambda^p_i (a^{pi}_{k\|s_{-p}} - a^{pi}_{i_p\|s_{-p}}) \right) \prod\limits_{q=\overline{1,n},q\neq p} x^q_{s_q} \leq 0, \\ k=\overline{1,m_p}, \lambda_1^p+\lambda_2^p+\cdots+\lambda^p_{k_p}=1,\ \lambda^{\mathbf{p}}>0, \\ x^q_1+x^q_2+\cdots+x^q_{m_q}=1, q=\overline{1,n}, q\neq p, \\ \mathbf{x^{-p}} \geq \mathbf{0}. \end{cases},$$

for $\mathbf{x^{-p}} \in \mathbf{R^{m-m_p}}$, $i_p = \overline{1,m_p}$ for which one of the optimal solution of the linear programming problem (3) is $\mathbf{e}^{x^p_i}$.

Let $U_p = \{i_p \in \{1,2,\ldots,m_p\} : X_{-p}(i_p)(\lambda^{\mathbf{p}}) \neq \emptyset\}$, $\mathbf{ep}^T = (1,\ldots,1) \in \mathbf{R}^{\mathbf{m_p}}$. In conformity with the optimality criterion of the simplex method $\forall i_p \in U_p$ and $\forall I_p \in 2^{U_p \setminus \{i_p\}}$ all the points of

$$\mathbf{Conv}\{\mathbf{e}^{x_k^p}, k \in I_p \cup \{i_p\}\} = \left\{ \mathbf{x} \in \mathbf{R}^{\mathbf{m_p}} : \begin{array}{l} \mathbf{ep}^T \mathbf{x}^p = 1, \\ \mathbf{x}^p \geq 0, \\ x_k^p = 0, k \notin I_p \cup \{i_p\} \end{array} \right\}$$

are optimal for parameters $\mathbf{x^{-p}} \in X_{-p}(i_p I_p)(\lambda^{\mathbf{p}}) \subset \mathbf{R}^{\mathbf{m}-\mathbf{m_p}}$, where $X_{-p}(i_p I_p)(\lambda^{\mathbf{p}})$ is a set of solutions of the system:

$$\begin{cases} \sum\limits_{s_{-p} \in S_{-p}} \left(\sum\limits_{i=\overline{1,k_p}} \lambda_i^p (a_{k\|s_{-p}}^{pi} - a_{i_p\|s_{-p}}^{pi}) \right) \prod\limits_{q=\overline{1,n}, q \neq p} x_{s_q}^q = 0, k \in I_p, \\ \sum\limits_{s_{-p} \in S_{-p}} \left(\sum\limits_{i=\overline{1,k_p}} \lambda_i^p (a_{k\|s_{-p}}^{pi} - a_{i_p\|s_{-p}}^{pi}) \right) \prod\limits_{q=\overline{1,n}, q \neq p} x_{s_q}^q \leq 0, k \notin I_p \cup \{i_p\}, \\ \lambda_1^p + \lambda_2^p + \cdots + \lambda_{k_p}^p = 1,\ \lambda^{\mathbf{p}} > 0, \\ \mathbf{er}^T \mathbf{x}^r = 1, r = \overline{1,n}, r \neq p, \\ \mathbf{x^r} \geq \mathbf{0}, r = \overline{1,n}, r \neq p. \end{cases}$$

Evidently $X_{-p}(i_p \emptyset)(\lambda^{\mathbf{p}}) = X_{-p}(i_p)(\lambda^{\mathbf{p}})$. Hence,

$$\mathbf{Gr_p}(\lambda^{\mathbf{p}}) = \bigcup_{i_p \in U_p, I_p \in 2^{U_p \setminus \{i_p\}}} \mathbf{Conv}\{\mathbf{e}^{x_k^p}, k \in I_p \cup \{i_p\}\} \times X_{-p}(i_p I_p)(\lambda^{\mathbf{p}}).$$

$$\mathbf{Gr_p} = \bigcup_{\lambda^{\mathbf{p}} \in \mathbf{\Lambda_p},\, \lambda^{\mathbf{p}} > \mathbf{0}} \mathbf{Gr_p}(\lambda^{\mathbf{p}}).$$

Finally,

$$\mathbf{PNE}(\Gamma''(\lambda)) = \bigcap_{p=1}^{n} \mathbf{Gr_p},$$

$$\bigcap_{p=1}^{n} \mathbf{Gr_p} = \bigcup_{\lambda \in \mathbf{\Lambda},\, \lambda > \mathbf{0}} \bigcup_{\substack{i_1 \in U_1, I_1 \in 2^{U_1 \setminus \{i_1\}} \\ \ldots \\ i_n \in U_n, I_n \in 2^{U_n \setminus \{i_n\}}}} X(i_1 I_1 \ldots i_n I_n)(\lambda),$$

where $X(i_1I_1 \dots i_nI_n)(\lambda) = \mathbf{PNE}(i_1I_1 \dots i_nI_n)(\lambda)$ is a set of solutions of the system:

$$\begin{cases} \sum\limits_{s_{-r}\in S_{-r}} \left(\sum\limits_{i=\overline{1,k_r}} \lambda_i^r (a_{k\|s_{-r}}^{ri} - a_{i_r\|s_{-r}}^{ri}) \right) \prod\limits_{q=\overline{1,n}, q\neq r} x_{s_q}^q = 0, k \in I_r, \\ \sum\limits_{s_{-r}\in S_{-r}} \left(\sum\limits_{i=\overline{1,k_r}} \lambda_i^r (a_{k\|s_{-r}}^{ri} - a_{i_r\|s_{-r}}^{ri}) \right) \prod\limits_{q=\overline{1,n}, q\neq r} x_{s_q}^q \leq 0, k \notin I_r \cup \{i_r\}, \\ \lambda_1^r + \lambda_2^r + \dots + \lambda_{k_r}^r = 1,\ \lambda > 0,\ r = \overline{1,n}, \\ \mathbf{er}^T \mathbf{x}^r = 1, \mathbf{x^r} \geq \mathbf{0}, r = \overline{1,n}, \\ x_k^r = 0, k \notin I_r \cup \{i_r\}, r = \overline{1,n}. \end{cases}$$

Theorem 11. $\mathbf{PNE}(\Gamma''(\lambda)) = \bigcap\limits_{p=1}^{n} \mathbf{Gr_p}$,

$$\bigcap_{p=1}^{n} \mathbf{Gr_p} = \bigcup_{\lambda\in\mathbf{\Lambda},\, \lambda>\mathbf{0}} \bigcup_{\substack{i_1 \in U_1, I_1 \in 2^{U_1\setminus\{i_1\}} \\ \dots \\ i_n \in U_n, I_n \in 2^{U_n\setminus\{i_n\}}}} X(i_1I_1 \dots i_nI_n)(\lambda).$$

The Theorem 11 is an extension of Theorem 7 to n-player game. The proof is performed above.

The following theorem is a corollary of Theorem 11.

Theorem 12. $\mathbf{PNE}(\Gamma''(\lambda))$ *consists of no more then*

$$(2^{m_1} - 1)(2^{m_2} - 1) \dots (2^{m_n} - 1)$$

components of the type $X(i_1I_1 \dots i_nI_n)(\lambda)$.

In game for which all the players have equivalent strategies **PNES** is partitioned in maximal number $(2^{m_1} - 1)(2^{m_2} - 1) \dots (2^{m_n} - 1)$ of components.

Generally, the components $X(i_1I_1 \dots i_nI_n)(\lambda)$ are non-convex in n-player game ($n \geq 3$).

An exponential algorithm for **PNES** computing in n-player game simply follows from the expression in Theorem 11. The algorithm requires to solve $(2^{m_1}-1)(2^{m_2}-1)\dots(2^{m_n}-1)$ finite systems of multilinear ($n-1$-linear) and linear equations and inequalities in m variables. The last problem is itself a difficult one.

Example 4. *It is considered a three-player extended* $2\times2\times2$ *(dyadic bi-criteria) game with matrices:*

$$a_{1**} = \begin{bmatrix} 9,6 & 0,0 \\ 0,0 & 3,2 \end{bmatrix},\ a_{2**} = \begin{bmatrix} 0,0 & 3,4 \\ 9,3 & 0,0 \end{bmatrix},$$

$$b_{*1*} = \begin{bmatrix} 8,3 & 0,0 \\ 0,0 & 4,6 \end{bmatrix},\ b_{*2*} = \begin{bmatrix} 0,0 & 4,3 \\ 8,6 & 0,0 \end{bmatrix},$$

$$c_{**1} = \begin{bmatrix} 12,6 & 0,0 \\ 0,0 & 2,4 \end{bmatrix},\ c_{**2} = \begin{bmatrix} 0,0 & 6,6 \\ 4,2 & 0,0 \end{bmatrix}.$$

$$\begin{aligned} F_1(\mathbf{x},\mathbf{y},\mathbf{z},\lambda^{\mathbf{1}}) = {} & ((9\lambda_1^1+6\lambda_2^1)y_1z_1+(3\lambda_1^1+2\lambda_2^1)y_2z_2)x_1+ \\ & +((9\lambda_1^1+3\lambda_2^1)y_2z_1+(3\lambda_1^1+4\lambda_2^1)y_1z_2)x_2, \end{aligned}$$

$$\begin{aligned} F_2(\mathbf{x},\mathbf{y},\mathbf{z},\lambda^{\mathbf{2}}) = {} & ((8\lambda_1^2+3\lambda_2^2)x_1z_1+(4\lambda_1^2+6\lambda_2^2)x_2z_2)y_1+ \\ & +((8\lambda_1^2+6\lambda_2^2)x_2z_1+(4\lambda_1^2+3\lambda_2^2)x_1z_2)y_2, \end{aligned}$$

$$\begin{aligned} F_3(\mathbf{x},\mathbf{y},\mathbf{z},\lambda^{\mathbf{3}}) = {} & ((12\lambda_1^3+6\lambda_2^3)x_1y_1+(2\lambda_1^3+4\lambda_2^3)x_2y_2)z_1+ \\ & +((4\lambda_1^3+2\lambda_2^3)x_2y_1+(6\lambda_1^3+6\lambda_2^3)x_1y_2)z_2. \end{aligned}$$

By applying substitutions: $\lambda_1^1=\lambda_1>0$ *and* $\lambda_2^1=1-\lambda_1>0$, $\lambda_1^2=\lambda_2>0$ *and* $\lambda_2^2=1-\lambda_2>0$, $\lambda_1^3=\lambda_3>0$ *and* $\lambda_2^3=1-\lambda_3>0$, *we obtain:*

$$F_1(\mathbf{x},\mathbf{y},\mathbf{z},\lambda_1) = ((6+3\lambda_1)y_1z_1 + (2+\lambda_1)y_2z_2)x_1 + \\ +((3+6\lambda_1)y_2z_1 + (4-\lambda_1)y_1z_2)x_2,$$

$$F_2(\mathbf{x},\mathbf{y},\mathbf{z},\lambda_2) = ((3+5\lambda_2)x_1z_1 + (6-2\lambda_2)x_2z_2)y_1 + \\ +((6+2\lambda_2)x_2z_1 + (3+\lambda_2)x_1z_2)y_2,$$

$$F_3(\mathbf{x},\mathbf{y},\mathbf{z},\lambda_3) = ((6+6\lambda_3)x_1y_1 + (4-2\lambda_3)x_2y_2)z_1 + \\ +((2+2\lambda_3)x_2y_1 + 6x_1y_2)z_2.$$

Totally, we have to consider $(2^2-1)(2^2-1)(2^2-1) = 27$ *components. Further, we will enumerate only nonempty components. Thus,* $\mathbf{PNE}(1\emptyset1\emptyset1\emptyset)(\lambda) = (1,0)\times(1,0)\times(1,0)$ *(for which the gains* $\langle 9,6\rangle, \langle 8,3\rangle, \langle 12,6\rangle$*) is the solution of the system:*

$$\begin{cases} (3+6\lambda_1)y_2z_1 + (4-\lambda_1)y_1z_2 - (6+3\lambda_1)y_1z_1 - (2+\lambda_1)y_2z_2 \leq 0, \\ (6+2\lambda_2)x_2z_1 + (3+\lambda_2)x_1z_2 - (3+5\lambda_2)x_1z_1 - (6-2\lambda_2)x_2z_2 \leq 0, \\ (2+2\lambda_3)x_2y_1 + 6x_1y_2 - (6+6\lambda_3)x_1y_1 - (4-2\lambda_3)x_2y_2 \leq 0, \\ \lambda_1,\ \lambda_2,\ \lambda_3 \in (0,\ 1), \\ x_1+x_2=1, x_1\geq 0, x_2=0, \\ y_1+y_2=1, y_1\geq 0, y_2=0, \\ z_1+z_2=1, z_1\geq 0, z_2=0. \end{cases}$$

$\mathbf{PNE}(\mathbf{1}\emptyset\mathbf{1}\{\mathbf{2}\}\mathbf{1}\{\mathbf{2}\})(\lambda) = (1,0)\times\begin{pmatrix} \frac{1}{3}\leq y_1\leq 1 \\ 1-y_1 \end{pmatrix}\times\begin{pmatrix} \frac{1}{3}\leq z_1\leq \frac{2}{5} \\ 1-z_1 \end{pmatrix}$
is the solution of the system:

$$\begin{cases} (3+6\lambda_1)y_2z_1 + (4-\lambda_1)y_1z_2 - (6+3\lambda_1)y_1z_1 - (2+\lambda_1)y_2z_2 \leq 0, \\ (6+2\lambda_2)x_2z_1 + (3+\lambda_2)x_1z_2 - (3+5\lambda_2)x_1z_1 - (6-2\lambda_2)x_2z_2 = 0, \\ (2+2\lambda_3)x_2y_1 + 6x_1y_2 - (6+6\lambda_3)x_1y_1 - (4-2\lambda_3)x_2y_2 = 0, \\ \lambda_1,\ \lambda_2,\ \lambda_3 \in (0,\ 1), \\ x_1+x_2=1, x_1\geq 0, x_2=0, \\ y_1+y_2=1, y_1\geq 0, y_2\geq 0, \\ z_1+z_2=1, z_1\geq 0, z_2\geq 0. \end{cases}$$

$\mathbf{PNE}(\mathbf{1}\emptyset\mathbf{2}\emptyset\mathbf{2}\emptyset)(\lambda) = (1,0)\times(0,1)\times(0,1)$ *and the gains* $\langle 3,2\rangle, \langle 4,3\rangle, \langle 6,6\rangle$ *are the solution of the system:*

$$\begin{cases}(3+6\lambda_1)y_2z_1+(4-\lambda_1)y_1z_2-(6+3\lambda_1)y_1z_1-(2+\lambda_1)y_2z_2\le 0,\\ -(6+2\lambda_2)x_2z_1-(3+\lambda_2)x_1z_2+(3+5\lambda_2)x_1z_1+(6-2\lambda_2)x_2z_2\le 0,\\ -(2+2\lambda_3)x_2y_1-6x_1y_2+(6+6\lambda_3)x_1y_1+(4-2\lambda_3)x_2y_2\le 0,\\ \lambda_1,\,\lambda_2,\,\lambda_3\in(0,\,1),\\ x_1+x_2=1, x_1\ge 0, x_2=0,\\ y_1+y_2=1, y_1=0, y_2\ge 0,\\ z_1+z_2=1, z_1=0, z_2\ge 0.\end{cases}$$

$$\mathbf{PNE(1\{2\}1\emptyset 1\{2\})}(\lambda)=\begin{pmatrix}\frac{1}{4}\le x_1\le 1\\ 1-x_1\end{pmatrix}\times\begin{pmatrix}1\\0\end{pmatrix}\times\begin{pmatrix}\frac{1}{3}\le z_1\le\frac{2}{5}\\ 1-z_1\end{pmatrix}$$

is the solution of the system:

$$\begin{cases}(3+6\lambda_1)y_2z_1+(4-\lambda_1)y_1z_2-(6+3\lambda_1)y_1z_1-(2+\lambda_1)y_2z_2=0,\\ (6+2\lambda_2)x_2z_1+(3+\lambda_2)x_1z_2-(3+5\lambda_2)x_1z_1-(6-2\lambda_2)x_2z_2\le 0,\\ (2+2\lambda_3)x_2y_1+6x_1y_2-(6+6\lambda_3)x_1y_1-(4-2\lambda_3)x_2y_2=0,\\ \lambda_1,\,\lambda_2,\,\lambda_3\in(0,\,1),\\ x_1+x_2=1, x_1\ge 0, x_2\ge 0,\\ y_1+y_2=1, y_1\ge 0, y_2=0,\\ z_1+z_2=1, z_1\ge 0, z_2\ge 0.\end{cases}$$

$$\mathbf{PNE(1\{2\}1\{2\}1\emptyset)}(\lambda)=\begin{pmatrix}\frac{1}{2}\le x_1\le\frac{2}{3}\\ 1-x_1\end{pmatrix}\times\begin{pmatrix}\frac{1}{3}\le y_1\le\frac{1}{2}\\ 1-y_1\end{pmatrix}\times(1,0)$$

is the solution of the system:

$$\begin{cases}(3+6\lambda_1)y_2z_1+(4-\lambda_1)y_1z_2-(6+3\lambda_1)y_1z_1-(2+\lambda_1)y_2z_2=0,\\ (6+2\lambda_2)x_2z_1+(3+\lambda_2)x_1z_2-(3+5\lambda_2)x_1z_1-(6-2\lambda_2)x_2z_2=0,\\ (2+2\lambda_3)x_2y_1+6x_1y_2-(6+6\lambda_3)x_1y_1-(4-2\lambda_3)x_2y_2\le 0,\\ \lambda_1,\,\lambda_2,\,\lambda_3\in(0,\,1),\\ x_1+x_2=1, x_1\ge 0, x_2\ge 0,\\ y_1+y_2=1, y_1\ge 0, y_2\ge 0,\\ z_1+z_2=1, z_1\ge 0, z_2=0.\end{cases}$$

$$\mathbf{PNE(1\{2\}1\{2\}1\{2\})}(\lambda)=$$

$$=\left\{\begin{pmatrix}\frac{1}{2}\le x_1\le 1\\ 1-x_1\end{pmatrix}\times\begin{pmatrix}\frac{1}{3}\le y_1\le\frac{1}{2}\\ 1-y_1\end{pmatrix}\times\begin{pmatrix}\frac{1}{3}\le z_1\le\frac{2}{5}\\ 1-z_1\end{pmatrix}\right\}\bigcup$$

$$\bigcup\left\{\begin{pmatrix} \frac{1}{4} \leq x_1 \leq \frac{2}{5} \\ 1-x_1 \end{pmatrix} \times \begin{pmatrix} 0 \leq y_1 \leq 1 \\ 1-y_1 \end{pmatrix} \times \begin{pmatrix} \frac{1}{3} \leq z_1 \leq \frac{2}{5} \\ 1-z_1 \end{pmatrix}\right\}\bigcup$$

$$\bigcup\left\{\begin{pmatrix} 0 \leq x_1 \leq \frac{1}{4} \\ 1-x_1 \end{pmatrix} \times \begin{pmatrix} 0 \leq y_1 \leq \frac{5x_1-2}{9x_1-3} \\ 1-y_1 \end{pmatrix} \times \begin{pmatrix} \frac{1}{3} \leq z_1 \leq \frac{2}{5} \\ 1-z_1 \end{pmatrix}\right\}\bigcup$$

$$\bigcup\left\{\begin{pmatrix} \frac{2}{5} \leq x_1 \leq \frac{1}{2} \\ 1-x_1 \end{pmatrix} \times \begin{pmatrix} \frac{5x_1-2}{9x_1-3} \leq y_1 \leq 1 \\ 1-y_1 \end{pmatrix} \times \begin{pmatrix} \frac{1}{3} \leq z_1 \leq \frac{2}{5} \\ 1-z_1 \end{pmatrix}\right\}$$

is the solution of the system:

$$\begin{cases} (3+6\lambda_1)y_2z_1 + (4-\lambda_1)y_1z_2 - (6+3\lambda_1)y_1z_1 - (2+\lambda_1)y_2z_2 = 0, \\ (6+2\lambda_2)x_2z_1 + (3+\lambda_2)x_1z_2 - (3+5\lambda_2)x_1z_1 - (6-2\lambda_2)x_2z_2 = 0, \\ (2+2\lambda_3)x_2y_1 + 6x_1y_2 - (6+6\lambda_3)x_1y_1 - (4-2\lambda_3)x_2y_2 = 0, \\ \lambda_1,\, \lambda_2,\, \lambda_3 \in (0,\, 1), \\ x_1 + x_2 = 1, x_1 \geq 0, x_2 \geq 0, \\ y_1 + y_2 = 1, y_1 \geq 0, y_2 \geq 0, \\ z_1 + z_2 = 1, z_1 \geq 0, z_2 \geq 0. \end{cases}$$

$$\mathbf{PNE(1\{2\}1\{2\}2\emptyset)}(\lambda) = \begin{pmatrix} \frac{1}{2} \leq x_1 \leq \frac{2}{3} \\ 1-x_1 \end{pmatrix} \times \begin{pmatrix} \frac{1}{3} \leq y_1 \leq \frac{1}{2} \\ 1-y_1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

is the solution of the system:

$$\begin{cases} (3+6\lambda_1)y_2z_1 + (4-\lambda_1)y_1z_2 - (6+3\lambda_1)y_1z_1 - (2+\lambda_1)y_2z_2 = 0, \\ (6+2\lambda_2)x_2z_1 + (3+\lambda_2)x_1z_2 - (3+5\lambda_2)x_1z_1 - (6-2\lambda_2)x_2z_2 = 0, \\ -(2+2\lambda_3)x_2y_1 - 6x_1y_2 + (6+6\lambda_3)x_1y_1 + (4-2\lambda_3)x_2y_2 \leq 0, \\ \lambda_1,\, \lambda_2,\, \lambda_3 \in (0,\, 1), \\ x_1 + x_2 = 1, x_1 \geq 0, x_2 \geq 0, \\ y_1 + y_2 = 1, y_1 \geq 0, y_2 \geq 0, \\ z_1 + z_2 = 1, z_1 = 0, z_2 \geq 0. \end{cases}$$

$$\mathbf{PNE(1\{2\}2\emptyset 1\{2\})}(\lambda) = \begin{pmatrix} 0 \leq x_1 \leq \frac{2}{5} \\ 1-x_1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} \frac{1}{3} \leq z_1 \leq \frac{2}{5} \\ 1-z_1 \end{pmatrix}$$

is the solution of the system:

$$\begin{cases} (3+6\lambda_1)y_2z_1+(4-\lambda_1)y_1z_2-(6+3\lambda_1)y_1z_1-(2+\lambda_1)y_2z_2=0,\\ -(6+2\lambda_2)x_2z_1-(3+\lambda_2)x_1z_2+(3+5\lambda_2)x_1z_1+(6-2\lambda_2)x_2z_2\leq 0,\\ (2+2\lambda_3)x_2y_1+6x_1y_2-(6+6\lambda_3)x_1y_1-(4-2\lambda_3)x_2y_2=0,\\ \lambda_1,\,\lambda_2,\,\lambda_3\in(0,\,1),\\ x_1+x_2=1,x_1\geq 0,x_2\geq 0,\\ y_1+y_2=1,y_1=0,y_2\geq 0,\\ z_1+z_2=1,z_1\geq 0,z_2\geq 0. \end{cases}$$

$\mathbf{PNE(2\emptyset 1\emptyset 2\emptyset)}(\lambda)=(0,1)\times(1,0)\times(0,1)$ *and the gains* $\langle 3,4\rangle,\langle 4,6\rangle,$ $\langle 4,2\rangle$ *is the solution of the system:*

$$\begin{cases} -(3+6\lambda_1)y_2z_1-(4-\lambda_1)y_1z_2+(6+3\lambda_1)y_1z_1+(2+\lambda_1)y_2z_2\leq 0,\\ (6+2\lambda_2)x_2z_1+(3+\lambda_2)x_1z_2-(3+5\lambda_2)x_1z_1-(6-2\lambda_2)x_2z_2\leq 0,\\ -(2+2\lambda_3)x_2y_1-6x_1y_2+(6+6\lambda_3)x_1y_1+(4-2\lambda_3)x_2y_2\leq 0,\\ \lambda_1,\,\lambda_2,\,\lambda_3\in(0,\,1),\\ x_1+x_2=1,x_1=0,x_2\geq 0,\\ y_1+y_2=1,y_1\geq 0,y_2=0,\\ z_1+z_2=1,z_1=0,z_2\geq 0. \end{cases}$$

$\mathbf{PNE(2\emptyset 1\{2\}1\{2\})}(\lambda)=\begin{pmatrix}0\\1\end{pmatrix}\times\begin{pmatrix}0\leq y_1\leq\frac{2}{3}\\1-y_1\end{pmatrix}\times\begin{pmatrix}\frac{1}{3}\leq z_1\leq\frac{2}{5}\\1-z_1\end{pmatrix}$ *is the solution of the system:*

$$\begin{cases} -(3+6\lambda_1)y_2z_1-(4-\lambda_1)y_1z_2+(6+3\lambda_1)y_1z_1+(2+\lambda_1)y_2z_2\leq 0,\\ (6+2\lambda_2)x_2z_1+(3+\lambda_2)x_1z_2-(3+5\lambda_2)x_1z_1-(6-2\lambda_2)x_2z_2=0,\\ (2+2\lambda_3)x_2y_1+6x_1y_2-(6+6\lambda_3)x_1y_1-(4-2\lambda_3)x_2y_2=0,\\ \lambda_1,\,\lambda_2,\,\lambda_3\in(0,\,1),\\ x_1+x_2=1,x_1=0,x_2\geq 0,\\ y_1+y_2=1,y_1\geq 0,y_2\geq 0,\\ z_1+z_2=1,z_1\geq 0,z_2\geq 0. \end{cases}$$

$\mathbf{PNE(2\emptyset 2\emptyset 1\emptyset)}(\lambda)=(0,1)\times(0,1)\times(1,0)$ *and the gains* $\langle 9,3\rangle,\langle 8,6\rangle,$ $\langle 2,4\rangle$ *is the solution of the system:*

$$\begin{cases} -(3+6\lambda_1)y_2z_1 - (4-\lambda_1)y_1z_2 + (6+3\lambda_1)y_1z_1 + (2+\lambda_1)y_2z_2 \leq 0, \\ -(6+2\lambda_2)x_2z_1 - (3+\lambda_2)x_1z_2 + (3+5\lambda_2)x_1z_1 + (6-2\lambda_2)x_2z_2 \leq 0, \\ (2+2\lambda_3)x_2y_1 + 6x_1y_2 - (6+6\lambda_3)x_1y_1 - (4-2\lambda_3)x_2y_2 \leq 0, \\ \lambda_1,\, \lambda_2,\, \lambda_3 \in (0,\, 1), \\ x_1 + x_2 = 1, x_1 = 0, x_2 \geq 0, \\ y_1 + y_2 = 1, y_1 = 0, y_2 \geq 0, \\ z_1 + z_2 = 1, z_1 \geq 0, z_2 = 0. \end{cases}$$

Thus the set of Pareto-Nash equilibria consists of eleven components.

7 Conclusions

The idea to consider **PNES** as an intersection of the graphs of efficient response mappings yields to a method of **PNES** computing, an extension of the method proposed in [16] for **NES** computing. Taking into account the computational complexity of the problem, the proposed exponential algorithms are pertinent.

The **PNES** in two-matrix mixed-strategy games may be partitioned into finite number of polytopes, no more then $(2^m - 1)(2^n - 1)$. The proposed algorithm examines, generally, a much more small number of sets of the type $X_{iI}^{jJ}(\lambda^{\mathbf{2}}) \times Y_{jJ}^{iI}(\lambda^{\mathbf{1}})$.

The **PNES** in multi-matrix mixed-strategy games may be partitioned into finite number of components, no more then $(2^{m_1} - 1) \ldots (2^{m_n} - 1)$, but they, generally, are non-convex and moreover non-polytopes. The algorithmic realization of the method is closely related with the problem of solving the systems of multi-linear ($n-1$-linear and simply linear) equations and inequalities, that itself represents a serious obstacle to efficient **PNES** computing.

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