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Robust Geometric Programming Approach to Profit Maximization with Interval Uncertainty

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Abstract

Profit maximization is an important issue to the firms that pursue the largest economic profit possible. In this paper, we consider the profit-maximization problem with the known Cobb-Douglas production function. Its equivalent geometric programming form is given. Then due to the presence of uncertainties in real world modeling, we have assumed interval uncertainties on the model parameters. The robust counterpart is not known to be considered as a geometric program and efficiently solvable using interior point algorithms. Thus using piecewise convex linear approximations, an approximate equivalent of the robust counterpart is given, which is in the form of a geometric programming problem. Finally an example is presented showing the impact of uncertainties.

Keywords: Economic Profit; Geometric Program; Robust Optimization.

1 Introduction

Economic profit is the difference between revenue from selling output and the cost of acquiring the factors necessary to produce it. A profit maximizing firm chooses both its inputs and outputs to achieve maximum economic profits. In other words, the firm seeks to maximize the difference between its total revenue and its total economic costs. If firms are strict profit maximizers, they will adjust those variables that can be controlled until it is impossible to increase profits further [9-14].

Most production functions in the profit maximization problem are represented as power functions. When the production function is represented as a power function, the profit-maximization problem can be treated as a geometric program, a class of nonlinear program which is efficiently solvable using interior point methods [8], while traditionally, the profit-maximization problem is solved by classical method of calculus [1, 5].

In real world applications, model parameters usually involve certain level of uncertainties and thus the original model does not apply anymore. Robust optimization is a new framework which takes into account the parameters uncertainties of the model and solves the underlying problem in the worst case. Several uncertainty sets have been considered in the literature. In this paper, we consider interval uncertainties on parameters of the model [2]. The robust counterpart is not known to be in the form of a tractable geometric programming problem [7]. Thus we give upper and lower piecewise convex linear approximations of it that are efficiently solvable using interior point methods [3, 8]. Finally an illustrative example is presented to show the importance of the model parameters uncertainties.

2 Mathematical Model and Robust Counterpart

Let us consider the following short-run profit maximization problem

$$\begin{array}{rl} \max & p(Ax_1^{\alpha}x_2^{\beta}) - v_1x_1 - v_2x_2 \\ s.t. & x_1 \leq k, \end{array}$$

where $Ax_1^{\alpha}x_2^{\beta}$ is the known Cobb-Douglas production function, p is the market price per unit, A is the scale of production, α and β are the output elasticities, x_i and v_i are ith input quantity and output price, respectively and k is a given constant that restricts the quantity of x_1 [4,9-14]. To solve this problem using efficient algorithms like interior point methods, we may write it as follows

$$\begin{aligned} \max \quad & \pi \\ s.t. \quad & pAx_1^{\alpha}x_2^{\beta} - v_1x_1 - v_2x_2 \geq \pi, \\ & x_1 \leq k, \end{aligned} \tag{1}$$

or in the geometric programming form

$$\begin{aligned} \min \quad & \pi^{-1} \\ s.t. \quad & \pi p^{-1}A^{-1}x_1^{-\alpha}x_2^{-\beta} + v_1p^{-1}A^{-1}x_1^{1-\alpha}x_2^{-\beta} + v_2p^{-1}A^{-1}x_1^{-\alpha}x_2^{1-\beta} \leq 1, \\ & k^{-1}x_1 \leq 1, \end{aligned} \tag{2}$$

where π, x_1, x_2 and p, A, v_1, v_2, k, α, β are variables and parameters respectively. Moreover, by the change of variables

$$z_1 = \log(\pi),\ z_2 = \log(x_1),\ z_3 = \log(x_2)$$

and taking logarithm of the objective function and constraints and finally using the following notation

$$lse(z_1, \dots, z_k) = v\log(e^{z_1} + \cdots + e^{z_k})$$

we have the following equivalent convex form of (1) which is efficiently solvable by Mehrotra's predictor-corrector interior point method [8]:

$$\begin{aligned} \min \quad & [-1 \quad 0 \quad 0]z \\ s.t.\ & lse([1 \quad -\alpha \quad -\beta]z + b_1,\ [0 \quad 1-\alpha \quad -\beta]z + b_2, \\ & \qquad [0 \quad -\alpha \quad 1-\beta]z + b_3) \leq 0, \\ & [0 \quad 1 \quad 0]z + b_4 \leq 0, \end{aligned} \tag{3}$$

where $z = (z_1 \quad z_2 \quad z_3)^T$ are variables and

$$b_1 = \log(p^{-1}A^{-1}),\ b_2 = \log(v_1p^{-1}A^{-1}),$$

$$b_3 = \log(v_2p^{-1}A^{-1}),\ b_4 = \log(k^{-1}).$$

Now suppose that parameters α, β, p, v_1, v_2, k are subject to interval uncertainties, namely

$$\begin{aligned}
\alpha - \varepsilon_1 \le \alpha \le \alpha + \varepsilon_1 \quad &\text{or} \quad \alpha + u_1\varepsilon_1 \quad |u_1| \le 1,\\
\beta - \varepsilon_2 \le \beta \le \beta + \varepsilon_2 \quad &\text{or} \quad \beta + u_2\varepsilon_2 \quad |u_2| \le 1,\\
p - \varepsilon_3 \le p \le p + \varepsilon_3 \quad &\text{or} \quad p + u_3\varepsilon_3 \quad |u_3| \le 1,\\
v_1 - \varepsilon_4 \le v_1 \le v_1 + \varepsilon_4 \quad &\text{or} \quad v_1 + u_4\varepsilon_4 \quad |u_4| \le 1,\\
v_2 - \varepsilon_5 \le v_2 \le v_2 + \varepsilon_5 \quad &\text{or} \quad v_2 + u_5\varepsilon_5 \quad |u_5| \le 1,\\
k - \varepsilon_6 \le k \le k + \varepsilon_6 \quad &\text{or} \quad k + u_6\varepsilon_6 \quad |u_6| \le 1.
\end{aligned}$$

In the sequel we show how these uncertainties affect $b_1, \ldots, b_4$. We have

$$b_1 = -\log p - \log A,\ b_2 = b_1 + \log v_1,\ b_3 = b_1 + \log v_2,\ b_4 = -\log k.$$

Let us consider the following case:

$$p - \varepsilon_3 \le p \le p + \varepsilon_3 \text{ implies } \log(p - \varepsilon_3) \le \log p \le \log(p + \varepsilon_3)$$

or

$$\log p - \log p + \log(p - \varepsilon_3) \le \log p \le \log p - \log p + \log(p + \varepsilon_3)$$

or

$$\log p + \log p^{-1} + \log(p - \varepsilon_3) \le \log p \le \log p + \log p^{-1} + \log(p + \varepsilon_3).$$

Thus we have

$$\log p + \log(1 - p^{-1}\varepsilon_3) \le \log p \le \log p + \log(1 + p^{-1}\varepsilon_3).$$

Since p, $\varepsilon_3 \ge 0$, then we have

$$1 + p^{-1}\varepsilon_3 \le \frac{1}{1 - p^{-1}\varepsilon_3} = (1 - p^{-1}\varepsilon_3)^{-1},$$

thus

$$\log(1 + p^{-1}\varepsilon_3) \le -\log(1 - p^{-1}\varepsilon_3).$$

Using this at the previous inequality we have

$$\begin{aligned}\log p + \log(1 - p^{-1}\varepsilon_3) \leq \log p \leq \log p + \log(1 + p^{-1}\varepsilon_3)\\ \leq \log p - \log(1 - p^{-1}\varepsilon_3)\end{aligned}$$

or

$$\log p - (-\log(1 - p^{-1}\varepsilon_3)) \leq \log p \leq \log p + (-\log(1 - p^{-1}\varepsilon_3)).$$

Thus the uncertainty for $\log p$ approximately is as follow

$$\log p - \delta_1 \leq \log p \leq \log p + \delta_1, \quad \delta_1 = -\log(1 - p^{-1}\varepsilon_3).$$

Analogously, for the other parameters we have

$$\begin{aligned}
&b_1 - \delta_1 \leq b_1 \leq b_1 + \delta_1,\ \delta_1 = -\log(1 - p^{-1}\varepsilon_3) \Rightarrow b_1 + u_7\delta_1,\ |u_7| \leq 1\\
&b_2 - \delta_2 \leq b_2 \leq b_2 + \delta_2,\ \delta_2 = \delta_1 - \log(1 - v_1^{-1}\varepsilon_4) \Rightarrow b_2 + u_8\delta_2,\ |u_8| \leq 1\\
&b_3 - \delta_3 \leq b_3 \leq b_3 + \delta_3,\ \delta_3 = \delta_1 - \log(1 - v_2^{-1}\varepsilon_5) \Rightarrow b_3 + u_9\delta_3,\ |u_9| \leq 1\\
&b_4 - \delta_4 \leq b_4 \leq b_4 + \delta_4,\ \delta_4 = -\log(1 - k^{-1}\varepsilon_6) \Rightarrow b_4 + u_{10}\delta_4,\ |u_{10}| \leq 1.
\end{aligned}$$

Now the approximate robust counterpart of (3) is as follows:

$$\begin{aligned}
&\min \quad \bar{c}^T t\\
&\sup_{u\in U} lse\begin{pmatrix}\begin{bmatrix}1 & -\alpha + u_1\varepsilon_1 & -\beta + u_2\varepsilon_2 & b_1 + u_7\delta_1\end{bmatrix} t,\\ \begin{bmatrix}0 & (1-\alpha) + u_1\varepsilon_1 & -\beta + u_2\varepsilon_2 & b_2 + u_8\delta_2\end{bmatrix} t,\\ \begin{bmatrix}0 & -\alpha + u_1\varepsilon_1 & (1-\beta) + u_2\varepsilon_2 & b_3 + u_9\delta_3\end{bmatrix} t\end{pmatrix} \leq 0,\\
&\sup_{u\in U} \left(\begin{bmatrix}0 & 1 & 0 & b_4 + u_{10}\delta_4\end{bmatrix} t\right) \leq 0,
\end{aligned} \tag{4}$$

where $\bar{c}^T = \begin{bmatrix}-1 & 0 & 0 & 0\end{bmatrix}$, $|u_i| \leq 1$, $t = \begin{pmatrix} z\\ w\end{pmatrix} \in \mathbb{R}^4$, $w = 1$.

In general it is not known whether (4) can be put in a geometric programming form, thus in the sequel we give an approximate model of (4). To do so, the first three-term constraint is approximated by two

two-term constraints as follows [3]:

$$
\begin{aligned}
\min \quad & c^T y \\
& \sup_{|u_i|\le 1} lse([1 \quad -\alpha+u_1\varepsilon_1 \quad -\beta+u_2\varepsilon_2 \quad b_1+u_7\delta_1 \quad 0]y, \\
& \quad [0 \quad 0 \quad 0 \quad 0 \quad 1]y) \le 0, \qquad\qquad (5)\\
& \sup_{|u_i|\le 1} lse([0 \quad (1-\alpha)+u_1\varepsilon_1 \quad -\beta+u_2\varepsilon_2 \quad b_2+u_8\delta_2 \quad -1]y, \\
& \quad [0 \quad -\alpha+u_1\varepsilon_1 \quad (1-\beta)+u_2\varepsilon_2 \quad b_3+u_9\delta_3 \quad -1]y) \le 0, \\
& \sup_{|u_i|\le 1} lse([0 \; 1 \quad 0 \quad b_4+u_{10}\delta_4 \quad 0]y) \le 0,
\end{aligned}
$$

where

$$
\begin{aligned}
& c^T = [-1 \quad 0 \quad 0 \quad 0 \quad 0],\, \bar{a}_1 = [1 \quad -\alpha \quad -\beta \quad b_1 \quad 0], \\
& \bar{a}_2 = [0 \quad 1-\alpha \quad -\beta \quad b_2 \quad -1], \bar{a}_3 = [0 \quad -\alpha \quad 1-\beta \quad b_3 \quad -1], \\
& \bar{a}_4 = [0 \quad 1 \quad 0 \quad b_4 \quad 0],\, B_1 = [0 \quad \varepsilon_1 \quad \varepsilon_2 \quad \delta_1 \quad 0]^T, \\
& B_2 = [0 \quad \varepsilon_1 \quad \varepsilon_2 \quad \delta_2 \quad 0]^T,\, B_3 = [0 \quad \varepsilon_1 \quad \varepsilon_2 \quad \delta_3 \quad 0]^T, \\
& B_4 = [0 \quad 0 \quad 0 \quad \delta_4 \quad 0]^T,\; |u_i| \le 1,\, y = \begin{pmatrix} t \\ s \end{pmatrix} \in \mathbb{R}^5.
\end{aligned}
$$

Since (5) still is not in the form of a problem which could be easily solved, thus we assume both the lower and upper convex piecewise linear approximations of the constraints to satisfy the same inequality [3]. For example the lower three-term approximation of a two-term constraint is as follows:

$$\sup_{u\in U} lse(x,y) = \max\{x,\, y,\, 0.5x+0.5y+0.693\}$$

Thus we consider to have

$$
\begin{aligned}
& \sup_{u\in U}([0 \quad (1-\alpha)+u_1\varepsilon_1 \quad -\beta+u_2\varepsilon_2 \quad b_2+u_8\delta_2 \quad -1]y) = \\
& \quad \sup_{u\in U}(\bar{a_2}y+u_1\varepsilon_1 y_2+u_2\varepsilon_2 y_3+u_8\delta_2 y_4) \le \\
& \bar{a_2}y+|\varepsilon_1 y_2|+|\varepsilon_2 y_3|+|\delta_2 y_4| = \bar{a_2}y+\sum_{i=1}^{5}|(B_2)_i y_i| \le 0,
\end{aligned}
$$

$$\sup_{u\in U}(0.5*[0 \quad (1-\alpha)+u_1\varepsilon_1 \quad -\beta+u_2\varepsilon_2 \quad b_2+u_8\delta_2 \quad -1]y+$$
$$0.5*[0 \quad -\alpha+u_1\varepsilon_1 \quad (1-\beta)+u_2\varepsilon_2 \quad b_3+u_9\delta_3 \quad -1]y)+0.693\leq$$
$$0.5\bar{a}_2y+0.5\bar{a}_3y+0.5\left(\sum_{i=1}^{5}|(B_2)_iy_i|+\sum_{i=1}^{5}|(B_3)_iy_i|\right)+0.693\leq 0,$$

and

$$\sup_{u\in U}([0 \quad -\alpha+u_1\varepsilon_1 \quad (1-\beta)+u_2\varepsilon_2 \quad b_3+u_9\delta_3 \quad -1]y)=$$
$$\sup_{u\in U}(\bar{a}_3y+u_1\varepsilon_1y_2+u_2\varepsilon_2y_3+u_9\delta_3y_4)\leq$$
$$\bar{a}_3y+|\varepsilon_1y_2|+|\varepsilon_2y_3|+|\delta_3y_4|=\bar{a}_3y+\sum_{i=1}^{5}|(B_3)_iy_i|\leq 0.$$

Moreover, to have more accurate results, we have used 25 term piecewise convex linear approximations that are derived using the algorithm in [3]. In Table 1, we have given several best lower piecewise convex linear approximations of $lse(x,y)$ function. It is worth to note that in [3] up to five term approximations are reported.

Furthermore, in the following table the errors of piecewise convex linear approximations up to 25 terms are given for two levels of uncertainties.

3 Example

Let us consider the following values for the parameters in (1)

$$p=20,\ A=40,\ \alpha=0.1,\ \beta=0.4,\ v_1=10,\ v_2=35,\ k=30.$$

Using Mehrotra's predictor-corrector interior point algorithm, the optimal objective value of (1) is 3399.55. However, if we take the following uncertainty set parameters

$$|u_i|\leq 0.5,\ \varepsilon_1=0.003,\ \varepsilon_2=0.007,\ \varepsilon_3=\varepsilon_4=\varepsilon_5=\varepsilon_6=0.5,$$

Table 1. Best lower piecewise linear approximations of $lse(x, y)$

r Number of terms	Error of approximation	The best lower r term piecewise linear approximation of $lse(x, y)$
3	0.223	$\max\{x, y, 0.5x + 0.5y + 0.693\}$
4	0.109	$\max\left\{\begin{matrix} x, 0.271x + 0.729y + 0.584, \\ 0.729x + 0.271y + 0.584, y \end{matrix}\right\}$
5	0.065	$\max\left\{\begin{matrix} x,\ 0.167x + 0.833y + 0.450, \\ 0.5x + 0.5y + 0.693, \\ 0.833x + 0.167y + 0.450,\ y \end{matrix}\right\}$
25	0.002076	$\max\left\{\begin{matrix} x,\ 0.0056x + 0.9944y + 0.0348,\ 0.0056y + 0.9944x + 0.0348, \\ 0.0195x + 0.9805y + 0.0962,\ 0.0195y + 0.9805x + 0.0962, \\ 0.0414x + 0.9586y + 0.1724, 0.0414y + 0.9586x + 0.1724, \\ 0.0709x + 0.9291y + 0.2560,\ 0.0709y + 0.9291x + 0.2560, \\ 0.1075x + 0.8925y + 0.3413,\ 0.1075y + 0.8925x + 0.3413, \\ 0.1506x + 0.8494y + 0.4238, 0.1506y + 0.8494x + 0.4238, \\ 0.1995x + 0.8005y + 0.4997,\ 0.1995y + 0.8005x + 0.4997, \\ 0.2534x + 0.7466y + 0.5660,\ 0.2534y + 0.7466x + 0.5660, \\ 0.3114x + 0.6886y + 0.6202, 0.3114y + 0.6886x + 0.6202, \\ 0.3725x + 0.6275y + 0.6603,\ 0.3725y + 0.6275x + 0.6603, \\ 0.4357x + 0.5643y + 0.6849,\ 0.4357y + 0.5643x + 0.6849, \\ 0.5x + 0.5y + 0.693,\ y \end{matrix}\right\}$

Table 2. Error of piecewise convex linear approximation

Errors	$\lvert u_i\rvert \leq 0.5$	$\lvert u_i\rvert \leq 1$
Error of 3 terms approximation	0.9184	0.9184
Error of 4 terms approximation	0.2876	0.3104
Error of 5 terms approximation	0.1986	0.1986
....	....	....
Error of 16 terms approximation	0.0138	0.0138
....	....	....
Error of 25 terms approximation	0.0055	0.0058

and solve its upper and lower approximations, then the optimal solutions of the upper and lower 25 term piecewise convex linear approximation of (5) are 2980.7 and 2964.3, respectively. Thus the lower bound for the optimal solution of (4) is 2964.3. As one can see, the range to which the optimal solution of the robust problem belongs, [2964.3, 2980.7], is significantly different than the original optimal objective value. Thus a slight uncertainty in the input parameter might lead to significant change of the optimal objective value. Moreover, if we solve the approximate robust model of (5) for the case, where

$$|u_i| \leq 1,\ \varepsilon_1 = 0.003,\ \varepsilon_2 = 0.007,\ \varepsilon_3 = \varepsilon_4 = \varepsilon_5 = \varepsilon_6 = 1,$$

then the values of the upper and lower 25 term piecewise convex linear approximations are 2797.6 and 2781.6, respectively. Thus the lower bound for the optimal solution of (4) is 2781.6. A similar observation as in the previous case holds here as well. We should note that all computations are done in MATLAB 7.8 and we have used cvx software package [6] to solve problem (5).

4 Conclusions

Extensive research specially in the last decade shows that robust optimization can alleviate sensitivity of a given problem to its data uncertainty by incorporating explicitly data uncertainty into the problem. In this paper, we consider the robust counterpart of the profit-maximization problem which is in the form of a geometric programming problem. Since it is not known in general that a robust geometric programming problem can be reformulated as a tractable optimization problem that interior point or other algorithms can efficiently solve, then using piecewise convex linear approximations, an approximate equivalent of the robust counterpart is given, which is in the form of a geometric programming problem. Moreover, an illustrative example is given which shows the importance and impact of the uncertainties in the model parameters for different level of uncertainties. Due to the presence of uncertainty in economical model parameters, the idea might be useful to be applied to other models.

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