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Linear discrete-time Pareto-Nash-Stackelberg control problem and principles for its solving

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Abstract

A direct-straightforward method for solving linear discrete-time optimal control problem is applied to solve control problem of a linear discrete-time system as a mixture of multi-criteria Stackelberg and Nash games. For simplicity, the exposure starts with the simplest case of linear discrete-time optimal control problem and, by sequential considering of more general cases, investigation finalizes with the highlighted Pareto-Nash-Stackelberg and set valued control problems. Different principles of solving are compared and their equivalence is proved.

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1 Introduction

Optimal control theory which appeared due to Lev Pontryagin [2] and Richard Bellman [3], as natural extension of calculus of variations, often doesn't satisfy all requirements and needs for modelling and solving problems of real dynamic systems and processes. A situation of this type occurs for problem of linear discrete-time system control by a decision process that evolves as Pareto-Nash-Stackelberg game with constraints – a mixture of hierarchical and simultaneous games [5, 6, 7, 8, 9]. For such system, the notion of optimal control evolves naturally to the notion of Pareto-Nash-Stackelberg type control and to the natural principle for solving the highlighted problem by applying a concept of Pareto-Nash-Stackelberg equilibrium [9] with a direct-straightforward principle for solving.

The direct method and principle for solving linear discrete-time optimal control problem is extended to control problem of a linear system in discrete time as a mixture of multi-criteria Stackelberg and Nash games [9]. The exposure starts with the simplest case of linear discrete-time optimal control problem [1] and, by sequential considering of more general cases, finalizes with the Pareto-Nash-Stackelberg and set valued control problems. The maximum principle of Pontryagin is formulated and proved for all the considered problems. Its equivalence with the direct-straightforward principle for solving is established.

2 Linear discrete-time optimal control problem

Consider the following problem [1][1]:

$$\begin{aligned} f(x,u) = \sum_{t=1}^{T} (c^t x^t + b^t u^t) \quad &\to \max, \\ x^t = A^{t-1} x^{t-1} + B^t u^t, \quad &t = 1,...,T, \\ D^t u^t \leq d^t, \quad &t = 1,...,T, \end{aligned} \tag{1}$$

where $x^0, x^t, c^t \in R^n$, $u^t, b^t \in R^m$, $A^{t-1} \in R^{n\times n}$, $B^t \in R^{n\times m}$, $d^t \in R^k$, $D^t \in R^{k\times n}$, $c^t x^t = \left\langle c^t, x^t \right\rangle$, $b^t u^t = \left\langle b^t, u^t \right\rangle$, $t = 1,...,T$, $u = (u^1, \ldots, u^T)$.

[1]Symbol T means discrete time horizon in this paper. Symbol of matrix translation is omitted. Left and right matrix multiplications are largely used. The reader is asked to understand by himself when column or row vector are used.

The problem (1) may be represented in the form:

$$
\begin{array}{lllllll}
Ex^1 & & -B^1u^1 & & & = A^0x^0, \\
-A^1x^1 + Ex^2 & & & -B^2u^2 & & = 0, \\
\quad \dots & & & \dots & & \dots \\
\quad -A^{T-1}x^{T-1} + Ex^T & & & & -B^Tu^T & = 0, \\
 & & D^1u^1 & & & \le d^1, \\
 & & & D^2u^2 & & \le d^2, \\
 & & & \dots & & \dots \\
 & & & & D^Tu^T & \le d^T, \\
c^1x^1 + c^2x^2 & + \dots + & b^1u^1 & + b^2u^2 + & \dots + b^Tu^t & \to \max .
\end{array}
$$

Its dual problem is

$$
\begin{array}{llllll}
p^1 & -p^2A^1 & & & & = c^1, \\
 & p^2 - p^3A^2 & & & & = c^2, \\
 & & \dots & & & \dots \\
 & & p^T & & & = c^T, \\
-p^1B^1 & & & +q^1D^1 & & = b^1, \\
 & -p^2B^2 & & & +q^2D^2 & = b^2, \\
 & & & & \dots & \dots \\
 & & -p^TB^T & & +q^TD^T & = b^T, \\
 & & & q^1 \ge 0, q^2 \ge 0, & q^T \ge 0, & \\
p^1A^0x^0 & & & +q^1d^1 & +q^2d^2 + \dots + q^Td^T & \to \min .
\end{array}
$$

From the constraints of dual problem it follows that the values of variables $p^1, p^2, \dots, p^T$ are calculated on the bases of recurrent relation:

$$
\begin{array}{lll}
p^T & = & c^T, \\
p^t & = & p^{t+1}A^t + c^t, \quad t = T-1, ..., 1.
\end{array}
\tag{2}
$$

So, the dual problem is equivalent to:

$$
\begin{array}{rcl}
q^1D^1 & = & p^1B^1 + b^1, \\
q^2D^2 & = & p^2B^2 + b^2, \\
 & \dots & \\
q^TD^T & = & p^TB^T + b^T, \\
q^t & \ge & 0, t = 1, \dots, T, \\
p^1A^0x^0 + q^1d^1 + q^2d^2 + \dots + q^Td^T & \to & \min .
\end{array}
$$

The dual of the last problem is:

$$
\begin{array}{rcl}
D^1u^1 & \leq & d^1, \\
D^2u^2 & \leq & d^2, \\
 & \dots & \\
D^Tu^T & \leq & d^T, \\
\sum_{t=1}^{T} \left\langle u^t, p^tB^T + b^t \right\rangle & \to & \max.
\end{array}
\tag{3}
$$

The solution of (3) may be found by solving T linear programming problems

$$
\begin{array}{rcl}
D^tu^t & \leq & d^t, \\
\left\langle u^t, p^tB^T + b^t \right\rangle & \to & \max,
\end{array}
$$

for $t = 1, \dots, T$. So, the solution of initial control problem (1) is identical with a sequence of solutions of T linear programming problems.

Similar results may be obtained by performing direct transformations of (1):

$$
\begin{array}{rcl}
x^1 & = & A^0x^0 + B^1u^1, \\
x^2 & = & A^1x^1 + B^2u^2 = A^1(A^0x^0 + B^1u^1) + B^2u^2 = \\
 & = & A^1A^0x^0 + A^1B^1u^1 + B^2u^2, \\
x^3 & = & A^2x^2 + B^3u^3 = A^2(A^1A^0x^0 + A^1B^1u^1 + B^2u^2) + B^3u^3 = \\
 & = & A^2A^1A^0x^0 + A^2A^1B^1u^1 + A^2B^2u^2 + B^3u^3, \\
 & \dots & \\
x^T & = & A^{T-1}x^{T-1} + B^Tu^T = \\
 & = & \prod_{t=0}^{T-1} A^tx^0 + \prod_{t=1}^{T-1} A^tB^1u^1 + \prod_{t=2}^{T-1} A^tB^2u^2 + \\
 & & + \dots + \prod_{t=T-1}^{T-1} A^tB^{T-1}u^{T-1} + B^Tu^T,
\end{array}
$$

and by subsequent substitution in the objective function:

$$
\begin{aligned}
f(x,u) = {} & c^1\left(A^0x^0 + B^1u^1\right) + c^2\left(A^1A^0x^0 + A^1B^1u^1 + B^2u^2\right) + \\
& + c^3\left(A^2A^1A^0x^0 + A^2A^1B^1u^1 + A^2B^2u^2 + B^3u^3\right) + \dots + \\
& + c^T\left(\prod_{t=0}^{T-1} A^t x^0 + \prod_{t=1}^{T-1} A^t B^1 u^1 + \prod_{t=2}^{T-1} A^t B^2 u^2 + \right. \\
& \left. + \dots + \prod_{t=T-1}^{T-1} A^t B^{T-1} u^{T-1} + B^T u^T \right) + \\
& + b^1u^1 + b^2u^2 + \dots + b^Tu^T = \\
= {} & \left(c^1 + c^2A^1 + c^3A^2A^1 + \dots + c^TA^{T-1}A^{T-2}\dots A^1\right)A^0x^0 + \\
& + \left(c^1B^1 + c^2A^1B^1 + c^3A^2A^1B^1 + \dots + \right. \\
& \left. + c^TA^{T-1}A^{T-2}\dots A^1B^1 + b^1\right)u^1 + \\
& + \left(c^2B^2 + c^3A^2B^2 + c^4A^3A^2B^2 + \dots + \right. \\
& \left. + c^TA^{T-1}A^{T-2}\dots A^2B^2 + b^2\right)u^2 + \\
& + \dots + \left(c^TB^T + b^T\right)u^T.
\end{aligned}
$$

Finally, the problem obtains the form

$$
\begin{aligned}
& f(u) = \\
& = \left(c^1 + c^2A^1 + c^3A^2A^1 + \dots + c^TA^{T-1}A^{T-2}\dots A^1\right)A^0x^0 + \\
& + \left(c^1B^1 + c^2A^1B^1 + c^3A^2A^1B^1 + \dots + \right. \\
& \left. + c^TA^{T-1}A^{T-2}\dots A^1B^1 + b^1\right)u^1 + \\
& + \left(c^2B^2 + c^3A^2B^2 + c^4A^3A^2B^2 + \dots + \right. \\
& \left. + c^TA^{T-1}A^{T-2}\dots A^2B^2 + b^2\right)u^2 + \\
& + \dots + \left(c^TB^T + b^T\right)u^T \to \max, \\
& D^tu^t \leq d^t, t = 1, \dots, T.
\end{aligned}
\tag{4}
$$

Obviously, (3) and (4) are identical. So, the solution of the last problem (4) is obtained as a sequence of solutions of T linear programming problems. Apparently, the complexity of such method is polynomial, but really it has pseudo-polynomial complexity because of possible exponential value of T on n.

Theorem 1. *Let (1) be solvable. The sequence $\bar{u}^1, \bar{u}^2, \dots, \bar{u}^T$ forms an optimal control if and only if $\bar{u}^t$ is the solution of linear programming* *problem*

$$\begin{array}{l}\left(c^tB^t + c^{t+1}A^tB^t + \cdots + c^TA^{T-1}A^{T-2}\dots A^tB^t + b^t\right)u^t \to \max,\\ D^tu^t \leq d^t,\end{array}$$

for $t = 1, \dots, T$.

Different particular cases may be established for (1).

Theorem 2. *If* $A^0 = A^1 = \cdots = A^{T-1} = A$, $B^1 = B^2 = \cdots = B^T = B$, *and (1) is solvable, then the sequence* $\bar{u}^1, \bar{u}^2, \dots, \bar{u}^T$ *forms an optimal control if and only if* $\bar{u}^t$ *is the solution of linear programming problem*

$$\begin{array}{l}\left(c^tB + c^{t+1}AB + c^{t+2}(A)^2B + \cdots + c^T(A)^{T-t}B + b^t\right)u^t \to \max,\\ D^tu^t \leq d^t,\end{array}$$

for $t = 1, \dots, T$.

Theorem 1 establishes a principle for solving (1). By considering Hamiltonian functions

$$H_t(u^t) = \left\langle p^tB^t + b^t, u^t\right\rangle, t = T, \dots, 1,$$

where $p^t, t = T, \dots, 1$ are defined by (2), as it is conjectured in [1] and proved above by two ways, the maximum principle of Pontryagin [2] holds.

Theorem 3. *Let (1) be solvable. The sequence* $\bar{u}^1, \bar{u}^2, \dots, \bar{u}^T$ *forms an optimal control if and only if*

$$H_t(\bar{u}^t) = \max_{u^t : D^tu^t \leq d^t} H_t(u^t), t = T, \dots, 1.$$

Evidently, Theorems 1 and 3 are equivalent.

3 Linear discrete-time Stackelberg control problem

Let us modify the problem (1) by considering the control of Stackelberg type [7], that is Stackelberg game with T players [7, 8, 5, 6]. In such game, at each stage t ($t = 1, \ldots, T$) the player t selects his strategy and communicates his and all precedent selected strategies to the following $t+1$ player. After all stage strategy selections, all the players compute their gains on the resulting profile. Let us name such type of system control as Stackelberg control, and the corresponding problem – linear discrete-time Stackelberg control problem. The described decision process may be formalized as it follows:

$$
\begin{aligned}
& f_1(x,u) = \sum_{t=1}^{T} \left(c^{1t}x^t + b^{1t}u^t\right) \underset{u^1}{\longrightarrow} \max, \\
& f_2(x,u) = \sum_{t=1}^{T} \left(c^{2t}x^t + b^{2t}u^t\right) \underset{u^2}{\longrightarrow} \max, \\
& \ldots \\
& f_T(x,u) = \sum_{t=1}^{T} \left(c^{Tt}x^t + b^{Tt}u^t\right) \underset{u^T}{\longrightarrow} \max, \\
& \\
& x^t = A^{t-1}x^{t-1} + B^t u^t, t = 1, ..., T, \\
& D^t u^t \le d^t, t = 1, ..., T,
\end{aligned}
\tag{5}
$$

where $x^0, x^t, c^{\pi t} \in R^n$, $u^t, b^{\pi t} \in R^m$, $A^{t-1} \in R^{n\times n}$, $B^t \in R^{n\times m}$, $d^t \in R^k$, $D^t \in R^{k\times n}$, $c^{\pi t}x^t = \left\langle c^{\pi t}, x^t\right\rangle$, $b^{\pi t}u^t = \left\langle b^{\pi t}, u^t\right\rangle$, $t, \pi = 1, ..., T$.

Formally, the set of strategies of player π ($\pi = 1, 2, \ldots, T$) is determined only by admissible solutions of the problem:

$$
\begin{aligned}
& f_\pi\left(x, u^\pi || u^{-\pi}\right) = \sum_{t=1}^{T} \left(c^{\pi t}x^t + b^{\pi t}u^t\right) \underset{u^\pi}{\longrightarrow} \max, \\
& \\
& x^\pi = A^{\pi-1}x^{\pi-1} + B^\pi u^\pi, \\
& D^\pi u^\pi \le d^\pi.
\end{aligned}
$$

In fact, as we can find out, the strategy sets of the players are interconnected and the game is not a simple normal form game. A situation similar with that in optimization theory may be established – there are problems without constraints and with constraints. So, the strategy (normal form) game may be named strategy game without constraints. Game which contains commune constraints on strategies may be named strategy game with constraints.

Player π ($\pi = 1, 2, \ldots, T$) decision problem is defined by the linear programming problem (5). Since the controlled system is one for all players, by performing the direct transformations as above, (5) is transformed into

$$\begin{array}{l}
f_\pi\left(u^\pi || u^{-\pi}\right) = \\
= \left(c^{\pi 1} + c^{\pi 2} A^1 + c^{\pi 3} A^2 A^1 + \cdots + \right. \\
\left. + \, c^{\pi T} A^{T-1} A^{T-2} \ldots A^1\right) A^0 x^0 + \\
+ \left(c^{\pi 1} B^1 + c^{\pi 2} A^1 B^1 + c^{\pi 3} A^2 A^1 B^1 + \cdots + \right. \\
\left. + \, c^{\pi T} A^{T-1} A^{T-2} \ldots A^1 B^1 + b^{\pi 1}\right) u^1 + \\
+ \left(c^{\pi 2} B^2 + c^{\pi 3} A^2 B^2 + c^{\pi 4} A^3 A^2 B^2 + \cdots + \right. \\
\left. + \, c^{\pi T} A^{T-1} A^{T-2} \ldots A^2 B^2 + b^{\pi 2}\right) u^2 + \\
+ \cdots + \left(c^{\pi T} B^T + b^{\pi T}\right) u^T \underset{u^\pi}{\rightarrow} \max, \pi = 1, \ldots, T, \\
D^t u^t \leq d^t, t = 1, \ldots, T.
\end{array} \tag{6}$$

From equivalence of (5) and (6) the proof of Theorem 4 follows.

Theorem 4. *Let (5) be solvable. The sequence $\bar{u}^1, \bar{u}^2, \ldots, \bar{u}^T$ forms a Stackelberg equilibrium control in (5) if and only if $\bar{u}^\pi$ is optimal solution of linear programming problem*

$$\begin{array}{l}
f_\pi(u^\pi) = \left(c^{\pi\pi} B^\pi + c^{\pi\pi+1} A^\pi B^\pi + c^{\pi\pi+2} A^{\pi+1} A^\pi B^\pi + \cdots + \right. \\
\left. + c^{\pi T} A^{T-1} A^{T-2} \ldots A^\pi B^\pi + b^{\pi\pi}\right) u^\pi \underset{u^\pi}{\rightarrow} \max, \\
D^\pi u^\pi \leq d^\pi,
\end{array}$$

for $\pi = 1, \ldots, T$.

There are various particular cases of (5) and Theorem 4. Theorem 5 presents one of such cases.

Theorem 5. *If $A^0 = A^1 = \cdots = A^{T-1} = A$, $B^1 = B^2 = \cdots = B^T = B$, and (5) is solvable, then the sequence $\bar{u}^1, \bar{u}^2, \ldots, \bar{u}^T$ forms a Stackelberg equilibrium control if and only if $\bar{u}^\pi$ is the solution of linear programming problem*

$$\begin{aligned}
&\left(c^{\pi\pi}B + c^{\pi\pi+1}AB + \cdots + c^{\pi T}(A)^{T-\pi}B + b^{\pi\pi}\right)u^\pi \underset{u^\pi}{\rightarrow} \max,\\
&D^\pi u^\pi \le d^\pi,
\end{aligned}$$

for $\pi = 1, \ldots, T$.

Theorem 4 establishes a principle for solving (5). The maximum principle of Pontryagin may be applied for solving (5) too. Let us consider the following recurrent relations

$$\begin{array}{rcl}
p^{\pi T} & = & c^{\pi T},\\
p^{\pi t} & = & p^{\pi t+1}A^t + c^{\pi t}, \quad t = T-1, ..., 1,
\end{array} \tag{7}$$

where $\pi = 1, \ldots, T$. Hamiltonian functions are defined as

$$H_{\pi t}\left(u^t\right) = \left\langle p^{\pi t}B^t + b^{\pi t}, u^t\right\rangle, t = T, \ldots, 1, \pi = 1, \ldots, T,$$

where $p^{\pi t}, t = T, \ldots, 1, \pi = 1, \ldots, T$, are defined by (7).

Theorem 6. *Let (5) be solvable. The sequence of controls $\bar{u}^1, \ldots, \bar{u}^T$ forms a Stackelberg equilibrium control if and only if*

$$H_{\pi\pi}\left(\bar{u}^\pi\right) = \max_{u^\pi : D^\pi u^\pi \le d^\pi} H_{\pi\pi}\left(u^\pi\right),$$

for $\pi = 1, \ldots, T$.

The proof of Theorem 6 may be provided by direct substitution of relations (7) in Hamiltonian functions and by comparing the final results with linear programming problems from Theorem 4. Obviously, Theorems 4 and 6 are equivalent.

From computational point of view, method for solving problem (5) established by Theorem 4 looks more preferable than the method established by Theorem 6.

4 Linear discrete-time Pareto-Stackelberg control problem

Let us modify the problem (5) by considering control of Pareto-Stackelberg type. At each stage a single player makes decision. Every player selects on his stage his strategy according to his criteria and communicates his choice and the precedent player choices to the following player. At the last stage, after all stage strategy selections, the players compute their gains. Such type of control is named Pareto-Stackelberg control, and the corresponding problem – the linear discrete-time Pareto-Stackelberg control problem.

The described decision process may be formalized in a following manner:

$$\begin{aligned}
&f_1(x,u) = \sum_{t=1}^{T} \left(c^{1t}x^t + b^{1t}u^t\right) \underset{u^1}{\longrightarrow} \text{ef}\max,\\
&f_2(x,u) = \sum_{t=1}^{T} \left(c^{2t}x^t + b^{2t}u^t\right) \underset{u^2}{\longrightarrow} \text{ef}\max,\\
&\dots\\
&f_T(x,u) = \sum_{t=1}^{T} \left(c^{Tt}x^t + b^{Tt}u^t\right) \underset{u^T}{\longrightarrow} \text{ef}\max,\\
&\\
&x^t = A^{t-1}x^{t-1} + B^t u^t, t = 1, ..., T,\\
&D^t u^t \le d^t, t = 1, ..., T,
\end{aligned} \tag{8}$$

where $x^0, x^t \in R^n$, $c^{\pi t} \in R^{k_\pi \times n}$, $u^t \in R^m$, $b^{\pi t} \in R^{k_\pi \times m}$, $A^{t-1} \in R^{n\times n}$, $B^t \in R^{n\times m}$, $d^t \in R^k$, $D^t \in R^{k\times n}$, $t, \pi = 1, ..., T$. Notation ef max means multi-criteria maximization.

The set of strategies of player π ($\pi = 1, \dots, T$) is determined formally by the problem

$$f_\pi\left(x, u^\pi||u^{-\pi}\right) = \sum_{t=1}^{T}\left(c^{\pi t}x^t + b^{\pi t}u^t\right) \underset{u^\pi}{\longrightarrow} \text{ef}\max,$$

$$\begin{array}{l} x^\pi = A^{\pi-1}x^{\pi-1} + B^\pi u^\pi, \\ D^\pi u^\pi \le d^\pi. \end{array}$$

By performing direct transformations as above, (8) is transformed into

$$\begin{array}{l} f_\pi\left(u^\pi||u^{-\pi}\right) = \\ = \left(c^{\pi 1} + c^{\pi 2}A^1 + c^{\pi 3}A^2A^1 + \cdots + \right. \\ \left. + c^{\pi T}A^{T-1}A^{T-2}\ldots A^1\right)A^0x^0 + \\ + \left(c^{\pi 1}B^1 + c^{\pi 2}A^1B^1 + c^{\pi 3}A^2A^1B^1 + \cdots + \right. \\ \left. + c^{\pi T}A^{T-1}A^{T-2}\ldots A^1B^1 + b^{\pi 1}\right)u^1 + \\ + \left(c^{\pi 2}B^2 + c^{\pi 3}A^2B^2 + c^{\pi 4}A^3A^2B^2 + \cdots + \right. \\ \left. + c^{\pi T}A^{T-1}A^{T-2}\ldots A^2B^2 + b^{\pi 2}\right)u^2 + \\ + \cdots + \left(c^{\pi T}B^T + b^{\pi T}\right)u^{\pi T} \underset{u^\pi}{\to} \text{ef}\max, \pi = 1, \ldots, T, \\ D^tu^t \le d^t, t = 1, \ldots, T. \end{array} \tag{9}$$

Equivalence of (8) and (9) proves the following Theorem 7.

Theorem 7. *Let (8) be solvable. The sequence $\bar{u}^1, \bar{u}^2, \ldots, \bar{u}^T$ forms a Pareto-Stackelberg equilibrium control in (8) if and only if $\bar{u}^\pi$ is efficient solution of multi-criteria linear programming problem*

$$\begin{array}{l} f_\pi(u^\pi) = \left(c^{\pi\pi}B^\pi + c^{\pi\pi+1}A^\pi B^\pi + c^{\pi\pi+2}A^{\pi+1}A^\pi B^\pi + \cdots + \right. \\ \left. + c^{\pi T}A^{T-1}A^{T-2}\ldots A^\pi B^\pi + b^{\pi\pi}\right)u^\pi \underset{u^\pi}{\to} \text{ef}\max, \\ D^\pi u^\pi \le d^\pi, \end{array}$$

for $\pi = 1, \ldots, T$.

As above, a particular case of (8) is examined in Theorem 7.

Theorem 8. *If $A^0 = A^1 = \cdots = A^{T-1} = A$, $B^1 = B^2 = \cdots = B^T = B$, and (8) is solvable, then the sequence $\bar{u}^1, \bar{u}^2, \ldots, \bar{u}^T$ forms a Pareto-Stackelberg equilibrium control if and only if $\bar{u}^\pi$ is the efficient solution* *of multi-criteria linear programming problem*

$$\left(c^{\pi\pi}B + c^{\pi\pi+1}AB + \cdots + c^{\pi T}(A)^{T-\pi}B + b^{\pi\pi}\right) u^{\pi} \underset{u^{\pi}}{\rightarrow} \text{ef}\max,$$
$$D^{\pi}u^{\pi} \leq d^{\pi},$$

for $\pi = 1, \ldots, T$.

Pontryagin maximum principle may be extended for (8). Let us consider the following recurrent relations

$$\begin{array}{rcl} p^{\pi T} & = & c^{\pi T}, \\ p^{\pi t} & = & p^{\pi t+1}A^{t} + c^{\pi t}, \quad t = T-1, ..., 1, \end{array} \tag{10}$$

where $\pi = 1, \ldots, T$. Hamiltonian vector-functions are defined as

$$H_{\pi t}\left(u^{t}\right) = \left\langle p^{\pi t}B^{t} + b^{\pi t}, u^{t}\right\rangle, t = T, \ldots, 1, \pi = 1, \ldots, T,$$

where $p^{\pi t}, t = T, \ldots, 1, \pi = 1, \ldots, T$ are defined by (10).

Theorem 9. *Let (8) be solvable. The sequence of controls* $\bar{u}^{1}, \ldots, \bar{u}^{T}$ *forms a Pareto-Stackelberg equilibrium control if and only if*

$$\bar{u}^{\pi} \in \underset{u^{\pi}: D^{\pi}u^{\pi} \leq d^{\pi}}{\text{Arg ef}\max} H_{\pi\pi}\left(u^{\pi}\right),$$

for $\pi = 1, \ldots, T$.

By direct substitution of (10) in Hamiltonian functions and by comparing the final results with multi-criteria linear programming problems from Theorem 7 the truth of Theorem 9 arises. Theorems 7 and 9 are equivalent.

It can be remarked especially that method of Pareto-Stackelberg control established by Theorems 7–9 needs solutions of multi-criteria linear programming problems.

5 Linear discrete-time Nash-Stackelberg control problem

Let us modify the problem (5) by considering the control of Nash-Stackelberg type with T stages and $\nu_1 + \nu_2 + \cdots + \nu_T$ players, where $\nu_1, \nu_2, \ldots, \nu_T$ are the numbers of players on stages $1, 2, \ldots, T$. Every player is identified by two numbers (indices) (τ, π), where τ is the number of stage on which player selects his strategy and $\pi \in \{1, 2, \ldots, \nu_\tau\}$ is his number at stage τ. In such game, at each stage τ the players $1, 2, \ldots, \nu_\tau$ play a Nash game by selecting simultaneously their strategies and by communicating his and all precedent selected strategies to the following $\tau + 1$ stage players. After all stage strategy selections, on the resulting profile all the players compute their gains. Such type of control is named Nash-Stackelberg control, and the corresponding problem – linear discrete-time Nash-Stackelberg control problem.

The described decision process may be modelled as it follows:

$$\begin{aligned}
&f_{\tau\pi}(x, u^{\tau\pi}||u^{-\tau\pi}) = \sum_{t=1}^{T}\left(c^{\tau\pi t}x^t + \sum_{\mu=1}^{\nu_t} b^{\tau\pi t\mu}u^{t\mu}\right) \underset{u^{\tau\pi}}{\longrightarrow} \max,\\
&\tau = 1, \ldots, T, \pi = 1, \ldots, \nu_\tau,\\
&x^t = A^{t-1}x^{t-1} + \sum_{\pi=1}^{\nu_t} B^{t\pi}u^{t\pi}, t = 1, ..., T,\\
&D^{t\pi}u^{t\pi} \le d^{t\pi}, t = 1, ..., T, \pi = 1, \ldots, \nu_t,
\end{aligned} \tag{11}$$

where

$x^0, x^t, c^{\tau\pi t} \in R^n$,
$u^t, b^{\tau\pi t\mu} \in R^m$,
$A^{t-1} \in R^{n\times n}$,
$B^{\tau\pi} \in R^{n\times m}$,
$d^{\tau\pi} \in R^k$,
$D^{\tau\pi} \in R^{k\times n}$,
$t, \tau = 1, \ldots, T$,
$\pi = 1, \ldots, \nu_\tau$,
$\mu = 1, \ldots, \nu_t$.

By performing direct transformations

$$
\begin{aligned}
x^1 &= A^0x^0 + \sum_{\pi=1}^{\nu_1} B^{1\pi}u^{1\pi}, \\
x^2 &= A^1x^1 + \sum_{\pi=1}^{\nu_2} B^{2\pi}u^{2\pi} = \\
&= A^1\left(A^0x^0 + \sum_{\pi=1}^{\nu_1} B^{1\pi}u^{1\pi}\right) + \sum_{\pi=1}^{\nu_2} B^{2\pi}u^{2\pi} = \\
&= A^1A^0x^0 + A^1\sum_{\pi=1}^{\nu_1} B^{1\pi}u^{1\pi} + \sum_{\pi=1}^{\nu_2} B^{2\pi}u^{2\pi}, \\
x^3 &= A^2x^2 + \sum_{\pi=1}^{\nu_3} B^{3\pi}u^{3\pi} = \\
&= A^2\left(A^1A^0x^0 + A^1\sum_{\pi=1}^{\nu_1} B^{1\pi}u^{1\pi} + \sum_{\pi=1}^{\nu_2} B^{2\pi}u^{2\pi}\right) + \\
&\quad + \sum_{\pi=1}^{\nu_3} B^{3\pi}u^{3\pi} = \\
&= A^2A^1A^0x^0 + A^2A^1\sum_{\pi=1}^{\nu_1} B^{1\pi}u^{1\pi} + A^2\sum_{\pi=1}^{\nu_2} B^{2\pi}u^{2\pi} + \\
&\quad + \sum_{\pi=1}^{\nu_3} B^{3\pi}u^{3\pi}, \\
&\dots
\end{aligned}
$$

$$
\begin{aligned}
x^T &= A^{T-1}x^{T-1} + \sum_{\pi=1}^{\nu_T} B^{T\pi}u^{T\pi} = \\
&= \prod_{t=0}^{T-1} A^t x^0 + \prod_{t=1}^{T-1} A^t \sum_{\pi=1}^{\nu_1} B^{1\pi}u^{1\pi} + \prod_{t=2}^{T-1} A^t \sum_{\pi=1}^{\nu_2} B^{2\pi}u^{2\pi} + \\
&\quad + \cdots + \prod_{t=T-1}^{T-1} A^t \sum_{\pi=1}^{\nu_{T-1}} B^{T-1\pi}u^{T-1\pi} + \sum_{\pi=1}^{\nu_T} B^{T\pi}u^{T\pi},
\end{aligned}
$$

and by subsequent substitution in the objective/cost functions of (11), the problem (11) is reduced to

$$
\begin{array}{l}
f(u^{\tau\pi}||u^{-\tau\pi}) = \\
= \left(c^{\tau\pi 1} + c^{\tau\pi 2}A^1 + c^{\tau\pi 3}A^2A^1 + \dots + \right. \\
\left. + c^{\tau\pi T}A^{T-1}A^{T-2}\dots A^1\right) A^0x^0 + \\
\\
+ \left(c^{\tau\pi 1}B^{11} + c^{\tau\pi 2}A^1B^{11} + c^{\tau\pi 3}A^2A^1B^{11} + \dots + \right. \\
\left. + c^{\tau\pi T}A^{T-1}A^{T-2}\dots A^1B^{11} + b^{\tau\pi 11}\right) u^{11} + \\
+ \left(c^{\tau\pi 1}B^{12} + c^{\tau\pi 2}A^1B^{12} + c^{\tau\pi 3}A^2A^1B^{12} + \dots + \right. \\
\left. + c^{\tau\pi T}A^{T-1}A^{T-2}\dots A^1B^{12} + b^{\tau\pi 12}\right) u^{12} + \\
+ \dots + \\
+ \left(c^{\tau\pi 1}B^{1\nu_1} + c^{\tau\pi 2}A^1B^{1\nu_1} + c^{\tau\pi 3}A^2A^1B^{1\nu_1} + \dots + \right. \\
\left. + c^{\tau\pi T}A^{T-1}A^{T-2}\dots A^1B^{1\nu_1} + b^{\tau\pi 1\nu_1}\right) u^{1\nu_1} + \\
\\
+ \left(c^{\tau\pi 2}B^{21} + c^{\tau\pi 3}A^2B^{21} + c^{\tau\pi 4}A^3A^2B^{21} + \dots + \right. \\
\left. + c^{\tau\pi T}A^{T-1}A^{T-2}\dots A^2B^{21} + b^{\tau\pi 21}\right) u^{21} + \\
+ \left(c^{\tau\pi 2}B^{22} + c^{\tau\pi 3}A^2B^{22} + c^{\tau\pi 4}A^3A^2B^{22} + \dots + \right. \\
\left. + c^{\tau\pi T}A^{T-1}A^{T-2}\dots A^2B^{22} + b^{\tau\pi 22}\right) u^{22} + \\
+ \dots + \\
+ \left(c^{\tau\pi 2}B^{2\nu_2} + c^{\tau\pi 3}A^2B^{2\nu_2} + c^{\tau\pi 4}A^3A^2B^{2\nu_2} + \dots + \right. \\
\left. + c^{\tau\pi T}A^{T-1}A^{T-2}\dots A^2B^{2\nu_2} + b^{\tau\pi 2\nu_2}\right) u^{2\nu_2} + \\
\\
+ \dots + \left(c^{\tau\pi T}B^{T\nu_T} + b^{\tau\pi T\nu_T}\right) u^{T\nu_T} \underset{u^{\tau\pi}}{\rightarrow} \max, \\
\tau = 1, \dots, T, \pi = 1, \dots, \nu_\tau, \\
D^{\tau\pi}u^{\tau\pi} \leq d^{\tau\pi}, \tau = 1, \dots, T, \pi = 1, \dots, \nu_\tau.
\end{array}
\tag{12}
$$

Evidently, (12) defines a strategic game for which Nash-Stackelberg equilibrium is also Nash equilibrium and it is simply computed as a sequence of solutions of $\nu_1 + \dots + \nu_T$ linear programming problems

$$
\begin{array}{l}
f(u^{\tau\pi}||u^{-\tau\pi}) = \\
= \left(c^{\tau\pi\tau}B^{\tau\pi} + c^{\tau\pi\tau+1}A^\tau B^{\tau\pi} + c^{\tau\pi\tau+2}A^{\tau+1}A^\tau B^{\tau\pi} + \dots + \right. \\
\left. + c^{\tau\pi T}A^{T-1}A^{T-2}\dots A^\tau B^{\tau\pi} + b^{\tau\pi\tau\pi}\right) u^{\tau\pi} \underset{u^{\tau\pi}}{\rightarrow} \max, \\
D^{\tau\pi}u^{\tau\pi} \leq d^{\tau\pi},
\end{array}
\tag{13}
$$

$\tau = 1, \dots, T, \pi = 1, \dots, \nu_\tau$.

Equivalence of (11) and (13) proves the following Theorem 10.

Theorem 10. *Let (11) be solvable. The sequence* $\bar{u}^{11}, \bar{u}^{12}, \ldots, \bar{u}^{T\nu_T}$ *forms a Nash-Stackelberg equilibrium control in (11) if and only if* $\bar{u}^{\tau\pi}$ *is optimal in linear programming problem (13), for* $\tau = 1, \ldots, T, \pi = 1, \ldots, \nu_\tau$.

Various particular cases of (11) may be examined in Theorem 10, e.g. Theorem 11.

Theorem 11. *If* $A^0 = A^1 = \cdots = A^{T-1} = A$, $B^{11} = B^{12} = \cdots = B^{T\nu_T} = B$, *and (11) is solvable, then the sequence* $\bar{u}^{11}, \bar{u}^{12}, \ldots, \bar{u}^{T\nu_T}$ *forms a Nash-Stackelberg equilibrium control if and only if* $\bar{u}^{\tau\pi}$ *is optimal in linear programming problem*

$$\begin{array}{l} f(u^{\tau\pi}||u^{-\tau\pi}) = \\ = \left(c^{\tau\pi\tau}B + c^{\tau\pi\tau+1}AB + c^{\tau\pi\tau+2}(A)^2B + \cdots + \right. \\ \left. + \, c^{\tau\pi T}(A)^{T-\tau}B + b^{\tau\pi\tau\pi}\right) u^{\tau\pi} \underset{u^{\tau\pi}}{\rightarrow} \max, \\ D^{\tau\pi}u^{\tau\pi} \leq d^{\tau\pi}, \end{array}$$

for $\tau = 1, \ldots, T, \pi = 1, \ldots, \nu_\tau$.

Pontryagin maximum principle may be extended for (11). Let us consider the following recurrent relations

$$\begin{array}{rcl} p^{\tau\pi T} & = & c^{\tau\pi T}, \\ p^{\tau\pi t} & = & p^{\tau\pi t+1}A^t + c^{\tau\pi t}, \quad t = T-1, ..., 1, \end{array} \tag{14}$$

where $\tau = 1, \ldots, T, \pi = 1, \ldots, \nu_\tau$. Hamiltonian functions are defined as

$$H_{\tau\pi t}\left(u^{\tau\pi}\right) = \left\langle p^{\tau\pi t}B^{\tau\pi} + b^{\tau\pi\tau\pi}, u^{\tau\pi}\right\rangle, t = T, \ldots, 1,$$

where $\tau = 1, \ldots, T, \pi = 1, \ldots, \nu_\tau$ and $p^{\tau\pi t}, t = T, \ldots, 1, \tau = 1, \ldots, T, \pi = 1, \ldots, \nu_\tau$ are defined by (14).

Theorem 12. *Let (11) be solvable. The sequence* $\bar{u}^{11}, \bar{u}^{12}, \ldots, \bar{u}^{T\nu_T}$ *forms a Nash-Stackelberg equilibrium control if and only if*

$$H_{\tau\pi t}\left(\bar{u}^{\tau\pi}\right) = \max_{u^{\tau\pi}: D^{\tau\pi}u^{\tau\pi} \leq d^{\tau\pi}} H_{\tau\pi t}\left(u^{\tau\pi}\right),$$

for $t = T, \ldots, 1, \tau = 1, \ldots, T, \pi = 1, \ldots, \nu_\tau$.

Theorems 10 and 12 are equivalent.

6 Linear discrete-time Pareto-Nash-Stackelberg control problem

Let us integrate the problems (8) and (11) by considering the control of Pareto-Nash-Stackelberg type with T stages and $\nu_1 + \cdots + \nu_T$ players, where $\nu_1, \ldots, \nu_T$ are the correspondent numbers of players on stages $1, \ldots, T$. Every player is identified by two numbers as above in Nash-Stackelberg control: τ – stage on which player selects his strategy and π – player number at stage τ. In such game, at each stage τ the players $1, 2, \ldots, \nu_\tau$ play a Pareto-Nash game by selecting simultaneously their strategies according to their criteria ($k_{\tau 1}, k_{\tau 2}, \ldots, k_{\tau \nu_\tau}$ are the numbers of criteria of respective players) and by communicating his and all precedent selected strategies to the following $\tau + 1$ stage players. After all stage strategy selections, all the players compute their gains on the resulting profile. Such type of control is named Pareto-Nash-Stackelberg control, and the corresponding problem – linear discrete-time Pareto-Nash-Stackelberg control problem.

The decision control process may be modelled as:

$$
\begin{aligned}
& f_{\tau\pi}(x, u^{\tau\pi} || u^{-\tau\pi}) = \sum_{t=1}^{T} \left(c^{\tau\pi t} x^t + \sum_{\mu=1}^{\nu_t} b^{\tau\pi t\mu} u^{t\mu} \right) \underset{u^{\tau\pi}}{\longrightarrow} \text{ef max}, \\
& \tau = 1, \ldots, T, \pi = 1, \ldots, \nu_\tau, \\
& x^t = A^{t-1} x^{t-1} + \sum_{\pi=1}^{\nu_t} B^{t\pi} u^{t\pi}, t = 1, ..., T, \\
& D^{t\pi} u^{t\pi} \le d^{t\pi}, t = 1, ..., T, \pi = 1, \ldots, \nu_t,
\end{aligned} \tag{15}
$$

where $x^0, x^t \in R^n$, $c^{\tau\pi t\mu} \in R^{k_{tp} \times n}$, $u^{\tau\pi} \in R^m$, $b^{\tau\pi t\mu} \in R^{k_{tp} \times n}$, $A^{t-1} \in R^{n \times n}$, $B^{\tau\pi} \in R^{n \times m}$, $d^{\tau\pi} \in R^k$, $D^{\tau\pi} \in R^{k \times n}$, $t, \tau = 1, \ldots, T, \pi = 1, \ldots, \nu_\tau, \mu = 1, \ldots, \nu_t$.

By performing similar direct transformation as above, (15) is reduced to a sequence of multi-criteria linear programming problems

$$\begin{array}{l} f(u^{\tau\pi}||u^{-\tau\pi}) = \\ = \left(c^{\tau\pi\tau}B^{\tau\pi} + c^{\tau\pi\tau+1}A^{\tau}B^{\tau\pi} + c^{\tau\pi\tau+2}A^{\tau+1}A^{\tau}B^{\tau\pi} + \dots + \right. \\ \left. + \, c^{\tau\pi T}A^{T-1}A^{T-2}\dots A^{\tau}B^{\tau\pi} + b^{\tau\pi\tau\pi}\right) u^{\tau\pi} \underset{u^{\tau\pi}}{\to} \text{ef max}, \\ D^{\tau\pi}u^{\tau\pi} \le d^{\tau\pi}, \end{array} \tag{16}$$

$\tau = 1, \dots, T, \pi = 1, \dots, \nu_\tau$.

Equivalence of (15) and (16) proves the following Theorem 13.

Theorem 13. *Let (15) be solvable. The sequence* $\bar{u}^{11}, \bar{u}^{12}, \dots, \bar{u}^{T\nu_T}$ *forms a Pareto-Nash-Stackelberg equilibrium control in (15) if and only if* $\bar{u}^{\tau\pi}$ *is an efficient solution of multi-criteria linear programming problem (16), for* $\tau = 1, \dots, T, \pi = 1, \dots, \nu_\tau$.

As a corollary, Theorem 14 follows.

Theorem 14. *If* $A^0 = A^1 = \dots = A^{T-1} = A$, $B^{11} = B^{12} = \dots = B^{T\nu_T} = B$, *and (15) is solvable, then the sequence* $\bar{u}^{11}, \bar{u}^{12}, \dots, \bar{u}^{T\nu_T}$ *forms a Pareto-Nash-Stackelberg equilibrium control if and only if* $\bar{u}^{\tau\pi}$ *is an efficient solution of multi-criteria linear programming problem*

$$\begin{array}{l} f(u^{\tau\pi}||u^{-\tau\pi}) = \\ = \left(c^{\tau\pi\tau}B + c^{\tau\pi\tau+1}AB + c^{\tau\pi\tau+2}(A)^2B + \dots + \right. \\ \left. + \, c^{\tau\pi T}(A)^{T-\tau}B + b^{\tau\pi\tau\pi}\right) u^{\tau\pi} \underset{u^{\tau\pi}}{\to} \text{ef max}, \\ D^{\tau\pi}u^{\tau\pi} \le d^{\tau\pi}, \end{array}$$

for $\tau = 1, \dots, T, \pi = 1, \dots, \nu_\tau$.

Pontryagin maximum principle may be generalized for (15). By considering recurrent relations

$$\begin{array}{rcl} p^{\tau\pi T} & = & c^{\tau\pi T}, \\ p^{\tau\pi t} & = & p^{\tau\pi t+1}A^t + c^{\tau\pi t}, \quad t = T-1, ..., 1, \end{array} \tag{17}$$

where $\tau = 1, \dots, T, \pi = 1, \dots, \nu_\tau$, Hamiltonian vector-functions are defined as

$$H_{\tau\pi t}\left(u^{\tau\pi}\right) = \left\langle p^{\tau\pi t}B^{\tau\pi} + b^{\tau\pi\tau\pi}, u^{\tau\pi}\right\rangle, t = T, \dots, 1,$$

where $\tau = 1, \dots, T, \pi = 1, \dots, \nu_\tau$ and $p^{\tau\pi t}, t = T, \dots, 1, \tau = 1, \dots, T, \pi = 1, \dots, \nu_\tau$. Remark the vector nature of (17) via (14).

Theorem 15. *Let (15) be solvable. The sequence* $\bar{u}^{11}, \bar{u}^{12}, \ldots, \bar{u}^{T\nu_T}$ *forms a Pareto-Nash-Stackelberg equilibrium control if and only if*

$$\bar{u}^{\tau\pi} \in \underset{u^{\tau\pi}: D^{\tau\pi} u^{\tau\pi} \le d^{\tau\pi}}{\text{Arg ef max}} H_{\tau\pi t}\left(u^{\tau\pi}\right),$$

for $t = T, \ldots, 1, \tau = 1, \ldots, T, \pi = 1, \ldots, \nu_T$.

Theorems 13 and 12 are equivalent.

7 Linear discrete-time set-valued optimal control problem

The state of controlled system is described above by a point. Indeed, real systems may be treated as n-dimension body, the state of which is described by a set of points in every time moment. Evidently, the initial state of the system is described by initial set $X^0 \subset R^n$. Naturally, the following problem arises

$$\begin{aligned} F(X,U) = \sum_{t=1}^{T}(c^t X^t + b^t U^t) & \quad \to \max, \\ X^t = A^{t-1} X^{t-1} + B^t U^t, & \quad t = 1, ..., T, \\ D^t U^t \le d^t, & \quad t = 1, ..., T, \end{aligned} \tag{18}$$

where $X^0, X^t \subset R^n$, $c^t \in R^n$, $U^t \subset R^m$, $b^t \in R^m$, $A^{t-1} \in R^{n\times n}$, $B^t \in R^{n\times m}$, $d^t \in R^k$, $D^t \in R^{k\times n}$, $c^t X^t = \left\langle c^t, X^t \right\rangle$, $b^t U^t = \left\langle b^t, U^t \right\rangle$, $t = 1, ..., T$. Linear set operations in (18) are defined obviously (see, e.g., [4]): $AX = \{Ax : x \in X\}$, $\forall X \subset R^n$, $\forall A \in R^{n\times n}$.

The objective set-valued map $F : X \times Y \multimap R$, $F(X,Y) \subset R$ represents a summation of intervals. That is, the optimization of the objective map in problem (11) needs interval arithmetic treatment.

By direct transformations, (18) is transformed into

$$\begin{aligned}
&F(U) = \\
&= \left(c^1 + c^2A^1 + c^3A^2A^1 + \dots + \right. \\
&\left. + c^TA^{T-1}A^{T-2}\dots A^1\right) A^0X^0 + \\
&+ \left(c^1B^1 + c^2A^1B^1 + c^3A^2A^1B^1 + \dots + \right. \\
&\left. + c^TA^{T-1}A^{T-2}\dots A^1B^1 + b^1\right) U^1 + \\
&+ \left(c^2B^2 + c^3A^2B^2 + c^4A^3A^2B^2 + \dots + \right. \\
&\left. + c^TA^{T-1}A^{T-2}\dots A^2B^2 + b^2\right) U^2 + \\
&+ \dots + \left(c^TB^T + b^T\right) U^T \to \max, \\
&D^tU^t \le d^t, t = 1, \dots, T.
\end{aligned} \tag{19}$$

The equivalence of the problems (18) and (19) and the form of objective map proves that optimal solution doesn't depend on initial point. The cardinality of every control set $U^1, \dots, U^T$ is equal to 1. Thus, the Theorem 1 is true for problem (18).

Theorem 16. *Let (18) be solvable. The sequence $\bar{u}^1, \bar{u}^2, \dots, \bar{u}^T$ forms an optimal control if and only if $\bar{u}^t$ is the solution of linear programming problem*

$$\begin{aligned}
&\left(c^tB^t + c^{t+1}A^tB^t + \dots + c^TA^{T-1}A^{T-2}\dots A^tB^t + b^t\right) u^t \to \max, \\
&D^tu^t \le d^t,
\end{aligned}$$

for $t = 1, \dots, T$.

Analogous conclusions are true for all problems and theorems considered above.

8 Concluding remarks

There are different types of processes control: optimal control, Stackelberg control, Pareto-Stackelberg control, Nash-Stackelberg control, Pareto-Nash-Stackelberg control, etc.

The direct-straightforward, dual and classical principles (Pontryagin and Bellman) may be applied for determining the desired control of dynamic processes. These principles are the bases for pseudo-polynomial methods, which are exposed as a consequence of theorems for linear discrete-time Pareto-Nash-Stackelberg control problems.

The direct-straightforward principle is applied for solving the problem of determining the optimal control of set-valued linear discrete-time processes. Pseudo-polynomial method of solving is constructed.

The results obtained for different types of set-valued control will be exposed in a future paper.

References

[1] S.A. Ashmanov, A.V. Timohov. *The optimization theory in problems and exercises,* Moscow, Nauka, 1991, pp. 142–143.

[2] L.S. Pontryagin, V.G. Boltyanskii, R.V. Gamkrelidze, E.F. Mishchenko. *Mathematical theory of optimal processes,* Moscow, Nauka, 1961 (in Russian).

[3] R. Bellman. *Dynamic Programming,* Princeton, New Jersey, Princeton University Press, 1957.

[4] T. Rockafellar. *Convex Analysis,* Princeton University Press, 1970.

[5] J. von Neuman, O. Morgenstern. *Theory of Games and Economic Behavior,* Annals Princeton University Press, Princeton, NJ, 1944, 2nd ed. 1947.

[6] J.F. Nash. *Noncooperative game,* Annals of Mathematics, Vol. 54, 1951, pp. 280–295.

[7] H. Von Stackelberg. *Marktform und Gleichgewicht (Market Structure and Equilibrium),* Springer Verlag, Vienna, 1934.

[8] G. Leitmann. *On Generalized Stackelberg Strategies,* Journal of Optimization Theory and Applications, Vol. 26, 1978, pp.637–648.

[9] V. Ungureanu. *Solution principles for simultaneous and sequential games mixture,* ROMAI Journal, Vol. 4, No.1, 2008, pp. 225–242.

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