Computer Science Journal of Moldova, vol.**21**, no.1(61), 2013

Median calculation for heterogeneous complex of abstract cubes*

Sergiu Cataranciuc

Abstract

Median problem for an arbitrary complex of abstract cubes is studied. First, a number of auxiliary results related to some special metric properties of abstract n-dimensional cube are presented. Basing on these results and results obtained in the study of homogeneous complexes [13], it is proved that median of an arbitrary complex of abstract cubes can be calculated without using metrics. Method which represents a generalization of method applied to homogeneous complex of abstract cubes is proposed.

Keywords: Abstract cube, cubic complex, median, transversal, group of homologies, convex set, class of parallel edges.

1 Introduction

In this paper we study median problem that was partially examined in [3], [5] for some particular case. For example, in a complex of abstract figures, which do not belong to any space, it is considered a finite metrizable set of points X, which is a part of a set of continuum power. Then we search the points from X with the following property: for any such point the sum of distances from this point to all other points from X to be *minimal*. These are the so-called **medians**. This problem is known as a practical one and even the corresponding applications [2], [3] are known.

We are interested to consider figures complex for which the searchable points **do not depend on the metric**, but only on the **weights** of the elements from X.

* This work was supported by the Institutional Project Ref. Nr. 11.817.08.47A

It is to mention that these simple formulations, as well as applications, require a strong theoretical foundation: *topology of multi-ary* relations [23] and their homology theory. Homology and co-homology groups for a complex of abstract cubes, as a particular case of complex of multi-ary relations, have been described and studied in [13], [15-17].

2 Complex of abstract cubes

In [14] abstract n-dimensional cube is defined. Definition is based on a complex of multi-ary relations defined on a set of elements X. Let X be an arbitrary set of elements and L – a finite subset from X. We form a finite sequence of Cartesian products $L = L^1$, $L^2 = L \times L, ..., L^{n+1} = = L^n \times L$ and consider the subset $R^m \subset L^m$, $1 \leqslant m \leqslant n+1$, called m-ary relation [23]. A case when these relations represent families of ordered sequences without repetitions of elements from L is studied in [17]. Any subsequence $(x_{j_1}, x_{j_2}, ..., x_{j_l})$ for sequence $(x_{i_1}, x_{i_2}, ..., x_{i_m} \in R^m)$, $1 \leqslant l \leqslant m$, which preserves an order of elements of $(x_{i_1}, x_{i_2}, ..., x_{i_m})$, is called a hereditary subsequence.

Definition 1. [17] *Family of relations* $\{R^1, R^2, ..., R^{n+1}\}$ *which satisfies the following conditions:*

I. $R^1 = L^1 = L$;

II. $R^{n+1} \neq \varnothing$;

III. *any hereditary subsequence with* l *elements of sequences from* R^m, $2 \leqslant m \leqslant n+1$, *belongs to the* l*-ary relation* R^l;

IV. *intersection of two sequences from* R^m, $1 \leqslant m \leqslant n$, *is a hereditary subsequence for each of them or is an empty set;*

is called ***finite complex of multi-ary relations*** *and denoted by* $\mathcal{R}^{n+1} = (R^1, R^2, ..., R^{n+1})$.

In general case, the sequences from sets R^m, $2 \leqslant m \leqslant n+1$, may also contain repetitions of elements. This situation leads to a complex, called generalized complex of multi-ary relations. This case is examined in papers [15], [16].

A sequence $(x_{i_0}, x_{i_1}, ..., x_{i_m}) \in R^{m+1}$, $0 \leqslant m \leqslant n$, can be examined as **abstract simplex** with dimension m. This simplex is denoted by $S_i^m = (x_{i_0}, x_{i_1}, ..., \ x_{i_m})$. Let $\mathcal{S}^m$ be a set of all m-dimensional simplexes, determined by the elements of multi-ary relation. Then the complex of multi-ary relations can be seen as a family of abstract simplexes of dimension $0 \leqslant m \leqslant n$, which we will denote by $K^n = = (\mathcal{S}^0, \mathcal{S}^1, \mathcal{S}^2, ..., \mathcal{S}^n)$. Any element from $\mathcal{S}^l$, $0 \leqslant m \leqslant n$, which is a hereditary subset of $S_i^m \in \mathcal{S}^m$, $0 \leqslant l \leqslant m \leqslant n$, is called **a face of dimension** l of the abstract simplex S_i^m. As it results from condition IV of the Definition 1, the intersection of any two simplexes represents an abstract simplex (face) or an empty set.

Definition 2. *A set of all simplexes from complex K^n, for which abstract simplex S_i^m is a common face, is called star of simplex S_i^m and is denoted by stS_i^m.*

Using abstract simplexes and their **vacuums**, the abstract n-dimensional cube as well as its vacuum is defined and studied in [14], [19].

Definition 3. *(Inductive definition of abstract n-dimensional cube)*

I. *Abstract 0-dimensional cube and abstract 1-dimensional cube coincide with abstract simplex of the same dimension. Vacuum of cube with size 0 (respectively 1) coincides with vacuum of simplex of the same dimension.*

II. *We consider two pairs of 0-dimensional cubes S_1^0, S_2^0 and S_3^0, S_4^0. The 2-ary relations of the pairs of cubes S_1^0, S_2^0 and S_3^0, S_4^0 determine existence of 1-dimensional cubes $S_1^1 = (S_1^0, S_2^0)$, $S_2^1 = (S_3^0, S_4^0)$, $S_3^1 = (S_1^0, S_3^0)$ $S_4^1 = (S_2^0, S_4^0)$. Further we consider 2- and 3-ary relations between these pairs of cubes, which determine a simplicial complex* [15]*, that leads to the existence of simplexes $S_5^1 = (S_1^0, S_4^0)$, $S_1^2 = (S_1^0, S_3^0, S_4^0)$, $S_2^2 = (S_1^0, S_2^0, S_4^0)$. To define notion of abstract 2-dimensional cube, we introduce firstly notion of* ***vacuum of the*** *corresponding* ***cube****, which will be denoted by $\overset{0}{I^2}$. This represents union of* *vacuums of simplexes* $\overset{0}{I}{}^2 = \overset{0}{S}{}_1^2 \cup \overset{0}{S}{}_2^2 \cup \overset{0}{S}{}_5^1$. *The abstract* 2*-dimensional cube is defined by its vacuum as follows:* $I^2 = \bigcup\limits_{i=1}^{4} S_i^1 \cup \overset{0}{I}{}^2$. *So, we obtain the simplicial complex that is called* ***2-dimensional pro-cube****. We will use notation* $I^2(\triangle)$. *Geometric reprezentation of 2-dimensional pro-cube is given in Figure 1.*

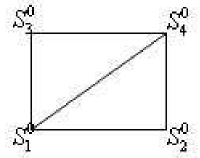

Figure 1. 2-dimensional pro-cube

III. *Assume that notion of* ***abstract*** i***-dimensional cube*** *and* ***pro-cube****,* I^i *and* $I^i(\Delta)$, $1 \leq i \leq n-1$, *as well as notion of cube's vacuum with the same dimension* $\overset{0}{I}{}^i$ *are known.*

IV. *Let's define notion of abstract n-dimensional cube by using* $(n-1)$*-dimensional cube. Let's consider 2n cubes* $I_1^{n-1}, I_2^{n-1}, ..., I_{2n}^{n-1}$ *with dimension* $(\mathrm{n}-1)$*, corresponding pro-cubes* $I_1^{n-1}(\Delta), I_2^{n-1}(\Delta), ...,$ $I_{2n}^{n-1}(\Delta)$ *and i-ary relations,* $2 \leqslant i \leqslant n$*, among vertexes of these cubes. These i-ary relations determine some simplexes, which form an abstract simplicial complex* [15]*. Thus,* ***vacuum of n-dimensional cube****, denoted by* $\overset{0}{I}{}^n$*, consists of a union of all vacuums of n-dimensional simplexes which don't intersect pro-cubes* $I_j^{n-1}(\triangle)$, $1 \leq j \leq 2n$*.* ***Abstract n-dimensional cube*** *is defined as* $I^n = \bigcup\limits_{i=1}^{2n} I_i^{n-1} \cup \overset{0}{I}{}^n$*. A simplicial complex* $I^n(\triangle)$ *will be called* ***pro-cube*** *of* I^n*.*

In what follows, we will consider that vacuums of abstract cubes are filled with elements of a set of cardinal of continuum (a set equivalent to the set of numbers from segment [0,1]).

Definition 4. *Non-empty and finite family of abstract cubes* $\mathbf{T}^n = \{I^m, 0 \leq m \leq n\}$, *which possesses the following properties:*

- ***I.*** *for any two abstract cubes* I^s, $I^t \in \mathbf{T}^n$, $0 \leqslant s, t \leqslant n$, *the following relation takes place:* $I^s \cap I^t \in \mathbf{T}^n$, *or* $I^s \cap I^t = \varnothing$ *;*
- ***II.*** *if* I^k *is the face of the m-dimensional cube* $I^m \in \mathbf{T}^n$, $0 \leqslant k < m \leqslant n$ *then* I^k *is an element of* $\mathbf{T}^n$*;*
- ***III.*** *family* $\mathbf{T}^n$ *contains at least one n-dimensional cube,*

is called ***abstract cubic complex with dimension*** n.

Homology groups $\square^m(\mathbf{T}^n, Z)$, $0 \leqslant m \leqslant n$, over a field of integer numbers Z for the complex of abstract cubes are defined in a similar way as the complex of multi-ary relations [13]. In terms of homology groups, we can affirm the following. If the complex of abstract cubes $\mathbf{T}^n$ respects the following conditions:

$$\square^0(\mathbf{T}^n, Z) \cong Z; \tag{1}$$

$$\square^1(\mathbf{T}^n, Z) \cong \square^2(\mathbf{T}^n, Z) \cong ... \cong \square^n(\mathbf{T}^n, Z) \cong 0, \tag{2}$$

then the corresponding complex is connected and acyclic.

In what follows we will study complexes of abstract cubes that satisfy the Definition 4, in which the homology group of rank zero is isomorphic to the group of integers Z (relation (1)), but the homology groups of rank $1, 2, 3, ..., n$ are isomorphic to zero (relation (2)). These complexes generalize the homogeneous complexes studied in [13] and are called heterogeneous complexes. Such a complex is represented in the Figure 2.

Definition 5. *Two edges, determined by 1-dimensional cubes* I_j^1 *and* I_k^1 *of the cubic complex* $\mathbf{T}^n$ *are called parallel if there is a sequence of 1-dimensional cubes* $I_j^1 = I_{\alpha_1}^1, I_{\alpha_2}^1, ..., I_{\alpha_t}^1 = I_k^1$, *so that every two neighboring cubes* $I_{\alpha_s}^1, I_{\alpha_{s+1}}^1$ *of this sequence represent the disjoint edges (opposite) of a 2-dimensional cube of* $\mathbf{T}^n$.

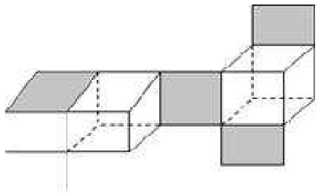

Figure 2. Heterogeneous complex of abstract cubes

Definition 6. *A maximal set of parallel edges of a complex of abstract cubes* $\mathbf{T}^n$ *is called class of parallel edges.*

According to the Definition 5 and Definition 6, in the connected and acyclic complex $\mathbf{T}^n$ every 1-dimensional cube, which is not the face of a cube with dimension $q \geqslant 2$ from $\mathbf{T}^n$, forms one class each.

It is to mention that in the case of homogeneous complexes of abstract cubes [13] every class of parallel edges contains at least 2^{n-1} of 1-dimensional cubes. For the complex represented in Figure 2 we have 11 classes of parallel edges and two of them contain only one element each.

3 Transversals of heterogeneous complex

Let $C_1, C_2, ..., C_m$ be classes of parallel edges of the complex $\mathbf{T}^n$. The set of abstract cubes of $\mathbf{T}^n$, which contains the edges of class C_i as faces, is called transversal of complex $\mathbf{T}^n$ and is denoted by $T(C_i)$, $1 \leqslant i \leqslant m$.

Theorem 1. *A transversal determined by a class of parallel edges of a connected and acyclic complex of abstract cubes* $\mathbf{T}^n$ *also represents complex of abstract cubes denoted by* $\mathbf{T}^q$*,* $1 \leqslant q \leqslant n$*.*

Proof: Let $T(C_i)$ be a transversal determined by a class of parallel edges C_i of the complex $\mathbf{T}^n$, and q – maximum dimension of abstract cubes of $\mathbf{T}^n$ which forms transversal $T(C_i)$. Thus, we have a complex of cubes $\mathbf{T}^q$, which satisfies the conditions I - III of the Definition 4. Taking into consideration the definition of class of parallel edges (Definition 6), it results that $\mathbf{T}^q$ is a connected complex. As the homology groups of the complex $\mathbf{T}^q$ are subgroups of homology groups of complex $\mathbf{T}^n$, $\mathbf{T}^q$ is an acyclic complex of abstract cubes. ■

The border of complex $\mathbf{T}_i^q$, determined by transversal $T(C_i)$, contains two maximal disjoint complexes of dimensions $(q-1)$ that do not contain any edge of C_i. We should denote these subcomplexes through $\mathbf{T}_{i(1)}^{q-1}$ and $\mathbf{T}_{i(2)}^{q-1}$. These complexes could be disconnected.

Let $V(T(C_i))$ be the vacuum of transversal $T(C_i)$, which is a union of vacuums of cubes of complex $\mathbf{T}_i^q$, which do not belong to subcomplexes $\mathbf{T}_{i(1)}^{q-1}$ and $\mathbf{T}_{i(2)}^{q-1}$.

Theorem 2. *Transversal $T(C_i)$, $1 \leqslant i \leqslant m$, of connected and acyclic abstract cubic n-dimensional complex, divides this complex through the vacuum of the transversal $T(C_i)$ in connected and acyclic two cubic abstract complexes of dimension q, $1 \leqslant q \leqslant n$.*

Proof of the theorem results immediately from the fact that $\mathbf{T}^n$ is a connected and acyclic complex, as well as from the definition of corresponding transversals.

Let $\mathbf{T}^n$ be the n-dimensional complex of abstract cubes which satisfies the Definition 4 and contains a cube $I^r \in \mathbf{T}^n$, $0 \leqslant r \leqslant n$, which is not a face of n-dimensional cubes of $\mathbf{T}^n$. We will call such complexes heterogeneous complexes.

Let's denote by $\mathcal{Q}^t$, $1 \leqslant t \leqslant n$, the family of maximal and connected homogeneous t-dimensional subcomplexes of $\mathbf{T}^n$. Obviously, in case of heterogeneous complex $\mathbf{T}^n$, we have:

1) $\mathcal{Q}^n \neq \varnothing$;

2) there is a value of t, $1 \leqslant t \leqslant n$, such that $\mathcal{Q}^t \neq \varnothing$ ($t \geqslant 1$, as it was mentioned above, we are studying only connected and acyclic complexes).

Thereby, in a heterogeneous complex any two maximal homogeneous n-dimensional subcomplexes are united by a sequence of homogeneous complexes $\mathbf{T}_1^{n_1}, \mathbf{T}_2^{n_2}, ..., \mathbf{T}_q^{n_q}$, where $1 \leqslant n_1, n_2, ..., n_q < n$, with the following properties:

a) $\mathbf{T}_j^{n_j} \in \mathcal{Q}^j, 1 \leqslant j \leqslant q$;

b) $n_i \neq n_{i+1}$, for any $i \in \{1, 2, .., q-1\}$.

In Figure 3 a geometric representation of a heterogeneous complex of dimension three is given. We have three families of homogeneous subcomplexes:

$$\mathcal{Q}^1 = \{\mathbf{T}_1^3, \mathbf{T}_2^3, \mathbf{T}_3^3\},$$

$$\mathcal{Q}^2 = \{\mathbf{T}_1^2, \mathbf{T}_2^2, \mathbf{T}_3^2, \mathbf{T}_4^2\},$$

$$\mathcal{Q}^3 = \{\mathbf{T}_1^1, \mathbf{T}_2^1, \mathbf{T}_3^1\}.$$

Let's note that the homogeneous 3-dimensional complexes $\mathbf{T}_1^3$ and $\mathbf{T}_2^3$ are united by sequence of homogeneous complexes with smaller dimensions: $\mathbf{T}_2^2, \mathbf{T}_1^1, \mathbf{T}_3^2$.

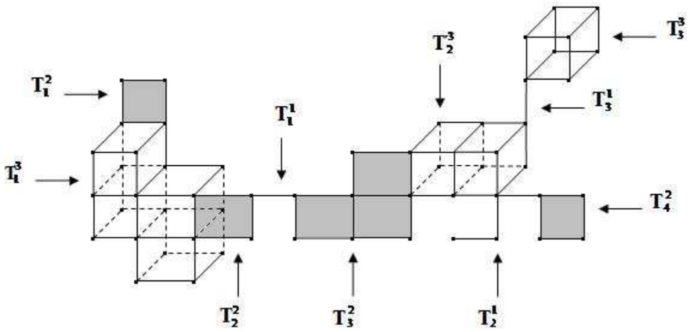

Figure 3. Heterogeneous complex with 3 families of homogeneous subcomplexes

The notion of an interior and a border of homogeneous complex of abstract cubes $\mathbf{T}^n$ was defined [13] and denoted respectively by int$\mathbf{T}^n$ and bd$\mathbf{T}^n$.

Let K^n be a heterogeneous complex of abstract cubes with families of homogeneous maximal connected complexes $\mathcal{Q}^1, \mathcal{Q}^2, ..., \mathcal{Q}^n$. In this case, for the complex K^n we use the notation: $K^n = (\mathcal{Q}^1, \mathcal{Q}^2, ..., \mathcal{Q}^n)$.

Definition 7. *A union* $\bigcup_{\mathbf{T}^n \in Q^n} int\mathbf{T}^n$ *is called an interior of complex* K^n *and is denoted by* $intK^n$*. Difference* $K^n \backslash intK^n$ *is called a border of this complex and is denoted by* bdK^n*.*

4 Representation of abstract cubes

Let I^n be an abstract n-dimensional cube with a set of vertexes $V = \{x_1, x_2, ..., x_{2^n}\} \subset X$, $cardX = r > 2^n$. We fix vertex x_1 and form the following sets:

$$\begin{aligned}
\Gamma_0 &= \{x_1\}, \\
\Gamma_1 &= \{x_i \in V : d(x_1, x_i) = 1\}, \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
\Gamma_k &= \{x_i \in V : d(x_1, x_i) = k\}, \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
\Gamma_n &= \{x_i \in V : d(x_1, x_i) = n\} = \{x_{2^n}\},
\end{aligned}$$

where d represents the Hamming metrics defined on this cube (see [21] and theorem 3 from [13]).

Without losing generality, we consider that x_{2^n} is the vertex of cube I^n for which $d(x_1, x_{2^n}) = n$.

Thus, the abstract n-dimensional cube I^n can be represented by the sequence of sets $\Gamma_0, \Gamma_1, ..., \Gamma_n$, which possesses the following properties:

a) $card\Gamma_i = C_n^i$;

b) any two distinct elements of the set Γ_i, $0 < i < n$, do not represent a 1-dimensional face of cube I^n,

c) $d(x_k, x_l)$ is an even number, for any two distinct elements x_k, x_l of the set Γ_i, $0 < i < n$.

Obviously, not every sequence of this type represents the n-dimensional cube.

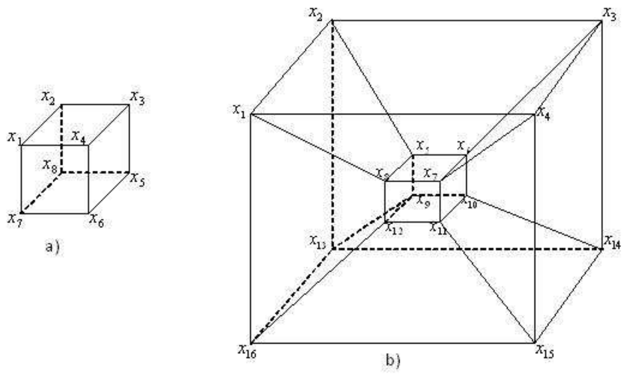

Figure 4. Geometric representation of 3-dimensional cube (a) and 4-dimensional cube (b)

In addition to classic geometric representations of n-dimensional cubes (Figure 4) we will also use the representation of those by respective sequence of sets $\Gamma_0, \Gamma_1, ..., \Gamma_n$ (Figure 5).

If for n-dimensional cube I^n there are known the sets $\Gamma_0, \Gamma_1, ..., \Gamma_n$, then we will write $I^n = (\Gamma_0, \Gamma_1, ..., \Gamma_n)$.

Lemma 1. *The number of edges that connect any vertex of* Γ_i *with vertexes of* Γ_{i-1}*,* $1 \leqslant i \leqslant n-1$*, is equal to* $N^i_{i-1} = i$ *and the number of edges that connect any vertex of* Γ_i *with vertexes of* Γ_{i+1} *is equal to* $N^i_{i+1} = n - i$*. (Obviously,* $N^0_1 = N^n_{n-1} = n$*.)*

Proof. Let x_j be the element of Γ_i, and $g_{i-1}(x_j)$ be the number of elements of Γ_{i-1} adjacent to x_j. On the n-dimensional cube $\mathbf{T}^n$ we introduce Hamming metrics, considering that the sequence $(0, 0, ..., 0)$ is attributed to vertex x_1. Then, any element of Γ_i will have a sequence α with i units. This element will be adjacent to those elements of Γ_{i-1} whose sequences differ from α by exactly one element. The number of such sequences is equal exactly to i (each sequence differs from α

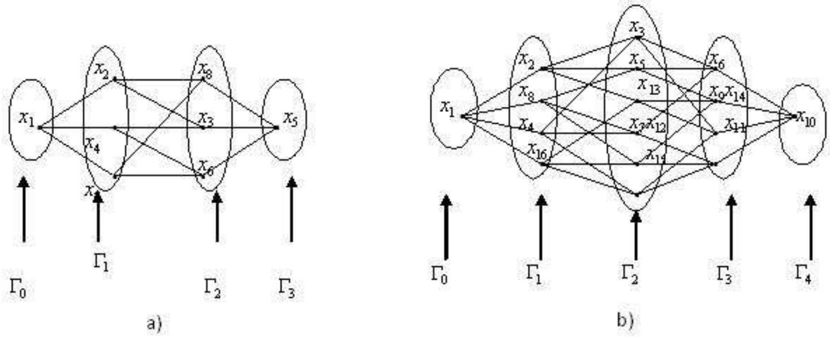

Figure 5. Representation of 3-dimensional cube by the set $\Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3$ (a) and 4-dimensional cube by the set $\Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ (b)

by the fact that only one from i units of α will be replaced by zero). Therefore, $q_{i-1}(x_j) = i$ and, as the x_j was chosen arbitrarily from Γ_i, we obtain $N^i_{i-1} = i$.

In the n-dimensional cube all vertexes have degree equal to n. Therefore, $N^i_{i+1} = n - i$.

If $i = 0$ and $i = n$ one must calculate only $N^0_1 = N^n_{n-1} = n$. ■

Now we will estimate the sum of distances between an arbitrary element $x \in \Gamma_k$ and the set Γ_{k+l} in case of abstract $n-$dimensional cube $I^n = (\Gamma_0, \Gamma_1, ..., \Gamma_n)$, $0 \leqslant k \leqslant n-1$, $1 \leqslant l \leqslant n-k$.

Lemma 2. *The sum of distances between the element $x \in \Gamma_k$ and all elements of set Γ_{k+l} is equal to*

$$\sum_{i=0}^{\min\{k,n-k-l\}} (l+2i) \cdot C^i_k \cdot C^{l+i}_{n-k}.$$

Proof. We choose an arbitrary vertex $x \in \Gamma_k$ and define Hamming metrics on the cube $I^n = (\Gamma_0, \Gamma_1, ..., \Gamma_n)$ so that the vertex $x_1 \in \Gamma_0$

is marked by the sequence (0, 0, ..., 0). Binary sequence which corresponds to vertex $x \in \Gamma_k$ will have k units and $(n-k)$ zeros. Without losing generality we will consider that it is the sequence $\widetilde{\alpha} = (\underbrace{11...1}_{k}\ \underbrace{00...0}_{n-k})$. Any element from Γ_{k+l} is marked by a sequence $\widetilde{\beta}$ which contains l units more than $\widetilde{\alpha}$. These l units can be obtained in two ways:

a) some l zeros from $\widetilde{\alpha}$ are replaced with units. The distance between the vertexes, to which $\widetilde{\alpha}$ and $\widetilde{\beta}$ correspond, will be equal to l. So the sequence $\widetilde{\beta}$ can be chosen in C^l_{n-k} modes;

b) some i units from $\widetilde{\alpha}$ are replaced by zeros, but some $(l+i)$ zeros are replaced by units (on condition $l+i \leqslant n-k$). As a result, we get a sequence $\widetilde{\beta}$ with $(k+l)$ units and the distances, between $\widetilde{\alpha}$ and $\widetilde{\beta}$ will be $l+2i$ (we apply Hamming distance, which is calculated according to the formula $\sum\limits_{i=1}^{n} |\alpha_i - \beta_i|$). Such a sequence $\widetilde{\beta}$ can be chosen in $C^i_k \cdot C^{l+i}_{n-k}$ modes.

As a result, we obtain that the sum of distances between the element $x \in \Gamma_k$ and all elements of set Γ_{k+1} is:

$$\sum_{i=0}^{\min\{k,n-k+l\}} (l+2i) \cdot C^i_k \cdot C^{l+i}_{n-k}. \quad \blacksquare$$

The sum from Lemma 2 is a constant, that characterizes the relation between the sets Γ_k and Γ_{k+l}. We will denote this constant by σ^{k+l}_k. Similarly it can be proved that the sum of distances between a fixed element $x \in \Gamma_{k+1}$ and all elements of the set Γ_k does not depend on the choice of x, so it is a constant value, which we denote by σ^k_{k+l}. It is easy to prove that $\sigma^{k+l}_k \neq \sigma^k_{k+l}$, for any values of k and l which verify the relations: $0 \leqslant k \leqslant n-l$, $1 \leqslant l \leqslant n$ and $k+l \neq n-k$. In case when $k+l = n-k$ we have the sets Γ_k and Γ_{n-k} for which $\sigma^{n-k}_k = \sigma^k_{n-k}$, i.e. $\sigma^{k+l}_k = \sigma^k_{k+l}$.

Corollary of Lemma 2 *For any set Γ_i, $1 \leqslant i \leqslant n$, of the n-dimensional cube $I^n = (\Gamma_0, \Gamma_1, ..., \Gamma_n)$, the sequence $\sigma^i_0, \sigma^i_1, ..., \sigma^i_{i-1}$ is decreasing.*

Let $K^n(I^n)$ be the n-dimensional complex of abstract cubes, determined by cube I^n, i.e. $K^n(I^n)$ is formed by I^n and all its faces. From Definition 5 and Definition 6 it results that the n-dimensional transversal divides the complex $K^n(I^n)$ into two connected subcomplexes. Let I^1 be a 1-dimensional face (an edge) of the cube $I^n = (\Gamma_0, \Gamma_1, ..., \Gamma_n)$, $\Gamma_0 = \{x_1\}$, $\Gamma_n = \{x_{2^n}\}$, incident to vertex x_{2^n}. We will denote by $T^n_{I^1}$ the transversal determined by the class of parallel edges, which contain the face I^1, and by $K(x_{2^n})$ – the subcomplex of $K^n(I^n)$, determined by $T^n_{I^1}$, which contains the vertex x_{2^n}. Respectively, by $K(\overline{x_{2^n}})$ we will denote the second subcomplex.

Lemma 3. *For any transversal $T^n_{I^1}$ and any set Γ_k, $\left[\frac{n}{2}\right] < k \leqslant n-1$, the following relation holds:*

$$|K(x_n) \cap \Gamma_k| > |K(\overline{x_n}) \cap \Gamma_k| \, .$$

Proof. Let I^1 be the 1-dimensional face (edge) of the n-dimensional cube I^n, incident to vertex x_{2^n}. We denote the extremities of edge I^1 by a and b, considering $b = x_{2^n}$. The transversal T_{I^1} divides the cube I^n into two $n-1$-dimensional cubes I^{n-1}_1 and I^{n-1}_2, so that the vertex a belongs to I^{n-1}_1 and the vertex $b = x_{2^n}$ – to cube I^{n-1}_2. Elements of set Γ_k are at the distance $(n-k)$ from the vertex b. The set Γ_k is formed by 2 subsets:

a) $S^{'}_k$ – the set of all vertexes of $(n-1)$-dimensional cube I^{n-1}_1, that is at the distance $(n-k)$ from the vertex b in the cube I^n.

b) $S^{''}_k$ – the set of all vertexes of I^{n-1}_2, that is at the distance $(n-k)$ from b in I^n;

Let's remind now that equality $d(b,z) = d(b,a) + d(a,z)$ holds in I^n for any vertex z of I^{n-1_1} that belongs to $S^{'_k}$. Taking into consideration that b ($b = x_{2^n}$) is vertex of I^{n-1}_2 we obtain the inequality $\left|S^{'}_k\right| > \left|S^{''}_k\right|$. This inequality confirms the lemma's affirmation. ■

Theorem 3. *For any set Γ_k of n-dimensional cube $I^n = (\Gamma_0, \Gamma_1, ..., \Gamma_n)$, $\left[\frac{n}{2}\right] < k \leqslant n-1$, the relation $\sigma^k_{k-l} > \sigma^k_n$ is true.*

Proof. On the basis of corollary from Lemma 2 it is sufficient to prove that for any set Γ'_k, $[n/2] < k \leqslant n-1$, the relation $\sigma^k_{k-l} > \sigma^k_n$ is true.

Evidently

$$\sigma^k_n = (n-k) \cdot |S_k| = (n-k) \cdot C^k_n.$$

According to Lemma 2 we obtain:

$$\sigma^k_{k-1} = \sum_{i=0}^{\min\{k-1,n-k\}} (1+2i) \cdot C^i_{k-1} \cdot C^{1+i}_{n-k+1}.$$

Since the theorem is formulated for the sets Γ_k with indexes $k > \left[\frac{n}{2}\right]$, we will have:

$$\min\{k-1, n-k\} = n-k.$$

So for σ^k_{k-1} we get the sum:

$$\sigma^k_{k-1} = \sum_{i=0}^{n-k} (1+2i) \cdot C^i_{k-1} \cdot C^{1+i}_{n-k+1}.$$

Examining expressions σ^k_n and σ^k_{k-1} and taking into consideration the condition $\left[\frac{n}{2}\right] < k \leqslant n-1$, we obtain $\sigma^k_{k-l} > \sigma^k_n$. ∎

Let $K^n(I^n)$ be an abstract cubic complex determined by n-dimensional cube $I^n = (\Gamma_0, \Gamma_1, ..., \Gamma_n)$, i.e. $K^n(I^n)$ is a complex formed by I^n and all its faces have dimensions k, $0 \leqslant k \leqslant n-1$. We choose a subcomplex $K^q \subset K^n(I^n)$, formed of faces of cube I^n, which are generated by the sets of vertexes $\Gamma_0, \Gamma_1, ..., \Gamma_q$, where $q > \left[\frac{n}{2}\right]$. We build the n-dimensional transversals through each edge I^1 incident to vertex x_1 (x_1 is the unique element of Γ_0) and denote by $K^{n-1}_{I^1}(I^n)$ subcomplex from $K^n(I^n)$ determined by the transversal T_{I^1} and containing at least half of elements of Γ_q. We denote by $Q^1(x_1)$ the family of all 1-dimensional cubes from I^n incident to vertex x_1. Further, the cube I^n will be called the cubic closing of the complex K^q.

Theorem 4. *If $I^n = (\Gamma_0, \Gamma_1, ..., \Gamma_n)$ is a cubic closing of complex of abstract cubes K^q, $[n/2] < q \leqslant n-1$, then*

a) $\left(\bigcap\limits_{I^1 \in Q^1(x_1)} K_{I^1}^{n-1}(I^n) \right) \cap K^q = \varnothing$;

b) $x_{2^n} \in \bigcap\limits_{I^1 \in Q^1(x_1)} K_{I^1}^{n-1}(I^n)$.

Proof. At first, we will prove the relation b) from the Theorem. We should mention that in the cubic closing I^n any transversal determined by the class of parallel edges, which contains an edge incident to vertex x_1, will also contain obligatory an edge incident to vertex x_{2^n}. On the basis of Lemma 3 and condition of the Theorem, it results that:

$$x_{2^n} \in \bigcap_{I^1 \in Q^1(x_1)} K_{I^1}^{q-1}.$$

Let us now prove the relation a). We should mention that in the n-dimensional cube any n-dimensional transversal divides it into two $(n-1)$-dimensional cubes, which are faces of I^n. For n-dimensional cube we can build exactly n transversals, hence, we obtain n pairs of $(n-1)$-dimensional cubes. We denote by $x_1, x_2, ..., x_{2^n}$ the vertexes of cube $I^n = (\Gamma_0 = \{x_1\}, \Gamma_1, ..., \Gamma_{n-1}, \Gamma_n = \{x_{2^n}\})$. Evidently, each transversal contains exactly one edge incident to vertex x_{2^n}. From each pair of $(n-1)$-dimensional cubes determined by transversals from I^n, we choose the cube which contains vertex x_{2^n} and forms a family of $(n-1)$-dimensional cubes $F_{I^n}^{n-1}(x_{2^n})$. The intersection of all cubes from $F_{I^n}^{n-1}(x_{2^n})$ contains only one vertex – vertex x_{2^n}.

The subcomplex $K_{I^1}^{n-1}(I^n)$ contains at least a half of the elements of set Γ_q. Since $q > [n/2]$, this subcomplex does not contain the vertex x_1 from K^q, but contains the vertex x_{2^n} from the cubic closing I^n of K^q. Hence, the affirmation a) is proved. ■

Theorem 5. *If $I^n = (\Gamma_0 = \{x_1\}, \Gamma_1, ..., \Gamma_{n-1}, \Gamma_n = \{x_{2^n}\})$ is the n-dimensional cube, and K^q is a subcomplex of $K^n(I^n)$ formed from faces of the cube I^n, that is generated by the sets $\Gamma_0, \Gamma_1, ..., \Gamma_q$, where*

$q > \left[\frac{n}{2}\right]$, then there exists a system of weights $p(x_i)$ of vertexes of complex K^q so that for any vertex x of K^n with function $f(x) = = \sum_{i=1}^{2^n} p(x_i) \cdot d(x, x_i)$ we obtain:

$$f(x) > f(x_{2^n}).$$

Proof. For vertexes of complex K^q we fix the weights:

$p(x_i) = 1$ for any element $x_i \in \Gamma_0 \cup \Gamma_1 \cup ... \cup \Gamma_{q-1}$;

$p(x_i) = M$ for any element $x_i \in \Gamma_q$, where M is a sufficiently big number.

Number M can be chosen so that it will be bigger than the sum of distances from x_{2^n} till all vertexes of set $\Gamma_0 \cup \Gamma_1 \cup ... \cup \Gamma_{q-1}$.

Basing on structure of the set Γ_q and the Theorem 4, we obtain the affirmation of the Theorem 5. ■

We will study further the complexes of abstract cubes $\mathbf{T}^n$ which possess the following properties:

1) any cube $I^k \in int\mathbf{T}^n$ belongs to at least 2^{n-k} n-dimensional cubes;

2) if x is a vertex from border $bd\mathbf{T}^n$, to which exactly t edges, $3 \leqslant t \leqslant n$, are incident and the union $\Gamma_0 \cup \Gamma_1 \cup ... \cup \Gamma_{q-1}$ exists on this border, then the cubic closing of this subcomplex belongs to $\mathbf{T}^n$;

3) homology group of rank zero of complex $\mathbf{T}^n$ is isomorphic with the group of integer numbers, i.e.

$$\square^0(\mathbf{T}^n, Z) \cong Z;$$

4) homology groups of rank $k = 1, 2, .., n$ of complex $\mathbf{T}^n$ are isomorphic with zero, i.e.

$$\square^1(\mathbf{T}^n, Z) \cong \square 2(\mathbf{T}^n, Z) \cong ... \cong \square^n(\mathbf{T}^n, Z) \cong 0;$$

5 Interpretation of complex $\mathbf{T}^n$ in a normed space

Similarly with the case of homogeneous tree of abstract cubes, we define metric d on the set of 0-dimensional cubes (vertexes) from K^n [13].

1-dimensional skeleton of complex $\mathbf{T}^n$ is a directed graph $G = (X; E)$ with the vertexes that correspond to 0-dimensional cubes, and the arcs that correspond to 1-dimensional cubes from $\mathbf{T}^n$.

Let $C_1, C_2, ..., C_m$ be the classes of parallel edges with lengths $d_1, d_2, ..., d_m$. We consider the space R_1^m over the field of real numbers with the norm $\|x\| = \sum\limits_{i=1}^{m} |x_i|$. We construct the segments with lengths $d_1, d_2, ..., d_m$ on axes of coordinates $OY_1, OY_2, ..., OY_m$ from origin $O \in R_1^m$. So the parallelepiped P^m is constructed univocally on these segments. The set of all k-dimensional faces of P^m forms a complex of parallelepipeds, which we will denote by $\mathcal{P}^k = \{P^k \subset P^m | 0 \leqslant k \leqslant m\}$. In case of $k = 1$ we obtain the complex $\mathcal{P}^1$, that represents a connected, metric and undirected graph. We will denote this graph by $H = (Y; V)$.

From modality of construction of complex $\mathcal{P}^m$ and definition of metric on complex, the following theorem results:

Theorem 6. *For the complex $\mathbf{T}^n$ exists an unequivocal application $\alpha : \mathbf{T}^n \longrightarrow \mathcal{P}^m$, that interprets $\mathbf{T}^n$ on the subcomplex of $\mathcal{P}^m$, so that $\alpha : G\mathcal{P}^1$ is an isometry.*

In R_1^m we denote by $Y = \{y_1, y_2, ..., y_m\} \subset \alpha(\mathbf{T}^n)$ the set of vertexes $\alpha(X)$ and consider the following function

$$f(y) = \sum_{i=1}^{m} p(y_i) \cdot \|y - y_i\|, \tag{4}$$

where y_i is the image $\alpha(x_i)$ and $p(y_i) = p(\alpha(x_i))$, $1 \leqslant i \leqslant n$.

We will study this function and prove that the point $y^* \in R_1^m$ such as

$$f(y^* = \min_{y \in R_1^m} f(y)) = \min_{y \in R_1^m} \sum_{i=1}^{m} p(y_i) \cdot \|y - y_i\|$$

is a median of graph $H = (Y; V)$.

6 Median calculation

For an arbitrary point $y = (y^1, y^2, ..., y^m) \in R_1^m$ and every point $y_j = = (y_j^1, y_j^2, ..., y_j^m) \in Y$, $1 \leqslant j \leqslant m$ we form the sets

$$\begin{aligned} J^+ &= \{j : y^i - y_j^i > 0\} \\ J^0 &= \{j : y^i - y_j^i = 0\} \\ J^- &= \{j : y^i - y_j^i < 0\}. \end{aligned}$$

similarly to [22].

We denote

$$\begin{aligned} A &= \sum_{j \in J^- \cup J^0} P(y_j), \\ B &= \sum_{j \in J^+} P(y_j), \\ C &= \sum_{j \in J^-} P(y_j), \\ D &= \sum_{j \in J^- \cup J^0} P(y_j). \end{aligned} \tag{1}$$

As in case of homogeneous complex of multi-ary relations [13], the following theorem holds:

Theorem 7. *The point $y_j \in Y \subset R_1^m$ minimizes the function (4) if and only if the following relations are satisfied:*

$$A \geqslant C \quad and \quad B \leqslant D. \tag{5}$$

We denote that the calculation of median does not depend on the edges length from classes $C_1, C_2, ..., C_m$. This suggests us the idea that the parallelepiped P^m could be replaced by a unitary cube in space R_1^m.

Thus, we can define the application $\beta : P^m \longrightarrow Q^m$, where Q^m is a unitary cube situated at the origin of coordinate system. As a result of application β, the graph H passes in a metric graph $\beta(H)$, which we will denote by $\mathcal{G} = (Z; W)$. This graph is a subgraph of 1-dimensional skeleton of the cube Q^m. As a result, we obtain that on the cube and respectively on the graph G the Hamming metric is defined. Hence, the application $\beta\alpha(C_i)$, $1 \leqslant i \leqslant n$, represents a set of edges C_i^1 in the constructed graph $\mathcal{G} = (Z; W)$. For the set of vertexes of the graph G we will keep the system of weights of the complex K^n, i.e. $p(z_i) = p(x_i)$, where $z_i = \beta\alpha(x_i), 1 \leqslant i \leqslant n$, but n represents the number of 0-dimensional cubes of K^n.

So we obtained the application:

$$\beta\alpha : G \longrightarrow \mathcal{G}.$$

Let's examine an arbitrary vertex z_i of the cube Q^m. We represent its coordinates by a sequence formed from 0 and 1. From coordinates of vertices z_i of the graph G we form a matrix A, similarly to the case of homogeneous complex of abstract cubes [13].

For the matrix A we calculate a resulting sequence $r = (r_1, r_2, ..., r_m)$ considering the following:

a) $r_j = 1$, if $\sum\limits_{i=1}^{n} z_i^j \cdot p(z_i) > 1/2 \cdot \sum\limits_{i=1}^{n} p(z_i)$;

b) $r_j = 0$, if $\sum\limits_{i=1}^{n} z_i^j \cdot p(z_i) < 1/2 \cdot \sum\limits_{i=1}^{n} p(z_i)$;

c) $r_j = 0$ or 1(indifferently), if $\sum\limits_{i=1}^{n} z_i^j \cdot p(z_i) = 1/2 \cdot \sum\limits_{i=1}^{n} p(z_i)$.

Theorem 8. *A vector $r = (r_1, r_2, ..., r_m)$ calculated according to the rules a)-c) represents a line of matrix A.*

Proof. Assume the opposite. Let there exists a vertex $r = (r_1, r_2, ..., r_m)$ in the cube Q^m, which does not belong to the graph $\mathcal{G} = \beta\alpha(\mathcal{G})$, but this vertex minimizes the function (4). Obviously, m faces of dimension $m-1$ correspond to r. Each of these faces contains the $(n-1)$-dimensional transversal of complex $\beta\alpha(\mathbf{T}^n)$. We should mention that r respects the relations (4.9) from [13]. In this complex any m transversal $\beta\alpha(T^{n-1}_{I^1_1}), ..., \beta\alpha(T^{n-1}_{I^m_1})$ has an empty intersection, because r does not belong to them. It is in contradiction to the Theorem 6. ■

References

[1] D. S. Hochbaum. *Heuristics for the fixed cost median problem*, Math. Programming, 22, 1982, pp. 148–162.

[2] N. Christofides. *Graph theory: an algorithmic approach,* Academic Press, 1975, 400p.

[3] P. Soltan, D. Zambiţchi, Ch. Prisăcaru. *Extremal problems on graphs and algorithms for their solution,* Chişinău, Ştiinţa, 1973, 90p. (in Russian).

[4] V. Soltan. *Introduction to axiomatic theory of convexity,* Chişinău, Ştiinţa, 1984, 221p. (in Russian).

[5] P. Soltan. *Extremal problems on convex sets,* Chişinău, Ştiinţa, 1976, 115p (in Russian).

[6] V. Boltyansky, P. Soltan. *Combinatorial geometry of various classes of convex sets,* Chişinău, Ştiinţa, 1978, 280p. (in Russian).

[7] K. Menger. *Untersuchungen uber allgemeine Metrik. I, II, III,* Math. Ann., nr.100, 1928, p.75–163 (in German).

[8] J. De. Groot. *Some special metrics in general topology,* Collog. Math., 6 (1958), pp. 283–286.

[9] A. D. Aleksandrov, V. A. Zalgaller. Two – dimensional manifolds of bounded curvature, *(Foundations of the intriusic geometry of surfaces)*, Trudy Math. Inst. Steklov., 63, Acad. Sci. USSR, Moscow – Leningrad, 1962, 262p. (in Russian).

[10] F. A. Toranzos. *Inmersion de espacios metricos convexos en* , Math. Notac, 21, Nos. 1-2 (1966/67), pp.29-53. (in Spanish).

[11] E. Soetens. *Convexity in Busemann spaces,* Bull. Soc. Math. Belg., 19, nr.2, 1967, pp. 194–213.

[12] T.T. Arkhipova, I. V. Sergienko. *On the formalization and solution of some problems of organizing the computing process in data processing systems,* Kibernetika (kiev), nr.5, 1973, pp. 11–18 (in Russian); English transl. in Cybernetics 9, 1973.

[13] S. Cataranciuc, P. Soltan. *Complex of abstact cubes and median problem,* Computer Science Journal of Moldova. Vol.19, nr.1(55) 2011, pp. 38–63.

[14] M. Bujac, S. Cataranciuc, P. Soltan. *On the division in cubes of abstract manifolds,* Buletinul Academiei de Ştiinţe a Rep. Moldova. Seria Matematica, nr. 2(51), Chişinău, 2006, pp. 29–34.

[15] S. Cataranciuc, P. Soltan. *Abstract complexes, their homologies and applications,* Buletinul Academiei de Ştiinţe a Rep. Moldova. Seria Matematica, nr.2(63), Chişinău, 2010, pp. 31–58.

[16] S. Cataranciuc. *G-complex of multi-ary relations,* Analele Ştiinţifice ale USM, Seria "Ştiinţe fizico-matematice", Chişinău, 2006, pp. 119–122 (in Romanian).

[17] S. Cataranciuc, P. Soltan. *Hypergraphs and their homologies,* Trends in the Development of the Information and Communication Technology in Education and Management. International Conf., March 20-21, 2003, Chişinău, pp. 294–300 (in Romanian).

[18] P. J. Hilton, S. Wylie. *Homology Theory: An Introduction to Algebraic Topology,* Cambridge University Press, New York 1960 xv+484p.

[19] M. Bujac. *Classification of abstract multidimensional orientable manifolds without borders,* Analele Ştiinţifice ale USM, Seria "Ştiinţe fizico-matematice", Chişinău, 2003, pp. 247–250 (in Romanian).

[20] V. Boltyanski, H. Martini, P. Soltan. *Excursions into Combinatorial Geometry,* Springer, Berlin-Heidelberg, 1997, 444p.

[21] W. Hamming Richard. *Error detecting and error correcting codes.* Bell. System Technical Journal Nr. 29(2), 1950, pp. 147-160.

[22] P.S. Soltan, K.F. Prisacaru. *The Steiner Problem on Graph.* Dokl. Acad. Nauk SSSR, 198(1971), Nr.1, p.46–49.

[23] A.G. Kurosh. *Lectures on general algebra,* Fizmatgiz, Moscow, 1962, 396p. (in Russian).

Sergiu Cataranciuc
Received February 2, 2013

State University of Moldova
60 A. Mateevici street, MD-2009, Chişinău, MD-2009,
E-mail: *s.cataranciuc@gmail.com*