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The Set of Pareto-Nash Equilibria in Multicriteria Strategic Games

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Abstract

The paper investigates the notion of Pareto-Nash equilibrium as continuation of the works [2–4]. Problems and basic theoretical results are exposed. Method of intersection of graphs of best response mappings [3] is applied to solve the dyadic two-criteria games.

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1 Introduction

Consider the noncooperative strategic form game:

$$\Gamma = \langle \mathbf{N}, \{\mathbf{X_p}\}_{p\in\mathbf{N}}, \{f_p^i(\mathbf{x})\}_{i=1}^{m_p}, p \in \mathbf{N}\rangle,$$

where

- $\mathbf{N} = \{1, 2, ..., n\}$ is a set of players;
- $\mathbf{X_p} \in \mathbf{R}^{k_p}$ is a set of strategies of player $p \in \mathbf{N}$;
- $k_p < +\infty$, $p \in \mathbf{N}$;
- and $\{f_p^i(\mathbf{x})\}_{i=1}^{m_p}$ are the p^{th} player cost functions defined on the Cartesian product $\mathbf{X} = \times_{p\in\mathbf{N}}\mathbf{X_p}$.

Remark that each player has to solve singly the multi-criteria parametric optimization problem, where the parameters are strategic choices of the others players.

To exclude uncertainty, the well known definitions 1–3 and the corresponding notations are presented.

Definition 1. *Strategy $x_p^{'}$ is "better" than $x_p^{''}$, if*

$$\{f_p^i(x_p^{'}, x_{-p})\}_{i=1}^{m_p} \geq \{f_p^i(x_p^{''}, x_{-p})\}_{i=1}^{m_p}, \forall x_{-p} \in \boldsymbol{X}_{-p},$$

and there exist at least one index $j \in \{1, ..., m_p\}$ and a joint strategy $x_{-p} \in \boldsymbol{X}_{-p}$ for which

$$f_p^j(x_p^{'}, x_{-p}) > f_p^j(x_p^{''}, x_{-p});$$

the last relationship is denoted as $x_p^{'} \succeq x_p^{''}$.

Player problem. The player p selects from his set of strategies the strategy $x_p^* \in \mathbf{X}_p$, $p \in \mathbf{N}$, for which all of his cost functions $\{f_p^i(x_p, x_{-p}^*)\}_{i=1}^{m_p}$ reach maximum values.

2 Pareto optimality

Definition 2. *Strategy x_p^* is named effective (optimal in the sense of Pareto), if there does not exist other strategy $x_p \in \boldsymbol{X}_p$ so that $x_p \succeq x_p^*$.*

Let us denote the set of effective strategies (solutions) of the player p by **ef** $\mathbf{X_p}$. Any two effective strategies are equivalent or incomparable.

Theorem 1. *If the sets $\boldsymbol{X_p} \in \boldsymbol{R}^{k_p}$, $p = \overline{1,n}$, are compact and the cost functions are continuous ($f_p^i(\boldsymbol{x}) \in C(\boldsymbol{X_p})$, $i = \overline{1,m_p}$, $p = \overline{1,n}$), then the sets **ef** $\boldsymbol{X_p}$, $p = \overline{1,n}$, are non empty (**ef** $\boldsymbol{X} \neq \emptyset$).*

The proof follows from the known results [1, 2].

Definition 3. *Every element $\boldsymbol{x}^* = (x_1^*, x_2^*, ..., x_n^*) \in$ **ef** $\boldsymbol{X} = \times_{p \in \boldsymbol{N}}$**ef** $\boldsymbol{X_p}$ is named effective or Pareto outcome (situation).*

3 Synthesis Function

Solution of multi-criteria problem may be found by applying synthesis function, which may be interpreted as unique cost function of the player p ($p = \overline{1,n}$):

$$F_p(x) = \sum_{i=\overline{1,m_p}} \lambda_i f_p^i(x_p, x_{-p}) \longrightarrow \max,$$

$$x_p \in \mathbf{X}_p,$$

$$\sum_{i=\overline{1,m_p}} \lambda_i = 1, \lambda_i \geq 0, i = \overline{1,m_p}.$$

Theorem 2. *If x_p^* is a solution of mono-criterion problem*

$$F_p(x) = \sum_{i=\overline{1,m_p}} \lambda_i f_p^i(x_p, x_{-p}) \longrightarrow \max, x_p \in \boldsymbol{X}_p$$

with $\lambda_i > 0$, $i = \overline{1,m_p}$, $\sum_{i=\overline{1,m_p}} \lambda_i = 1$, then x_p^* is the eficient point for the given $x_{-p} \in \boldsymbol{X}_{-p}$.

The theorem's proof follows from the sufficient Pareto condition with linear synthesis function [1, 2].

4 Pareto-Nash equilibriun

Consider the convex game Γ for which the sets of strategies are convex and the cost functions are concave in relation to respective player strategies, when the strategies of the other players are fixed.

Definition 4. *The point $\boldsymbol{x}^* = (x_1^*, x_2^*, ..., x_n^*) \in \boldsymbol{X}$ is a Pareto-Nash equilibrium, if and only if for any player p the relations*

$$F_p(x_p, x_{-p}^*) \leq F_p(x_p^*, x_{-p}^*) \equiv F_p(x^*), \forall x_p \in \boldsymbol{X}_p,$$

are verified.

As a corollary of the precedent two theorems follow.

Theorem 3. *If the sets $\boldsymbol{X_p}$, $p = \overline{1,n}$, of the convex game Γ are compact and the functions $\{f_p^i(\boldsymbol{x})\}_{i=1}^{m_p}$ are continuous on $\boldsymbol{X} = \times_{p\in \boldsymbol{N}}\boldsymbol{X_p}$, then the convex game Γ has the Pareto-Nash equilibrium.*

Proof of the Theorem 3 follows from the known result [3].

The definition 4 may be formulated in other equivalent form:

Definition 5. *The point $\boldsymbol{x}^* = (x_1^*, x_2^*, ..., x_n^*) \in \boldsymbol{X}$ is a Pareto-Nash equilibrium, if and only if*

$$F(x^*) = \left(\max_{x_1\in \boldsymbol{X}_1} F_1(x_1, x_{-1}^*), ..., \max_{x_n\in \boldsymbol{X}_n} F_n(x_n, x_{-n}^*)\right),$$

where $(x_p, x_{-p}^*) \equiv (x_1^*, x_2^*, ..., x_{p-1}^*, x_p, x_{p+1}^*, ..., x_n^*)$, $p = \overline{1,n}$.

So, the Pareto-Nash equilibrium requires from each player to choose his own strategy as the Pareto best response to the strategies chosen by other players.

Let us denote the graph of the mapping

$$Argmax_{x_p\in\mathbf{X_p}} F_p(x_p, x_{-p}) : \mathbf{X_{\text{-}p}} \longrightarrow \mathbf{X_p}$$

by

$$Gr_p = \{(x_p, x_{-p}) \in \mathbf{X} : x_{-p} \in \mathbf{X_{\text{-}p}}, x_p = argmax_{y_p\in\mathbf{X_p}} F_p(y_p, x_{-p})\}.$$

In such notation, by [4], the set of Pareto-Nash equilibrium is:

$$\mathbf{PNE} = \bigcap_{p=\overline{1,n}} Gr_p,$$

where $\mathbf{X_{-p}} = \times_{i\in\mathbf{N}\setminus\{p\}}\mathbf{X}_i$.

As an illustration of previous notions and method of PNE set determination, let us consider the following example.

Example 1. *Consider the discrete game of two players. Each player has two strategies and two cost functions. The players have to maximize the values of both cost functions. The values of the cost functions are associated with the matrix elements:*

$$A = \begin{pmatrix} 4,3 & 7,7 \\ 6,6 & 8,4 \end{pmatrix},$$

$$B = \begin{pmatrix} 5,-1 & 2,4 \\ 4,3 & 6,2 \end{pmatrix}.$$

First of all the sets of effective strategies $\boldsymbol{ef}\ \boldsymbol{X}$ *and* $\boldsymbol{ef}\ \boldsymbol{Y}$ *are determined. Elements of* $\boldsymbol{ef}\ \boldsymbol{X}$ *and* $\boldsymbol{ef}\ \boldsymbol{Y}$ *are included in angle brackets.*

$$A = \begin{pmatrix} 4,3 & \langle 7,7 \rangle \\ 6,6 & \langle 8,4 \rangle \end{pmatrix},\ B = \begin{pmatrix} 5,-1 & \langle 2,4 \rangle \\ \langle 4,3 \rangle & \langle 6,2 \rangle \end{pmatrix}.$$

$$\boldsymbol{PNE} = \boldsymbol{ef}\ \boldsymbol{X} \bigcap \boldsymbol{ef}\ \boldsymbol{Y} = \{(1,2),(2,2)\}$$

with the costs $\{((7,7),(2,4)),((8,4),(6,2))\}$.

Theorem 4. *If the sets* $\boldsymbol{X_p}$, $p = \overline{1,n}$, *in the convex game* Γ *are compact and the functions* $F_p(x)$ *are continuous on* $\boldsymbol{X} = \times_{p \in \boldsymbol{N}} \boldsymbol{X_p}$, *then the convex game* Γ *has the Pareto-Nash equilibrium.*

The proof follows from [4, 2, 5] and the theorems 1–3, also.

5 Dyadic two-criteria game with mixed strategies

Consider dyadic two-criteria game with mixed strategies. The sets of strategies are:

$$\mathbf{X} = \{(x_1, x_2) : x_1 + x_2 = 1, x_1 \geq 0, x_2 \geq 0\},$$

$$\mathbf{Y} = \{(y_1, y_2) : y_1 + y_2 = 1, y_1 \geq 0, y_2 \geq 0\}.$$

The cost functions are bilinear, i.e. the functions are linear for fixed opponent strategy:

$$f_1^1(\mathbf{x},\mathbf{y}) = \mathbf{x}^T A\mathbf{y},$$
$$f_1^2(\mathbf{x},\mathbf{y}) = \mathbf{x}^T B\mathbf{y},$$
$$f_2^1(\mathbf{x},\mathbf{y}) = \mathbf{x}^T C\mathbf{y},$$
$$f_2^2(\mathbf{x},\mathbf{y}) = \mathbf{x}^T D\mathbf{y},$$

where $\mathbf{x},\mathbf{y} \in \mathbf{R^2}$, $A, B, C, D \in \mathbf{R^{2\times 2}}$. For each player consider the synthesis function:

$$F_1(x) = \lambda_1 f_1^1(x) + \lambda_2 f_1^2(x) \longrightarrow \max,$$

$$F_2(x) = \mu_1 f_2^1(x) + \mu_2 f_2^2(x) \longrightarrow \max .$$

By applying substitutions: $\lambda_1 = \lambda > 0$, and $\lambda_2 = 1 - \lambda > 0$, $\mu_1 = \mu > 0$ and $\mu_2 = 1 - \mu > 0$, we obtain:

$$F_1(\mathbf{x},\mathbf{y}) = \lambda f_1^1(\mathbf{x},\mathbf{y}) + (1-\lambda) f_1^2(\mathbf{x},\mathbf{y}) = \lambda \mathbf{x}^T A\mathbf{y} + (1-\lambda)\mathbf{x}^T B\mathbf{y},$$

$$F_2(\mathbf{x},\mathbf{y}) = \mu f_2^1(\mathbf{x},\mathbf{y}) + (1-\mu) f_2^2(\mathbf{x},\mathbf{y}) = \mu \mathbf{x}^T C\mathbf{y} + (1-\mu)\mathbf{x}^T D\mathbf{y}.$$

By applying obvious transformations:

$$x_1 = x, x_2 = 1 - x, 1 \geq x \geq 0,$$

$$y_1 = y, y_2 = 1 - y, 1 \geq y \geq 0,$$

the second equivalent game is obtained:

$$F_1(x,y) = (\alpha(\lambda)y + \beta(\lambda))x + \alpha_0(\lambda)y + \beta_0(\lambda),$$

$$F_2(x,y) = (\gamma(\mu)x + \delta(\mu))y + \gamma_0(\mu)x + \delta_0(\mu),$$
$$x, y \in [0,1], \lambda, \mu \in [0,1],$$

where:

$$\alpha(\lambda) = (a_{11} - a_{12} - a_{21} + a_{22} - b_{11} + b_{12} + b_{21} - b_{22})\lambda + b_{11} - b_{12} - b_{21} + b_{22},$$

$$\beta(\lambda) = (a_{12} - a_{22} - b_{12} + b_{22})\lambda + b_{12} - b_{22},$$

$$\alpha_0(\lambda) = (a_{21} - a_{22} - b_{21} + b_{22})\lambda + b_{21} - b_{22},$$

$$\beta_0(\lambda) = (a_{22} - b_{22})\lambda + b_{22},$$

$$\gamma(\mu) = (c_{11} - c_{12} - c_{21} + c_{22} - d_{11} + d_{12} + d_{21} - d_{22})\mu + d_{11} - d_{12} - d_{21} + d_{22},$$

$$\delta(\mu) = (c_{21} - c_{22} - d_{21} + d_{22})\mu + d_{21} - d_{22},$$

$$\gamma_0(\mu) = (c_{12} - c_{22} - d_{12} + d_{22})\mu + d_{12} - d_{22},$$

$$\delta_0(\mu) = (c_{22} - d_{22})\mu + d_{22}.$$

The graphs of Pareto best response mappings are:

$$\mathbf{Gr_1} = \begin{cases} (1, y), & \text{if } \alpha(\lambda)y + \beta(\lambda) > 0, \\ (0, y) & \text{if } \alpha(\lambda)y + \beta(\lambda) < 0, \\ [0,1] \times y, & \text{if } \alpha(\lambda)y + \beta(\lambda) = 0, \end{cases}$$

$$\mathbf{Gr_2} = \begin{cases} (x, 1), & \text{if } \gamma(\mu)x + \delta(\mu) > 0, \\ (x, 0), & \text{if } \gamma(\mu)x + \delta(\mu) < 0, \\ x \times [0,1], & \text{if } \gamma(\mu)x + \delta(\mu) = 0. \end{cases}$$

The solutions of equations $\alpha(\lambda)y + \beta(\lambda) = 0$ and $\gamma(\mu)x + \delta(\mu) = 0$ are $y(\lambda) = -\frac{\beta(\lambda)}{\alpha(\lambda)}$ and $x(\mu) = -\frac{\delta(\mu)}{\gamma(\mu)}$. Vertical asymptotes of the respective hyperboles are determined by relations $\alpha(\lambda) = 0$ and $\gamma(\mu) = 0$ and they are denoted by λ_α and μ_γ, respectively.

If the solution λ_α does not belong to the interval $(0, 1)$, then y belongs to the interval with extremities $[y(0), y(1)]$. If the extremity value is negative, it is replaced by 0, if it is greater than 1, it is replaced by 1.

If λ_α belongs to the interval $(0, 1)$, then the graph $\mathbf{Gr_1}$ will be represented by two rectangles and one edge ($[(0,0),(0,1)]$ or $[(1,0),(1,1)]$) of the square $[0,1] \times [0,1]$ or two edges and one rectangle of the square $[0,1] \times [0,1]$. Other graphs are possible in the case when λ_α does not belong to the interval $(0, 1)$ and they are described below.

Similar reasoning is applied for μ_γ and graph $\mathbf{Gr_2}$.

For the first player and his $\mathbf{Gr_1}$ the following cases are possible also:

1. If $\alpha(\lambda) > 0$, $\beta(\lambda) < 0$, $\alpha(\lambda) > -\beta(\lambda)$, then
$\mathbf{Gr_1} = [(0,0),(0,y(\lambda))] \bigcup [(0,y(\lambda)),(1,y(\lambda))] \bigcup [(1,y(\lambda)),(1,1)]$;

2. If $\alpha(\lambda) < 0$, $\beta(\lambda) > 0$, $-\alpha(\lambda) > \beta(\lambda)$, then
$\mathbf{Gr_1} = [(0,1),(0,y(\lambda))] \bigcup [(0,y(\lambda)),(1,y(\lambda))] \bigcup [(1,y(\lambda)),(1,0)]$;

3. If $\alpha(\lambda) > 0$, $\beta(\lambda) < 0$, $\alpha(\lambda) = -\beta(\lambda)$, then
$\mathbf{Gr_1} = [(0,0),(0,1)] \bigcup [(0,1),(1,1)]$;

4. If $\alpha(\lambda) < 0$, $\beta(\lambda) > 0$, $-\alpha(\lambda) = \beta(\lambda)$, then
$\mathbf{Gr_1} = [(0,1),(1,1)] \bigcup [(1,1),(1,0)]$;

5. If $\alpha(\lambda) > 0$, $\beta(\lambda) = 0$, then $\mathbf{Gr_1} = [(0,0),(1,0)] \bigcup [(1,0),(1,1)]$;

6. If $\alpha(\lambda) < 0$, $\beta(\lambda) = 0$, then $\mathbf{Gr_1} = [(0,1),(0,0)] \bigcup [(0,0),(1,0)]$;

7. If $\alpha(\lambda) > 0$, $\beta(\lambda) < 0$, $\alpha(\lambda) < -\beta(\lambda)$ or $\alpha(\lambda) < 0$, $\beta(\lambda) < 0$ or $\alpha(\lambda) = 0$, $\beta(\lambda) < 0$, then $\mathbf{Gr_1} = [(0,0),(0,1)]$;

8. If $\alpha(\lambda) < 0$, $\beta(\lambda) > 0$, $-\alpha(\lambda) < \beta(\lambda)$ or $\alpha(\lambda) > 0$, $\beta(\lambda) > 0$ or $\alpha(\lambda) = 0$, $\beta(\lambda) > 0$, then $\mathbf{Gr_1} = [(1,0),(1,1)]$;

9. If $\alpha(\lambda) = 0$, $\beta(\lambda) = 0$, then $\mathbf{Gr_1} = [0,1] \times [0,1]$.

For the second player the following cases are possible:

1. If $\gamma(\mu) > 0$, $\delta(\mu) < 0$, $\gamma(\mu) > -\delta(\mu)$, then
$\mathbf{Gr_2} = [(0,0),(x(\mu),0)] \bigcup [(x(\mu),0),(x(\mu),1)] \bigcup [(x(\mu),1),(1,1)]$;

2. If $\gamma(\mu) < 0$, $\delta(\mu) > 0$, $-\gamma(\mu) > \delta(\mu)$, then
$\mathbf{Gr_2} = [(0,1),(x(\mu),1)] \bigcup [(x(\mu),1),(x(\mu),0)] \bigcup [(x(\mu),0),(1,0)]$;

3. If $\gamma(\mu) > 0$, $\delta(\mu) < 0$, $\gamma(\mu) = -\delta(\mu)$, then
$\mathbf{Gr_2} = [(0,0),(1,0)] \bigcup [(1,0),(1,1)]$;

4. If $\gamma(\mu) < 0$, $\delta(\mu) > 0$, $-\gamma(\mu) = \delta(\mu)$, then
$\mathbf{Gr_2} = [(0,1),(1,1)] \bigcup [(1,1),(1,0)]$;

5. If $\gamma(\mu) > 0$, $\delta(\mu) = 0$, then $\mathbf{Gr_2} = [(0,0),(0,1)] \bigcup [(0,1),(1,1)]$;

6. If $\gamma(\mu) < 0$, $\delta(\mu) = 0$, then $\mathbf{Gr_2} = [(0,1),(0,0)] \bigcup [(0,0),(1,0)]$;

7. If $\gamma(\mu) > 0$, $\delta(\mu) < 0$, $\gamma(\mu) < -\delta(\mu)$ or $\gamma(\mu) < 0$, $\delta(\mu) < 0$ or $\gamma(\mu) = 0$, $\delta(\mu) < 0$, then $\mathbf{Gr_2} = [(0,0),(1,0)]$;

8. If $\gamma(\mu) < 0$, $\delta(\mu) > 0$, $-\gamma(\mu) < \delta(\mu)$ or $\gamma(\mu) > 0$, $\delta(\mu) > 0$ or $\gamma(\mu) = 0$, $\delta(\mu) > 0$, then $\mathbf{Gr_2} = [(0,1),(1,1)]$;

9. If $\gamma(\mu) = 0$, $\delta(\mu) = 0$, then $\mathbf{Gr_2} = [0,1] \times [0,1]$.

Note. For drawing the graphs, the expressions $\alpha(0)$, $\beta(0)$, $y(0)$ and $\alpha(1)$, $\beta(1)$, $y(1)$ are calculated for the first player and $\gamma(0)$, $\delta(0)$, $x(0)$ and $\gamma(1)$, $\delta(1)$, $x(1)$ for the second player. When the player's graph depends only on one of the matrix, it is constructed exactly as in the case of Nash equilibrium [2]. If the expressions $y(0)$ and $y(1)$ do not depend on parameter λ and $y(0) = y(1)$, the graph of the first player will be all the square $[0,1] \times [0,1]$. The similar argument is true for the second player.

Based on the above, $\mathbf{Gr_1}$ and $\mathbf{Gr_2}$ can be drawn.

The set of Pareto-Nash equilibria (**PNE**) is obtained as the intersection of the player's graphs, that is $\mathbf{PNE} = \mathbf{Gr_1} \bigcap \mathbf{Gr_2}$.

Example 2. *Consider the following matrices:*

$$(A,B) = \begin{pmatrix} 4,3 & 7,7 \\ 6,6 & 8,4 \end{pmatrix}; \; (C,D) = \begin{pmatrix} 5,-1 & 2,4 \\ 4,3 & 6,2 \end{pmatrix}.$$

After simplifications, the synthesis functions of the players are:

$$F_1(x,y) = [(5\lambda - 6)y - 4\lambda + 3]x + (4\lambda + 10)y + 4\lambda + 4,$$

$$F_2(x,y) = [(11\mu - 6)x - 3\mu + 1]y + (2\mu + 6)x + 4\mu + 2.$$

In conformity with the described method, the following 4 steps are provided:

1. $\alpha(\lambda) = 5\lambda - 6$, $\beta(\lambda) = -4\lambda + 3$, $y(\lambda) = \frac{4\lambda - 3}{5\lambda - 6}$; $\gamma(\mu) = 11\mu - 6$, $\delta(\mu) = -3\mu + 1$, $x(\mu) = \frac{3\mu - 1}{11\mu - 6}$.

2. *The values* λ_α *and* μ_γ *are the solutions of equations* $\alpha(\lambda) = 0$ *and* $\gamma(\mu) = 0$*, respectively.* $\lambda_\alpha = \frac{6}{5} \notin (0,1)$ *and* $\mu_\gamma = \frac{6}{11} \in (0,1)$.

3. *The values on the interval extremities are calculated:*

   (a) $y(0) = \frac{1}{2}$, $\alpha(0) = -6 < 0$, $\beta(0) = 3 > 0$ *and* $-\alpha(0) > \beta(0)$ *- the case 2;* $y(1) = -1$, $\alpha(1) = -1 < 0$ *and* $\beta(1) = -1 < 0$ *- the case 7. The lines are drawn and the interval between them is hatched. The following result is obtained*

   $$\boldsymbol{Gr_1} = \textit{Rectangle} : [(0,0),(0,y(0)),(1,y(0)),(1,0)] \bigcup$$
   $$[(0,0),(0,1)],$$

   where $y(0) = \frac{1}{2}$.

   (b) $x(0) = \frac{1}{6} \in (0,1)$, $\gamma(0) = -6 < 0$, $\delta(0) = 1 > 0$ *and* $-\gamma(0) > \delta(0)$ *- the case 2;* $x(1) = \frac{2}{5} \in (0,1)$, $\gamma(1) = 5 > 0$, $\delta(0) = -2 < 0$ *and* $\gamma(0) > -\delta(0)$ *- the case 1. The respective lines are drawn and the interval between the respective sides of the square is hatched* $[0,1] \times [0,1]$*. The following result is obtained*

   $$\boldsymbol{Gr_2} = \textit{Rectangle} : [(0,0),(0,1),(x(0),1),(x(0),0)] \bigcup$$
   $$\textit{Rectangle} : [(1,0),(1,1),(x(0),1),(x(0),0)] \bigcup$$
   $$[(0,0),(1,0)],$$

   where $x(0) = \frac{1}{6}$ *and* $x(1) = \frac{2}{5}$.

4. *By determining the intersection of the graphs obtained above, the following set of Pareto-Nash equilibrium in mixed strategies is obtained:*

$$\boldsymbol{PNE} = [(0,1),(0,0)] \bigcup$$
$$[(0,0),(1,0)] \bigcup$$
$$\textit{Rectangle} : \left[(0,0), \left(0, \frac{1}{2}\right), \left(\frac{1}{6}, \frac{1}{2}\right), \left(\frac{1}{6}, 0\right)\right] \bigcup$$
$$\textit{Rectangle} : \left[\left(\frac{2}{5}, 0\right), \left(\frac{2}{5}, \frac{1}{2}\right), \left(1, \frac{1}{2}\right), (1,0)\right].$$

6 Wolfram Mathematica Program for Two Criteria Dyadic Games with Mixed Strategies

The method of graph intersection was realized as Wolfram Mathematica Program for Two Criteria Dyadic Games with Mixed Strategies. The program was published on Wolfram Demonstration Project [6]. It may be used online, after the installation of CDF player. The program code may be downloaded at the same address [6], also. The results obtained in the example 2 may be tested online at the same address [6].

7 Concluding remarks

By applying the generalization of the well known notions and by applying the combination of the synthesis function method and the method of intersection of best response graph, the conditions for the Pareto-Nash solutions existence are deduced. The method for determining Pareto-Nash Equilibrium Set in dyadic two criteria games with mixed strategy is clarified from elaboration to final Wolfram Mathematica Program publication. Illustration examples are presented for easier reading. Since the investigated problems have an exponential complexity, a further development of the method in games with bigger dimensions, with implication of the computer science technologies, will be welcome.

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