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On Convexity Preserving C^1 Hermite Spline Interpolation

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Abstract

The aim of this paper is to present a general approach to the problem of shape preserving interpolation. The problem of convexity preserving interpolation using C^1 Hermite splines with one free generating function is considered.

1 Introduction

It is well known that the problems concerning nonnegativity, monotonicity, or convexity preserving interpolation have received considerable attention during the last two decades, because of their interest in computer aided design and in other practical applications. One can construct a convexity preserving interpolant by increasing the degree of interpolating polynomials (see, e.g., [1]- [3]), by adding new mesh points and increasing the number of pieces (see, for example, [4], [8]), by constraining the derivatives for Hermite interpolants to meet conditions which imply the desired properties (see [6], [7]) or using different type of nonclassical splines with free parameters (see, e.g. [5], [10], [11]). The reader is referred to the survey paper [9] for a large set of references. A review of the approaches mentioned above shows as it is mentioned in [2] that, in fact, we have the following common scheme:

a) A set of piecewise functions in which interpolants are thought is choosen and interpolants are expressed using a set of parameters (e.g. the derivatives at knots);

b) The nonnegativity, monotonicity, or convexity conditions are rewritten using these parameters and admissible domains are derived;

c) A theory for checking the feasibility of the problem is developed and, eventually, an algorithm is provided.

In the same paper [2] this situation was qualified as schizophrenic one. In the present paper the idea of generating function of the spline introduced in [12] is suggested to use for the problems concerning shape preserving interpolation. This idea was succesfully used in [13]-[16] for constructing different types of interpolating splines and for studying their convergence properties in a general case.

2 A family of C^1 Hermite interpolants

Let us assume that a mesh $\Delta : a = x_0 < x_1 < ... < x_N = b$ is given and at the knots of this mesh values $f_i = f(x_i)$ and $f'_i = f'(x_i), i = 0(1)N$ which are supposed to come from an unknown differentiable function f are given. The problem of constructing an interpolant S, such that $S(x_i) = f_i, S'(x_i) = f'(x_i), i = 0(1)n$, and $S \in C^1[a, b]$ is considered.

Let us consider the following family of C^1 Hermite interpolants introduced in [16]: on $[x_i, x_{i+1}]$

$$S(x) = f_i + h_i f'_i(2t - t^2)/2 + h_i f'_{i+1} t^2/2 + h_i \gamma_i \nu(t), \tag{1}$$

where

$$h_i = x_{i+1} - x_i, \quad t = (x - x_i)/h_i, \quad \gamma_i = (f_{i+1} - f_i)/h_i - (f'_i + f'_{i+1})/2$$

and the function ν called generating function in what follows is to be from the following set of functions

$$L = \{\phi \in C^1[0, 1] : \phi(0) = 0; \phi(1) = 1; \phi'(0) = \phi'(1) = 0\}. \tag{2}$$

There are no problems to verify that the set of generating functions L is the convex and closed one and that splines from the family (1) solve the problem under consideration. It was shown in [16] that these splines are third-order accurate if $f_i', i = 0(1)n$ are known exactly (except cubic Hermite splines, which are fourth-order accurate).

In order to illustrate that splines of different type can be represented in the form (1) we present some examples of generating functions below. So, from the set of generating functions L the following functions are:

1. The function
$$\nu(t) = 3t^2 - 2t^3 \tag{3}$$
lead us to well known cubic Hermite splines.

2. Quadratic Hermite splines are obtained if the following generating function
$$\nu(t) = \begin{cases} \frac{t^2}{\tau}, & t \in [0, \tau] \\ 1 - \frac{(1-t)^2}{(1-\tau)}, & t \in [\tau, 1] \end{cases} \tag{4}$$
where $\tau \in (0, 1)$ is the additional knot on the mesh Δ, is used. The second derivative of this function has a discontinuity at the point τ.

3. Splines generated by the following function
$$\nu(t) = (1 - \cos(\pi t))/2 \tag{5}$$
seems to be the new ones. We did not meet the splines of this form in the literature before.

4. The following two generating functions:
$$\nu(t) = \begin{cases} \frac{3t^2}{(1+2\tau)}, & t \in [0, \tau] \\ 1 - \frac{3(1-t)^2(1+\tau)}{(1-\tau)(1+2\tau)} + \frac{2(1-t)^3}{(1-\tau)^2(1+2\tau)}, & t \in [\tau, 1] \end{cases} \tag{6}$$
and
$$\nu(t) = \begin{cases} \frac{3(\tau-2)t^2}{\tau(2\tau-3)} + \frac{2t^3}{\tau^2(2\tau-3)}, & t \in [0, , \tau] \\ 1 + \frac{3(1-t)^2}{(2\tau-3)}, & t \in [\tau, 1] \end{cases} \tag{7}$$
give us blend parabolic-cubic splines. In the contrast to parabolic splines the second derivative of these ones has no more a discontinuity at the point τ.

5. Using the generating function

$$\nu(t) = \begin{cases} \frac{(3+p-p\tau)t^2}{\tau} + \frac{(2p\tau^2-p\tau-3\tau-p-3)t^3}{3\tau^2}, & t \in [0,\tau] \\ 1+p(1-t)^2 + \frac{(2p\tau-3p-3)(1-t)^3}{3(1-\tau)}, & t \in [\tau,1] \end{cases} \tag{8}$$

we get cubic splines with one additional knot. In this case there are two free parameters for each piece - p and τ. It is easy to see that this function is a generalisation of the previous two. So, the function (7) is obtained if $p = 3/(2\tau-3)$ and, correspondingly, we get the generating function (6) for $p = -3(1+\tau)/(1+2\tau)(1-\tau)$.

6. Examples of generating function given below

$$\nu(t) = \frac{nt^2}{(n-2)} - \frac{2t^n}{(n-2)}, \quad n > 2 \tag{9}$$

$$\nu(t) = \frac{n_2 t^{n_1}}{(n_2-n_1)} - \frac{n_1 t^{n_2}}{(n_2-n_1)}, \quad n_2 > n_1 \geq 2 \tag{10}$$

show that polynomial splines with variable degree can be represented in the same way. Here, n, n_1 and n_2 are free parameters.

7. Finally, an example of generating function

$$\nu(t) = \frac{((p+3)t^2 - (p+2)t^3)}{(1+pt(1-t))}, \quad p > -4, \tag{11}$$

which leads us to rational splines is given. In this case the free parameter p is a parameter of rationality.

As it follows from the examples given above we can generate a lot of splines of the form (1), which have different analytical representation. At the same time we can study properties of all these splines in a general setting using the representation (1).

3 Convexity preserving C^1 Hermite interpolation

Following [1] for any integer i arbitrary but fixed we say that data are increasing on $[x_i, x_{i+1}]$ if

$$f'_i > 0; \quad f'_{i+1} > 0, \quad \delta_i > 0, \tag{12}$$

convex on $[x_i, x_{i+1}]$ if

$$f'_i < \delta_i < f'_{i+1} \tag{13}$$

and increasing and convex if both (12) and (13) are fulfilled.

Let us suppose that data are convex(concave) on $[x_i, x_{i+1}]$ and let us introduce $\lambda_i = (\delta_i - f'_i)/(f'_{i+1} - f'_i)$. It is obvious that $0 < \lambda_i < 1$ for convex(concave) data. The next formula for the second derivative of the spline follows from (1).

$$S''(x) = (f'_{i+1} - f'_i)/h_i + \gamma_i \nu''(t)/2h_i \tag{14}$$

Then there are no problems to prove the following theorem.

Theorem 1 *The spline (1) with a generating function $\nu \in L$ preserves convexity(concavity) of data in $[x_i, x_{i+1}]$ if and only if*

$$\min_t \nu''(t) > -2/(2\lambda_i - 1) \quad for \quad \lambda_i \in (1/2, 1)$$

and

$$\max_t \nu''(t) < -2/(2\lambda_i - 1) \quad for \quad \lambda_i \in (0, 1/2)$$

The proof of the theorem is omitted since it is the elementary one.

Conditions of convexity preserving for splines generated by a certain generating function $\nu \in L$ can be obtained now directly from the theorem stated above. If the generating function has no free parameters we get rigid conditions, concerning initial data, when the resulting interpolant preserves convexity(concavity). So, in the case of the function (3) we get that the Hermite C^1 cubic interpolant preserves convexity(concavity) if $1/3 < \lambda_i < 2/3$. For the generating function (5) the corresponding condition is $1/2 - 2/\pi^2 < \lambda_i < 1/2 + 2/\pi^2$. As it can be seen in this case the condition is weaker then the previous one.

If the generating function contains free parameter(s) the resulting curve may be modified choosing value(s) of parameter(s). For the parabolic Hermite splines generated by the function (4) we get that these splines preserve convexity(concavity) of data if the additional knot τ on the mesh is choosen so, that $\tau > 1 - 2\lambda_i$ for $\lambda_i \in (0, 1/2)$ and $\tau < 2(1 - \lambda_i)$ for $\lambda_i \in (1/2, 1)$. As it was mentioned above the second derivative of these splines has an additional point of discontinuity at τ. This may become an undesirable fact in many cases. It can be avoided if other generating functions are used. So, if $\lambda_i \in (0, 1/2)$ then the generating function (6) can be used and the additional knot in this case must satisfy the condition $\tau > 1 - 3\lambda_i$. If $\lambda_i \in (1/2, 1)$ we get a required interpolant using the generating function (7) with the addtitional knot which satisfies the condition $\tau < 3(1 - \lambda)$.

Finally, let us consider the case when generating function of variable degree is used. If $\lambda_i \in (0, 1/2)$ and the function (9) is used to generate an interpolant in order to preserve the convexity(concavity) of the data the degree n of the function ν must satisfy the condition $n > 1/\lambda_i$. In the case when $\lambda_i \in (1/2, 1)$ the following generating function can be used

$$\nu(t) = 1 - \frac{n(1-t)^2}{(n-2)} + \frac{2(1-t)^n}{(n-2)}, \quad n > 2,$$

where free parameter n (the degree of function) must be selected according to the following condition $n > 1/(1 - \lambda_i)$.

Summarising it can be said that, in fact, we are in the position to construct convexity-preserving C^1 Hermite interpolants for any set of initial data which are convex using splines of the form (1) with generating functions from (2).

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