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A form of multidimensional averiging functions satisfying the property of associativity

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Abstract

In this paper multidimensional averiging functions are studied in assumption that they are continuous, monotonic and satisfy the property of associativity. It is proved that such functions are very "closed" to functions MAX() and MIN().

1 Introduction

As a rule, the estimation process results in obtaining a sequence of numbers. Because each number is the "measurement" of one essence by an expert, but in reality we are interested in a general image which is a combination of all opinions, it often resorts to an aggregation of given numbers, that is to obtaining the smaller secquence, but which concentrates already the general opinion of many experts.

Definition 1 *The aggregation is called a total one if the obtained sequence consists of a single element.*

The aggregation procedure is often based on the definition of a function, called aggregation function, which is applied to sequence obtained as a rezult of estimation. In this case the following algorithm is applied. Let $f(x_1, x_2, \ldots, x_n)$ be the aggregation function and let $A = \{a_1, a_2, \ldots, a_{j+1}, \ldots, a_{j+n}, a_{j+n+1}, \ldots, a_m\}$ be the sequence for aggregation. From sequence A a subsequence of n numbers $\{a_{j+1}, \ldots, a_{j+n}\}$ is selected, applying the function $f(x_1, x_2, \ldots, x_n)$ we obtain

$$a'_{j+1} = f(a_{j+1}, \ldots, a_{j+n}).$$

Replacing this subsequence by a'_{j+1} a new sequence $A' = \{a_1, a_2, \dots, a'_{j+1}, \dots, a_{m-n+1}\}$ is obtained. Iteratively applying this algorithm, we gradually decrease the initial sequence.

Definition 2 *The function $f(x_1, x_2, \dots, x_n)$ is called correct aggregation function relatively to sequence $A = \{a_1, a_2, \dots, a_m\}$ if iteratively applying it to this sequence a total aggregation is obtained.*

Statement 1 *For the function $f(x_1, x_2, \dots, x_n)$ to be correct aggregation function relatively to sequence $A = \{a_1, a_2, \dots, a_m\}$ the condition $(m-1)|(n-1)$ must be satisfied.*

Proof. Because at each stage of aggregation the sequence decreases with $(n-1)$ elements and at termination of iterative aggregation process a sequence of one element is obtained it rezults that $(m-1) = k(n-1)$, where k is a natural number, hence $(m-1)|(n-1)$.

Statement 2 *The number of classes of correct aggregation functions is equal to the number of divisors of $(m-1)$, where m is the number of elements of sequence which must be aggregated.*

Proof. Because $(m-1)|(n-1)$, where n is the number of aggregation function variables, it rezults that reuniting all the functions of n variables into one class we obtain so many classes as many divisors $(m-1)$ has.

Statement 3 *Aggregation functions of two variables are correct relatively to every sequence.*

Proof. Because in this case we have $n = 2$ and because 1 is the divisor of every number we have $(m-1)|1$ and hence functions of two variables are correct relatively to every sequence.

The aggregation function is a simmertical one if every arrangement of sequence $x_1, x_2, \dots, x_n$ does not change the value of the function, that is

$$f(x_1, x_2, \dots, x_n) = f(x_{i1}, x_{i2}, \dots, x_{in}).$$

The aggregation function satisfies a complete associativity if the following relationships are satisfied

$$f(f(x_1, x_2, \ldots, x_n), x_{n+1}, \ldots, x_{2n-1}) =$$

$$= f(x_1, f(x_2, \ldots, x_{n+1}), \ldots, x_{2n-1}) = \ldots$$

$$= f(x_1, x_2, \ldots, x_{n-1}, f(x_n, x_{n+1}, \ldots, x_{2n-1})).$$

If only some of the relationships are satisfied we have a partial associativity.

Definition 3 *Aggregation procedure is called averiging aggregation if the aggregation function is an averiging one.*

Definition 4 *Function* $f(x_1, x_2, \ldots, x_n)$ *is called averiging function if*

$$m \leq f(x_1, x_2, \ldots, x_k) \leq M$$

where $m = \min(x_1, x_2, \ldots, x_n)$ *and* $M = \max(x_1, x_2, \ldots, x_n)$.

2 The axioms involved in aggregation process

In the aggregation process some axioms can be involved which are the cristallization of some requirement imposed to the aggregation process. We shall express this axioms in the form of properties of averiging functions [1].

Let $f(x_1, x_2, \ldots, x_n)$ be an averiging function. In the assumption that 0 and 1 are respectively the minimal and the maximal values admited at the process of estimation, we have the following axiom.

Axiom 1 $f(0, 0, \ldots, 0) = 0$ *and* $f(1, 1, \ldots, 1) = 1$.

If we have that the vectors x and y are connected by relationship $x \geq y$ this fact would mean that the object reprezented by vector x in the opinion of some experts is better than the object reprezented by the vector y and hence, because an aggregation is an union of opinions expressed by all experts, it is natural that aggregation performed on the vector x to be better than aggregation performed on the vector y and hence the following axiom is natural.

Axiom 2 *$f(x_1, x_2, \ldots, x_n) \geq f(y_1, y_2, \ldots, y_n)$ for $x \geq y$.*

This axiom is so called axiom of monotony.

Making abstraction of some critical moments, we shall assume that a very small variation of opinions of experts has a small influence on the general opinion about the given object. This moment is expressed in the following axiom.

Axiom 3 *The averiging function is a continuous one.*

This is the axiom of continuity.

3 Properties of associative averiging functions

Statement 4 *There are no strictly monotonic associative averiging functions, that is, if the vectors x and y are connected by relationship $x > y$ then we have*

$$f(x_1, x_2, \ldots, x_n) > f(y_1, y_2, \ldots, y_n).$$

Proof. We assume that there are such functions. Then for the sequence $x_1 \leq x_2 \leq \ldots < x_n$ we have

$$f(x_1, x_2, \ldots, x_n) < f(f(x_1, x_2, \ldots, x_n), x_n, \ldots, x_n) =$$

$$= f(x_1, x_2, \ldots, f(x_n, x_n, \ldots, x_n)) = f(x_1, x_2, \ldots, x_n).$$

The obtained contradiction proves the assertion.

We make the following notation

$$f(0, \ldots, 0, \underbrace{x, \ldots, x, x}_{i}) = f_x^i.$$

For monotonic functions we have that $f_x^i \leq f_x^j$ for $i \leq j$.

Statement 5 *If the function is associative and monotonic then* $f_x^i = f_x^j$.

Proof. To prove this assertion it is sufficient to show that $f_x^1 = f_x^i$ for every $i > 1$. From associativity we obtain for f_x^1 the following relationships:

$$f_x^1 = f(0, \dots, 0, f_x^{n-1}, x) =$$
$$= f(0, \dots, 0, f_x^{n-2}, x, x) = \dots = f(f_x^1, x, x, \dots, x)$$

and for f_x^i the relationships which follow below:

$$f_x^i = f(0, \dots, 0, \underbrace{f_x^1, x, \dots, x}_{i}) =$$

$$= f(0, \dots, 0, \underbrace{f_x^2, x, \dots, x}_{i-1}) = \dots = f(0, 0, \dots, f_x^i)$$

From these relationships we can observe that

$$f_x^1 = f(0, \dots, 0, f_x^{n-1}, x) \geq f(0, 0, \dots, f_x^{i-1}, x) = f_x^i$$

Hence we obtain $f_x^1 = f_x^i$.

Statement 6 *If the function is associative, continuous and monotonic and* $f_1^1 = \alpha$*, then* $f(0, 0, \dots, 0, z) = z$ *for every* z *that belongs to segment* $[0, \alpha]$.

Proof. From associativity we have the relationship

$$f(0, \dots, 0, x) = f(0, 0, \dots, f(0, \dots, 0, x)).$$

Because the function $f(x_1, x_2, \dots, x_n)$ is continuous and varying x from 0 to 1, the function takes values from 0 to α, which means that substituting $f(0, \dots, 0, x)$ by z we have a variation of z from 0 to α and therefore we obtain equality $f(0, \dots, 0, z) = z$ for $\forall z \in [0, \alpha]$.

Statement 7 *If the function is associative, simmetrical, continuous and monotonic and* $f_1^1 = a$*, then* $f(0,\ldots,0,\alpha) = \alpha$ *and* $f(\alpha,\ldots,\alpha,1) = \alpha$.

Proof. The equality $f(0,0,\ldots,0,\alpha) = \alpha$ is obtained from Statement 6. From associativity and simmetry we obtain

$$f(\alpha,\alpha,\ldots,\alpha,1) = f(\alpha,\alpha,\ldots,f(\alpha,0,\ldots,0),1) =$$
$$= f(\alpha,\alpha,\ldots,\alpha,f(0,0,\ldots,0,1)) = f(\alpha,\alpha,\ldots,\alpha,\alpha) = \alpha$$

Statement 8 *If* $f(\underbrace{\alpha,\ldots,\alpha}_{n-1},1) = \alpha$ *and* $f(x_1,\ldots,x_n)$ *is associative and simmetrical then* $f(\underbrace{\alpha,\alpha,\ldots,\alpha}_{k},1,\ldots,1) = \alpha$ *for every* $1 \le k \le n-1$.

Proof. From associativity we obtain

$$f(\underbrace{\alpha,\alpha,\ldots,\alpha}_{k},1,\ldots,1) = f(\underbrace{\alpha,\alpha,\ldots,\alpha}_{k-1} f(\alpha,\alpha,\ldots,\alpha),1,\ldots,1) =$$

$$= f(\underbrace{\alpha,\alpha,\ldots,\alpha}_{k},f(\alpha,\alpha,\ldots,\alpha,1)1,\ldots,1) = f(\underbrace{\alpha,\alpha,\ldots,\alpha}_{k+1},1,\ldots,1)$$

Repeating this procedure $(n-k-1)$ times we derive the equality

$$f(\underbrace{\alpha,\alpha,\ldots,\alpha}_{k},1,\ldots,1) = f(\underbrace{\alpha,\alpha,\ldots,\alpha}_{n-1},1) = \alpha$$

Therefore we obtain that

$$f(\underbrace{\alpha,\alpha,\ldots,\alpha}_{k},1,\ldots,1) = \alpha$$

for every $1 \le k \le n-1$.

Statement 9 *If* $f_z^i = z \forall z \in [0, \alpha]$ *and* $f(x_1, x_2, \ldots, x_n)$ *is an monotonic function then* $f(x_1, x_2, \ldots, x_n) = \max(x_1, x_2, \ldots, x_n)$*, for values* x_i *which dont́ exceed* α*.*

Proof. We make the notation $M = \max(x_1, x_2, \ldots, x_n)$. Then we have the relationship

$$M = f(0, \ldots, 0, M) \leq f(x_1, x_2, \ldots, x_n) \leq f(M, M, \ldots, M) = M$$

Therefore $f(x_1, x_2, \ldots, x_n) = \max(x_1, x_2, \ldots, x_n)$.

Statement 10 *If the function is associative, simmetrical, continuous and monotonic and* $f_1^1 = \alpha$*, then* $f(x_1, x_2, \ldots, x_n) = \alpha$ *for every sequence which has values both greater than* α *and smaller than* α*.*

Proof. Replacing in $f(x_1, x_2, \ldots, x_n)$ the values greater than α with α, and the values smaller than α with 0 we obtain the value $f(0, \ldots, 0, \alpha, \ldots, \alpha)$. Performing such a procedure under the condition that instead of the values smaller than α we substitute the value α, and instead of the values greater than α we substitute the value 1 we obtain the value $f(\alpha, \ldots, \alpha, 1, \ldots, 1)$; as a rezult we can write the following relationships

$$\alpha = f(0, \ldots, 0, \alpha, \ldots, \alpha) \leq f(x_1 x_2 \ldots x_n) \leq f(\alpha, \ldots, \alpha, 1, \ldots, 1) = \alpha$$

Hence we obtain the equality

$$f(x_1, x_2, \ldots, x_n) = \alpha$$

Statement 11 *If the function is associative, continuous, monotonic and*

$$f(1, \ldots, 1, \alpha) = \alpha,$$

then $f(1, 1, \ldots, 1, z) = z$ *for every* z *that belongs to the segment* $[\alpha, 1]$*.*

Proof. From associativity we have the relationship

$$f(1,\ldots,1,x) = f(1,1,\ldots,f(1,\ldots,1,x)).$$

Because the function $f(x_1, x_2, \ldots, x_n)$ is continuous and varying x from α to 1, the function takes values from α to 1, which means that substituting $f(1,\ldots,1,x)$ by z we have a variation of z from α to 1 and therefore we obtain equality $f(1,\ldots,1,z) = z$ for $\forall z \in [\alpha, 1]$.

Statement 12 *If $f(1,\ldots,1,z) = z \forall z \in [\alpha,1]$ and $f(x_1, x_2, \ldots, x_n)$ is a monotonic function then $f(x_1, x_2, \ldots, x_n) = \min(x_1, x_2, \ldots, x_n)$, for values x_i which exceed α.*

Proof. We make the notation $m = \min(x_1, x_2, \ldots, x_n)$. Then we have the relationship

$$m = f(1,\ldots,1,m) \geq f(x_1, x_2, \ldots, x_n) \geq f(m, m, \ldots, m) = m$$

Therefore $f(x_1, x_2, \ldots, x_n) = \min(x_1, x_2, \ldots, x_n)$, for values x_i which exceed α.

Theorem 1 *The associative, simmetrical, monotonic and continuous averiging functions have the following form:*

$$med(x_1,\ldots,x_n,\alpha) = \begin{cases} \max x_1,\ldots,x_n), \textit{if } \max(x_1,\ldots,x_n) \leq \alpha \\ \alpha, \textit{if } \min(x_1,\ldots,x_n) \leq \alpha \leq \max(x_1,\ldots,x_n) \\ \min(x_1,\ldots,x_n), \textit{if } \min(x_1,\ldots,x_n) \geq \alpha \end{cases}$$

Proof. We make the following notation:

$$f(0,0,\ldots,1) = \alpha$$

Then from the Statement 6 we have that $f(0,0,\ldots,0,z) = z$ for every z that belongs to the segment $[0,\alpha]$. It follows from the Statement 9 that $f(x_1, x_2, \ldots, x_n) = \max(x_1, x_2, \ldots, x_n)$, for values x_i which dont́ exceed α. From the Statement 10 it follows that

$f(x_1, x_2, \ldots, x_n) = \alpha$ for every sequence which has values both greater than α and smaller than α. From the Statement 7 and the Statement 8 we deduce that $f(1, \ldots, \alpha) = \alpha$, and from the Statement 11 we obtain that $f(1, \ldots, z) = z$ for every z which belongs to the segment $[\alpha, 1]$. Further we apply the Statement 12 and we obtain that $f(x_1, x_2, \ldots, x_n) = \min(x_1, x_2, \ldots, x_n)$, for values x_i which exceed α.

References

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